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# APM 421/ MAT 1723 fall 2005, incomplete lecture notes

These notes probably have many misprints and typos. If something looks wrong, it probably is!

# 1. Basics of Hilbert spaces

1.1. **introductory material.** definition of a Hilbert space, inner product, and other fundamental notions, orthogonality, existence of orthogonal projections, characterization of bounded linear functionals.... existence of orthonormals bases; characterization of separable Hilbert spaces as those admitting a finite or countable orthonormal basis

1.2. bounded and unbounded operators. In these notes all operators are linear. That is, every operator S that we will consider satisfies

$$S(a\psi + b\phi) = aS\psi + bS\phi.$$

for all  $a, b \in \mathbb{C}$  and  $\psi, \phi \in \mathcal{H}$ . We always assume that this holds, generally without explicitly mentioning it. And when we write things like "bounded operator" we always mean "bounded linear operator."

A bounded operator S from  $\mathcal{H}$  to  $\mathcal{H}'$  is a map  $\mathcal{H} \to \mathcal{H}'$  for which there exists a constant C such that

(1) 
$$\|S\psi\|_{\mathcal{H}'} \le C \|\psi\|_{\mathcal{H}}$$

for all  $\psi \in \mathcal{H}$ .

An unbounded operator from  $\mathcal{H}$  to  $\mathcal{H}'$  is an operator from a subspace  $D(A) \subset \mathcal{H}$  to  $\mathcal{H}'$ . The subspace D(A) is called the domain of A. It is part of the definition of the operator; if  $A_1$ and  $A_2$  are unbounded operators such that  $D(A_1) \neq D(A_2)$  but  $A_1 = A_2$  on  $D(A_1) \cap D(A_2)$ , then  $A_1, A_2$  are different operators.

If we say, "A is an operator", then A could be either bounded or unbounded. If A is bounded then  $D(A) = \mathcal{H}$ .

An operator on  $\mathcal{H}$  (bounded or unbounded) is an operator from  $\mathcal{H}$  to itself.

**example** Let  $\mathcal{H}$  be a Hilbert space, and let D be a subspace of  $\mathcal{H}$  such that  $D(A) \neq \mathcal{H}$ . Define

$$D(A) = D, \quad A\psi = \psi \text{ for } \psi \in D(A).$$

For examples of this sort, our terminology is rather unsatisfactory. According to our definitions, A is an unbounded operator. It cannot be a bounded operator, because for a bounded operator, the domain is *by definition* the whole Hilbert space  $\mathcal{H}$ . So, instead of writing things like, "A is bounded", we can only write things like "A is clearly bounded in the sense that  $\|A\psi\|_{\mathcal{H}} \leq \|\psi\|_{\mathcal{H}}$  for all  $\psi$  in the domain of A, and hence A extends to a bounded operator on all of  $\mathcal{H}$ ."

(This last fact about extending to a bounded operator on all of  $\mathcal{H}$  is a consequence of the Hahn-Banach Theorem. If D(A) is dense and  $||A\psi||_{\mathcal{H}} \leq C ||\psi||_{\mathcal{H}}$  for all  $\psi$  in the domain of A, then an easy argument shows that A extends to a *unique* bounded operator on  $\mathcal{H}$ .

An operator A is closed if its graph

$$\{(\phi, A\psi) \in \mathcal{H} \oplus \mathcal{H} : \psi \in D(A)\}$$

is closed in  $\mathcal{H} \oplus \mathcal{H}$ ; in other words, A is closed if:

$$\psi_n \in D(A), \psi_n \to \psi, A\psi_n \to \phi$$

imply that

 $\psi \in D(A)$  and  $A\psi = \phi$ .

The nullspace of A, denoted N(A), is defined by

$$N(A) = \{ \psi \in D(A) : A\psi = 0 \}.$$

The range of A, denoted R(A), is defined by

$$N(A) = \{A\psi : \psi \in D(A)\}.$$

An operator A, bounded or unbounded, is said to be one-to-one if  $N(A) = \{0\}$ . If A is one-to-one we define  $A^{-1}$  by

$$D(A^{-1}) = R(A), \quad A^{-1}\psi = \phi \iff \psi = A\phi.$$

An operator A is said to be invertible if  $A^{-1}$  exists and is bounded.

Using the open mapping theorem (a basic result in functional analysis) one can show the following:

**Lemma 1.** Suppose that A is a closed operator on  $\mathcal{H}$ , bounded or unbounded, and that  $N(A) = \{0\}$ . Then there exists some C such that

$$||A^{-1}\psi|| \le C||\psi|| \quad for \ all \ \psi \in D(A^{-1})$$

if and only if A is onto, i.e.  $D(A^{-1}) = \mathcal{H}$ .

### 1.3. operator adjoints.

1.3.1. *definitions.* Let A be an operator from  $\mathcal{H}$  to  $\mathcal{H}'$ . If D(A) is dense then we define  $A^* : \mathcal{H}' \to \mathcal{H}$  by

(2)  $D(A^*) = \{ \psi \in \mathcal{H}' : \exists C < \infty \text{ such that } \langle \psi, A\phi \rangle_{\mathcal{H}'} \le C \|\phi\|_{\mathcal{H}} \text{ for all } \phi \in D(A) \}$ 

and  $A^*\psi$  is defined by requiring that

$$\langle A^*\psi, \phi \rangle_{\mathcal{H}} = \langle \psi, A\phi \rangle_{\mathcal{H}'}$$
 for all  $\phi \in D(A)$ .

It is easy to check that, in order for  $A^*\psi$  to be well-defined, it is necessary and sufficient that D(A) be dense in  $\mathcal{H}$ .

One can also check that the following definition is equivalent to the one we have given above:

(3) 
$$D(A^*) = \{ \psi \in \mathcal{H}' : \exists \zeta \in \mathcal{H} \text{ such that } \langle \psi, A\phi \rangle_{\mathcal{H}'} \le \langle \zeta, \phi \rangle_{\mathcal{H}} \text{ for all } \phi \in D(A) \}.$$

If D(A) is dense then such  $\zeta$  is necessarily unique and we define  $A^*\psi = \zeta$ .

The adjoint of an operator is always closed. One can also prove that

**Lemma 2.** An unbounded operator A has a closed extension if and only if  $D(A^*)$  is dense. When these equivalent conditions are satisfied, the closure of A (i.e. the smallest closed extension) is precisely  $A^{**} := (A^*)^*$ .

An example of an operator that does not have a closed extension is given by

$$\mathcal{H} = \ell^2 = \{ x = (x_1, x_2, \ldots) : \sum |x_n|^2 < \infty \} \text{ with the standard inner product}$$
$$D(A) = \{ x : \exists N \text{ such that } x_n = 0 \text{ for all } n \ge N \}$$

and

$$Ax = \left(\sum_{n} x_n, 0, 0, 0, \ldots\right).$$

One can check that in this example, the closure in  $\mathcal{H} \oplus \mathcal{H}$  of the graph of A is not the graph of an operator, since it contains the subspace

span{
$$(0, e_1)$$
}  $\subset \mathcal{H} \oplus \mathcal{H}$  where  $e_1 := (1, 0, 0, 0, \ldots) \in \mathcal{H}$ .

And if one computes the domain of  $A^*$ , one finds that  $D(A^*) = (\operatorname{span}\{e_1\})^{\perp}$ . This hints at why the above lemma is true.

1.3.2. some examples. Let  $\mathcal{H} = L^2(I)$  with the standard inner product, for  $I := [0, 1] \subset \mathbb{R}$ . We consider several operators on  $\mathcal{H}$ . We define

$$D(A_1) = \{ \psi \in \mathcal{H} : \psi' \in \mathcal{H}, \psi(0) = \psi(1) = 0 \}$$
$$D(A_2) = \{ \psi \in \mathcal{H} : \psi' \in \mathcal{H}, \psi(0) = \psi(1) \}$$
$$D(A_3) = \{ \psi \in \mathcal{H} : \psi' \in \mathcal{H} \}$$

And for  $\psi \in D(A_i)$  we define

$$A_j\psi = i\hbar\psi', \qquad j = 1, 2, 3.$$

Note: when we say that  $\psi' \in \mathcal{H}$ , it means that there exists some  $\phi \in \mathcal{H}$  and some  $c \in \mathbb{C}$  such that

$$\psi(x) = c + \int_0^x \phi(s) \ ds$$

for almost every  $x \in I$ . When we make statements about pointwise values of  $\psi$ , they are understood to be statements about the function  $\tilde{\psi}(x) = c + \int_0^x \phi(y) \, dy$ . This function is continuous. In fact, we can estimate the modulus of continuity:

$$|\tilde{\psi}(x_1) - \tilde{\psi}(x_2)| \le \int_{x_1}^{x_2} |\phi(y)| \, dy \le \left(\int_{x_1}^{x_2} |\phi(y)|^2 \, dy\right)^{1/2} \left(\int_{x_1}^{x_2} 1^2 \, dy\right)^{1/2} \le \|\phi\|_{\mathcal{H}} \, |x_2 - x_1|^{1/2}.$$

In this sense, it is reasonable to talk about pointwise values of a function whose derivative belongs to  $\mathcal{H}$ .

We compute the adjoints of these operators. The start of the computation is the same; we will only need to consider different i separately towards the end.

First, recall the definition

 $D(A_j^*) = \{ \psi \in \mathcal{H} : \exists \eta \in \mathcal{H} \text{ such that } \langle \psi, A_j \phi \rangle = \langle \eta, \phi \rangle \text{ for all } \phi \in D(A_j) \}.$ 

Fix  $\psi \in D(A_i^*)$  and let  $\eta$  be as above, so that

$$\int_0^1 \bar{\psi}(i\hbar\phi') \ dx = \int_0^1 \bar{\eta}\phi \ dx$$

for all  $\phi \in D(A_j)$ . (Thus  $\eta = A_j^* \psi$ .) We let

(4) 
$$\zeta(x) = \int_0^x \eta(y) \, dy = \int_0^x A_j^* \psi(y) \, dy$$

so that  $\zeta \in \mathcal{H}$  (in fact  $\zeta$  is continuous) and  $\zeta' = \eta$ . Note also that  $\zeta(0) = 0$ . Integration by parts (which can be justified in this setting) yields

$$\int_{0}^{1} \bar{\eta}\phi \, dx = \int_{0}^{1} \bar{\zeta}'\phi \, dx = (\bar{\zeta}\phi)_{0}^{1} - \int_{0}^{1} \bar{\zeta}\phi' \, dx$$

Comparing this with the above, we find that  $\psi \in D(A_j)$  if and only if

(5) 
$$\int_{0}^{1} (i\hbar\bar{\psi})\phi' \, dx = (\zeta\phi)_{0}^{1} - \int_{0}^{1} \bar{\zeta}\phi' \, dx$$

for all  $\phi \in D(A_j)$ .

adjoint of  $A_1$ :

If  $\phi \in D(A_1)$  then  $\phi(0) = \phi(1) = 0$ , so (5) reduces to

(6) 
$$-\int_0^1 (\overline{i\hbar\psi})\phi' \, dx = -\int_0^1 \overline{\zeta}\phi' \, dx \quad \text{for all } \phi \in D(A_1).$$

Next, we claim that

$$\{\phi' : \phi \in D(A_1)\} = \{f \in \mathcal{H} : \int_0^1 f \, dx = 0\}.$$

This is easy. on the one hand, if  $\phi \in D(A_1)$  then  $\int_0^1 \phi'(y) \, dy = \phi(1) - \phi(0) = 0$ . Conversely, if f is any function such that  $\int_0^1 f(y) \, dy = 0$ , then  $\phi(x) := \int_0^x (f(y) \, dy)$  defines an element of  $D(A_1)$ . Thus (5) becomes:

(7) 
$$\int_0^1 \overline{(i\hbar\psi - \zeta)} f \, dy = 0$$

for all  $f \in \mathcal{H}$  such that  $\int_0^1 f(y) \, dy = 0$ .

This says that  $i\hbar\psi - \zeta \perp f$  whenever f is orthogonal to all constants. This condition is satisfied if and only if  $i\hbar\psi - \zeta$  is a constant function on (0, 1), so that

$$i\hbar\psi = \zeta + \text{const}$$

Recalling that  $\zeta' = \eta = A_1^* \psi$ , we conclude that  $\psi \in D(A_1^*)$  if and only if  $\psi$  has a derivative in  $\mathcal{H}$ , so that  $\psi \in D(A_3)$ ; and moreover  $A_1 \psi = i\hbar \psi'$ . Thus  $A_1^* = A_3$ .

adjoint of  $A_2$ : If  $\phi \in D(A_2)$  then  $\phi(0) = \phi(1)$ . In addition,  $\zeta(0) = 0$  by construction. So (5) reduces to

(8) 
$$-\int_0^1 (\overline{i\hbar\psi})\phi' \, dx = \zeta(1)\phi(1) - \int_0^1 \overline{\zeta}\phi' \, dx, \qquad \text{all } \phi \in D(A_2).$$

Suppose that (8) holds. Then by taking  $\phi$  to be a nonzero constant, we conclude that  $\zeta(1) = 0 = \zeta(0)$ . Then arguing as above, we see that (8) is equivalent to condition (7), which in turn is equivalent to the condition that  $\psi' \in \mathcal{H}$ .

It follows that  $D(A_2^*) \subset D(A_3)$ , and that  $i\hbar\psi = \zeta + \text{ constant}$ . Then the condition  $\zeta(0) = \zeta(1)$  implies that  $\psi(0) = \psi(1)$ , so that  $D(A_2^*) \subset D(A_2)$ .

On the other hand, if  $\psi \in D(A_2)$  then (8) holds with  $\zeta = \psi - \psi(0)$  (so that  $\zeta(0) = \zeta(1) = 0$ .)

adjoint of  $A_3$ : By similar arguments one can check that  $A_3^* = A_1^{\cdot}$ . This can also be proved by showing that  $A_1$  is closed, so that  $A_1 = ($  closure of  $A_1) = A_1^{**} = A_3^*$ , using the work we have already done above.

## 2. STATEMENT OF SPECTRAL THEOREM

2.1. **statement.** The spectral theorem can be stated in a number of ways that are in some sense equivalent but in many ways are really quite different. Here is the one that we will focus on:

**Theorem 1** (Spectral Theorem). Suppose that A is a self-adjoint operator on a Hilbert space  $\mathcal{H}$ . Then there exists a unique family  $\{E_{\lambda}\}_{\lambda \in \mathbb{R}}$  of orthogonal projection operators satisfying the following:

(9) 
$$E_{\lambda}E_{\mu} = E_{\mu}E_{\lambda} = E_{\lambda} \quad for \ \lambda \le \mu$$

(10) 
$$E_{\lambda} = \text{s-lim}_{\mu \searrow \lambda} E_{\mu}$$

(11) 
$$\operatorname{s-lim}_{\lambda \to -\infty} E_{\lambda} = 0, \quad \operatorname{s-lim}_{\lambda \to \infty} E_{\lambda} = I$$

and finally

(12) 
$$A = \int_{-\infty}^{\infty} \lambda dE_{\lambda} = \text{s-lim}_{N \to \infty} \int_{-N}^{N} \lambda dE_{\lambda}, \quad with$$

$$D(A) = \{ \psi \in \mathcal{H} : \int_{-\infty}^{\infty} \lambda^2 d\langle \psi, E_{\lambda} \psi \rangle < \infty \}.$$

 $E_{\lambda}$  is interpreted as the projection onto the subspace of  $\mathcal{H}$  generated by the set of all eigenvectors (and "generalized eigenvectors", about which more later) associated with eigenvalues less than or equal to  $\lambda$ .

 $\{E_{\lambda}\}_{\lambda\in\mathbb{R}}$  is sometimes called the *spectral family* of projection operators for A.

2.2. physical interpretation of the theorem. Suppose that we have a physical system (such as a hydrogen atom for example) that is described by a quantum mechanical model, i.e., a Hilbert space  $\mathcal{H}$  and various self-adjoint operators on  $\mathcal{H}$  corresponding to physical attributes of the system that we can observe and measure.

For now let us temporarily forget mathematics and think of an observable not as a selfadjoint operator, but rather as something that we can measure in a laboratory. Let A denote such an observable. We imagine that in our laboratory, we can prepare the Hilbert space to be in state  $\psi$ , then perform a measurement to get a number, and repeat if we like.

Given any real number  $\lambda$ , we define a new observable<sup>1</sup>, which we denote  $E_{\lambda}$ , in the following way: Every time we perform an experiment and measure A, we will say that  $E_{\lambda} = 1$  if A is measured to be less than or equal to  $\lambda$ , and  $E_{\lambda} = 0$  otherwise. Then if we perform numerous experiments and average, we will find that the expected value of  $E_{\lambda}$  is the probability that A is less than or equal to  $\lambda$ .

It is clear that we can measure A if and only if we can measure  $E_{\lambda}$  for every  $\lambda$ .

We now remember our mathematics and think of A and  $\{E_{\lambda}\}$  as self-adjoint operators. (As the notation suggests, the observables  $\{E_{\lambda}\}$  will turn out to be the spectral family from the statement of Theorem 1.) Based on what we have said we want the physical interpretation of  $E_{\lambda}$  to be, if  $\psi$  is a state (i.e. a nonzero element of  $\mathcal{H}$ ), then  $\langle E_{\lambda} \rangle_{\psi}$  should be the probability that a measurement of the system in state  $\psi$  will yield a value less than or equal to  $\lambda$ . Similarly, the probability that, for the system in state  $\psi$ , a measurement of A falls in the interval  $(\mu, \lambda]$  should correspond to  $\langle E_{\lambda} - E_{\mu} \rangle_{\psi}$ , or equivalently  $\langle E_{\lambda} \rangle_{\psi} - \langle E_{\mu} \rangle_{\psi}$ . These statements, which make sense in view of the spectral theorem, constitute a precise formulation of the measurement postulate in quantum mechanics.

We now demonstrate that (12) is consistent with the above physical considerations. Indeed, it should be the case that

$$< A >_{\psi} = \text{expected valued of } \psi$$

$$= \lim_{n \to \infty} \sum_{i=-\infty}^{\infty} \mu_i^n (\text{Probability that } \frac{i-1}{n} < A \le \frac{i}{n})$$

$$= \lim_{n \to \infty} \sum_{i=-\infty}^{\infty} \mu_i^n (< E_{i/n} >_{\psi} - < E_{(i-1)/n} >_{\psi}).$$

where for each  $i, n, \mu_i^n$  is some point in the interval  $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ . When we write down the definitions in a moment, we will see that the right-hand side is essentially an instance of the integral appearing in (12), so that the above equality becomes

$$\langle A \rangle_{\psi} = \int_{-\infty}^{\infty} \lambda \ d \langle E_{\lambda} \rangle_{\psi} .$$

The spectral theorem is ultimately a theorem about self-adjoint operators, and reasoning about laboratory measurements is completely irrelevant to the proof of the theorem. But

<sup>&</sup>lt;sup>1</sup>still using the word "observable" in the naive sense of a physical quantity that we can measure

the theorem helps justify the postulate that self-adjoint operators are interpreted as physically observable quantities and it makes possible a precise statement of the measurement hypothesis. It also facilitates the study of dynamics and related questions.

2.3. definition of terms. We now define all the terms and notation appearing in the statement of the theorem.

An operator P is a projection operator if  $P^2\psi = P\psi$  for all  $\psi \in \mathcal{H}$ . An orthogonal projection operator means the same thing as a symmetric projection operator. The reason for the word "orthogonal" is that P is symmetric if and only if the nullspace of P is orthogonal to the range of P, i.e.

$$\{\psi \in \mathcal{H} : P\psi = 0\} \perp \{P\psi : \psi \in \mathcal{H}\}.$$

Every orthogonal projection operator is bounded. (In fact it is easy to check that ||P|| = 1 for every such operator). Note that the statement that  $E_{\lambda}$  is a projection operator is redundant, since it follows from (9).

 $A = \lim_{\sigma \to \sigma_0} A_{\sigma}$  means that  $A_{\sigma} \to A$  in the operator norm, i.e. that  $\lim_{\sigma \to \sigma_0} ||A - A_{\sigma}|| = 0$ .

For bounded operators  $A = \text{s-lim}A_{\sigma}$  means that  $||A_{\lambda}\psi - A\psi|| \to 0$  for every  $\psi \in \mathcal{H}$ . It is easy to see that if  $A = \lim_{\sigma \to \sigma_0} A_{\sigma}$  then  $A = \text{s-lim}_{\sigma \to \sigma_0} A_{\sigma}$ . The converse is not true:

**Exercise 1**: construct a sequence of bounded operators  $A_n$  such that s- $\lim_{n\to\infty} A_n = 0$  but  $A_n$  does not converge to zero in the operator norm.

For unbounded operators,  $A = s - \lim_{n \to \infty} A_n$  means that

if  $\psi \in D(A)$  then there exists  $n_0$  such that  $\psi \in D(A_n)$  for all  $n \ge n_0$ ,

and  $\lim_{n\to\infty} ||A\psi - A_n\psi|| = 0$  for every  $\psi \in D(A)$ . The expression  $A\psi - A_n\psi$  makes sense for sufficiently large *n* by the condition on the domains. The condition about the domains is automatically satisfies if *A* is unbounded and  $A_n$  is bounded for every *n*.

The integral  $\int_a^b \lambda dE_\lambda$  is understood in the following way. Let P denote a partition of the interval [a, b], that is, a sequence  $a = \lambda_0 < x_1 < \ldots < \lambda_K = b$ . For a partition P, let |P| denote the size of the largest subinterval  $|P| = \max_{i=1}^K |\lambda_i - \lambda_{i-1}|$ . Then

$$\int_{a}^{b} \lambda \, dE_{\lambda} = \lim_{|P| \to 0} \sum \mu_{i} (E_{\lambda_{i}} - E_{\lambda_{i-1}}).$$

Here  $\mu_i$  denotes a point in the interval  $[\lambda_{i-1}, \lambda_i]$ . When we write such an integral, we are implicitly asserting that this limit exists and is independent of the particular sequence of partitions P considered and of the points  $\mu_i$  chosen. (The limit is understood to be in the operator norm, consistent with the notational conventions introduced above.)

The integral appearing in the definition of D(A) is defined in an analogous way, i.e.

$$\int_{-\infty}^{\infty} \lambda^2 d\langle \psi, E_{\lambda} \psi \rangle = \lim_{|P| \to 0} \sum \mu_i^2 (\langle \psi, E_{\lambda_i} \psi \rangle - \langle \psi, E_{\lambda_{i-1}} \psi \rangle).$$

It is not hard to check from (9) that  $\lambda \to \langle \psi, E_{\lambda} \psi \rangle$  is a bounded, nondecreasing function for every fixed  $\psi$ . Thus this integral has the general form

(13) 
$$\int_{-\infty}^{\infty} f(\lambda) \, dg(\lambda) = \lim_{|P| \to 0} \sum f(\mu_i) (g(\lambda_i) - g(\lambda_{i-1})).$$

It is a theorem from real analysis that such an integral is well-defined (that is, the limit exists, though it may be infinite) whenever g is a nondecreasing function and f is a nonnegative, continuous (though possibly unbounded) function. Thus the definition of D(A) makes sense.

Incidentally, the integral appearing in (13) is called a Stieltjes integral. The integral  $\int \lambda dE_{\lambda}$  in (12) should be viewed as an operator-valued version of the Stieltjes integral. One can also define  $\int f(\lambda) dE_{\lambda}$  for more general functions f. We will discuss some of this later.

2.4. examples. In all the following examples one can see directly that  $E_{\lambda}$  is the projection onto the subspace of  $\mathcal{H}$  associated with  $(-\infty, \lambda] \cap \sigma(A)$ , where  $\sigma(A)$  denotes the spectrum of A.

In finite-dimensional cases and some infinite-dimensional examples,  $\sigma(A)$  is the collection of all eigenvalues of A, and "subspace associated with" means, the corresponding eigenspaces. In general, the situation is more complicated in the infinite-dimensional setting. Later on we will give precise definitions of terms such as spectrum.

2.4.1. some finite-dimensional examples. First, let A be the self-adjoint operator on the Hilbert space  $\mathcal{H} = \mathbb{C}^n$ . We know from linear algebra that for such an operator, there are real eigenvalues  $\{\lambda_1, \ldots, \lambda_n\}$  (repeated according to multiplicity) and an orthonormal basis  $\{v_1, \ldots, v_n\}$  of  $\mathbb{C}^n$  consisting of eigenvectors of A, so that

$$\langle v_i, v_j \rangle = \delta_{ij}, \qquad Av_i = \lambda_i v_i.$$

And A is completely determined by these eigenvalues and eigenvectors. In fact

(14) 
$$Aw = \sum_{i=1}^{n} \langle v_j, w \rangle \lambda_j v_j.$$

We next give a different description of A, in terms of a spectral family as in Theorem 1. It will be clear that (in the finite dimensional case) this is just a different way of encoding all the information about eigenvalues and eigenvectors.

For each j, let  $P_j$  be the orthogonal projection operator onto the span of the jth eigenvector  $v_j$ , so that

$$P_j w = \langle v_j, w \rangle v_j.$$

Then for every  $\lambda \in \mathbb{R}$ , define

$$E_{\lambda} := \sum_{\{j:\lambda_j \le \lambda\}} P_j.$$

We claim that this family  $E_{\lambda}$  is the spectral family of projection operators for A.

**Exercise 2**: Verify that  $\{E_{\lambda}\}$  is a spectral family for the operator A:

**a.** Prove (in 1-2 lines) that  $P_j P_k = 0$  if  $j \neq k$  and  $P_j^2 = P_j$ . Use this to give a concise (1-2 additional lines) proof of (9).

**b.** Prove (in one short line) that each  $P_j$  is symmetric, and deduce (in one short line) that each  $E_{\lambda}$  is symmetric.

c. Note (without writing anything down) that (10), (11) are obvious.

**d.** Use the definition of the integral to verify, by taking the limit of the sequence of approximating sums, that

$$A = \int_{-\infty}^{\infty} \lambda dE_{\lambda}$$

where A is defined in (14).

**Exercise 3**: A very concrete example: suppose that A is the self-adjoint operator on  $\mathcal{H} = \mathbb{C}^4$  represented in the standard basis by the 4 by 4 matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

What is the spectral family  $\{E_{\lambda}\}$  for A?

(Please just write down the answer, without giving any more justification than an appeal to Exercise 2. This is all that is needed, and if you try to justify it in detail, you will only end up repeating in this concrete setting the arguments from Exercise 2.)

2.4.2. an infinite-dimensional examples with discrete spectrum. Even in infinite-dimensional Hilbert spaces, as long as A is an operator with a complete, orthonormal basis of eigenvectors, then the situation is a lot like the finite-dimensional case, in that we can represent A either in some way analogous to (14), or via a spectral family as in Theorem 1.

For example, let I be the unit interval  $I = (0, 1) \subset \mathbb{R}$ , and let  $\mathcal{H} = L^2(I)$  be the space of square integrable functions  $I \to \mathbb{C}$ . Define

$$D(A) = \{ \psi \in \mathcal{H} : \psi' \in \mathcal{H}, \psi(0) = \psi(1) \}$$

and

$$(A\psi) = i\psi'(x).$$

Let  $e_j(x) = e^{-2\pi i j x}$ , for  $j \in \mathbb{Z}$ , and note that

$$Ae_j = 2\pi j e_j$$

Thus each  $e_j$  is an eigenfunction, with eigenvalue  $\lambda_j = 2\pi j$ . It is a basic fact from fourier series (mentioned several times in this class) that  $\{e_j\}_{j=-\infty}^{\infty}$  form a complete orthonormal basis for  $\mathcal{H}$ . Thus we have found a complete orthonormal basis of  $\mathcal{H}$  consisting of eigenvectors of A, putting us in the situation discussed above.

We now discuss different ways of using this spectral information (eigenvalues and eigenvectors) to represent A. The analog of (14) is the formula

$$A\psi = \sum_{j=-\infty}^{\infty} \langle e_j, \psi \rangle \lambda_j e_j$$

for  $\psi \in D(A)$ . This is easy to check if one knows the relevant background from analysis.

And (*exactly* as in the finite-dimensional case) we can rewrite the above in terms of a spectral family: define

$$P_j w = \langle e_j, \psi \rangle e_j$$

Then for every  $\lambda \in \mathbb{R}$ , define

$$E_{\lambda} := \sum_{\{j:\lambda_j \le \lambda\}} P_j.$$

(Note, these are exactly the same formulas as before.) Then one can check that  $\{E_{\lambda}\}$  is a spectral family for A. The verification is similar to that in the finite-dimensional case, except that one has to be a bit careful about domains. I am not going to call this an exercise, but I suggest that you think about it.....

2.4.3. an infinite-dimensional examples with continuous spectrum. Now let  $\mathcal{H} = L^2(\mathbb{R})$ , and let define an operator A by

$$D(A) = \{ \psi \in \mathcal{H} : \int_{\mathbb{R}} x^2 |\psi(x)|^2 \, dx < \infty \},$$
$$(A\psi)(x) = x\psi(x).$$

This is an operator that does not have any eigenvalues and eigenfunctions; that is, for every real number  $\lambda$ , there is no solution  $\psi \in \mathcal{H}$  of the equation  $A\psi = \lambda\psi$ . However, we will later see (after we define "spectrum") that the spectrum of A is the whole real line  $\mathbb{R}$ .

Because there are no eigenvalues and eigenfunctions, we cannot hope to write A in a form similar to (14). However, we can represent it in terms of a spectral family.

For each  $\lambda \in \mathbb{R}$ , define an operator  $E_{\lambda}$  on  $\mathcal{H}$  by:

$$(E_{\lambda}\psi)(x) = \begin{cases} \psi(x) & \text{if } x \leq \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Thus  $E_{\lambda}$  is the multiplication operator associated with the multiplier  $m_{\lambda}(x) = \begin{cases} 1 & \text{if } x \leq \lambda, \\ 0 & \text{otherwise.} \end{cases}$ 

**Exercise 4**: Verify that  $\{E_{\lambda}\}$  is a spectral family for the operator A. In other words, verify that conditions (9) through (12) are satisfied.

Verifying conditions (9) through (11) should be quick and easy. The hard part is using the definition of the integral to check that (12) holds.

Note that for this operator A, recalling that A is normally understood as the "position" operator, the physical interpretation from Section 2.2 reduces to this: If the system is in state  $\psi$  (normalized so that  $\|\psi\| = 1$ ) then the probability of measuring the position in the interval (a, b] is given by

$$\langle \psi, (E_b - E_a)\psi \rangle = \int_a^b |\psi|^2 dx.$$

This is of course consistent with our earlier discussions.

### 3. Sketch of the proof of the Spectral Theorem

There are a number of proofs of the spectral theorem. Most of them start by proving the theorem first for bounded symmetric operators, and then deducing the general result from the bounded case. The general case can also be deduced from a similar spectral theorem that applies to unitary operators. This is the approach followed by von Neumann,<sup>2</sup> who gave the first proof of the theorem, and it is the one I will sketch below.

### 3.1. the spectral theorem for bounded symmetric operators.

 $<sup>^{2}</sup>$ Von Neumann's motivation was precisely to provide a firm mathematical foundation for quantum mechanics.

3.1.1. motivation. To understand the idea, let us first consider a self-adjoint operator A on a finite dimensional Hilbert space  $\mathcal{H} = \mathbb{C}^n$ . Given arbitrary  $\lambda \in \mathbb{R}$ , we will construct the operator  $E_{\lambda}$  corresponding to the projection onto the subspace of  $\mathcal{H}$  spanned by all eigenvectors with eigenvalues  $\leq \lambda$ . Moreover, we would like to have a proceduce for constructing  $E_{\lambda}$  that we can use in the infinite-dimensional setting.

If A is a self-adjoint operator on  $\mathbb{C}^n$ , then we know from linear algebra that there exists a unitary operator U such that  $D := UAU^*$  is diagonal, with real entries  $\lambda_1, \ldots, \lambda_n$  along the diagonal. (Here  $U^*$  is the complex conjugate of the transpose of U. Because U is unitary,  $U^* = U^{-1}$ .) For any polynomial  $p(x) = \sum_{k=0}^{M} a_k x^k$ , it is clear that

$$p(A) = p(U^*DU) = U^* p(D) U.$$

And P(D) is a diagonal matrix with  $p(\lambda_1), \ldots, p(\lambda_n)$  along the diagonal.

We might suppose that if  $u : \mathbb{R} \to \mathbb{R}$  is any function that can be approximated well by polynomials, then (by some sort of approximation procedure) we should be able to define u(A), and we should have the identity

$$u(A) = U^* u(D) \ U$$

for A as above.

Suppose we are given  $\lambda \in \mathbb{R}$  and we want to construct  $E_{\lambda}$ . We will try to prove that the function  $e_{\lambda} : \mathbb{R} \to \mathbb{R}$ , defined by

(15) 
$$e_{\lambda}(x) = \begin{cases} 1 & \text{if } x \leq \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

can be approximated sufficiently well by polynomials, so that  $e_{\lambda}(A)$  makes sense. If we can do this, we can define  $E_{\lambda} = e_{\lambda}(A)$ , and we will have

$$E_{\lambda} = e_{\lambda}(A) = U^* e_{\lambda}(D) U_{\lambda}$$

with  $e_{\lambda}(D)$  a diagonal matrix with 1s and 0s along the diagonal, 1s for the eigenvectors with eigenvalue  $\leq \lambda$ , and 0s for the other eigenvectors. Thus, recalling (from linear algebra) that the rows of U form an orthonormal basis of  $\mathcal{H}$  consisting of eigenvectors of A, we see that  $E_{\lambda}$  is exactly the projection operator that we seek.

Note that, although our reasoning involved diagonalizing matrices, the construction of  $E_{\lambda}$  that we have suggested in the end does not require that we know how to diagonalize A, only that we have a consistent way of defining functiong u(A) for a large class of functions including  $e_{\lambda}$  as defined above.

3.1.2. *a crucial proposition.* Thus a main point in the proof of the spectral theorem in the case of bounded operators (but infinite-dimensional Hilbert spaces) is the following:

**Proposition 1.** Let A be a bounded symmetric operator on a Hilbert space  $\mathcal{H}$ , and suppose that

(16) 
$$m \|\psi\|^2 \le \langle \psi, A\psi \rangle \le M \|\psi\|^2 \quad \text{for all } \psi \in \mathcal{H}$$

for  $-\infty < m \leq M < \infty$ . Let  $C_u(m, M)$  denote the class of upper semicontinuous functions on [m, M]. Then for any  $u \in C_u(m, M)$  there exists a unique bounded, symmetric operator u(A). Moreover, for any  $u, v \in C_u(m, M)$  and  $a \in \mathbb{R}$ , the map  $u \mapsto u(A)$  is:

homogeneous: (au)(A) = au(A)

additive: (u+v)(A) = u(A) + v(A)multiplicative: (uv)(A) = u(A)v(A)monotone:  $u \ge v \Longrightarrow u(A) \ge v(A)$ 

Finally, the mapping  $u \mapsto u(A)$  is continuous in the following senses: first

if 
$$u_n(x) \searrow u(x)$$
 for all  $x \in [m, M]$ , then  $u(A) = \operatorname{s-lim}_{n \to \infty} u_n(A)$ .

And second,

$$\|u(A)\| \le \max_{x \in m, M} |u(x)|,$$

so that in particular, if  $u_n \to u$  uniformly in [m, M], then  $\lim_{n\to\infty} ||u(A) - u_n(A)|| = 0$ .

If A, B are symmetric operators, then  $A \ge B$  means that  $\langle \psi, A\psi \rangle \ge \langle \psi, B\psi \rangle$  for all  $\psi$ . Thus condition (16) can be written as  $mI \le A \le MI$ , or still more concisely as  $m \le A \le M$ .

The idea of the proof is as follows.

1. First, check that homgeneity, additivity etc hold when u, v are polynomial functions. All of these except monotonicity are almost immediate. To establish monotonicity, it suffices to show that if p is a polynomial such that  $p(x) \ge 0$  for  $x \in [m, M]$ , then  $p(A) \ge 0$ . This can be done by factoring p to write p(A) as a product of nonnegative operators, then using Lemma 5 below.

To factor p as a product of nonnegative operators, we claim that any polynomial p with real coefficients which is nonnegative on [m, M] can be written in the form

(17) 
$$p(\lambda) = c(\lambda - a_1) \cdots (\lambda - a_j)(b_1 - \lambda) \cdots (b_k - \lambda)q_i(\lambda) \cdots q_\ell(\lambda)$$

with c a nonnegative number,  $a_i \leq m$ ,  $b_i \geq M$ , and  $q_i$  a nonnegative quadratic for all *i*. The point is that, first, there can be no real roots of odd multiplicity in the interval (m, M); and second, if  $z_i$  is a complex root, then so is  $\bar{z}_i$ , and hence  $q_i(\lambda) := (\lambda - z_i)(\lambda - \bar{z}_i)$  is a nonnegative quadratic.

Having factored p as in (17), it is not hard to check that, for  $m \leq A \leq M$ , each factor  $(A - a_i I), (b_i I - A)$  and  $q_i(A)$ , is a positive operator.

**2**. Recall that a function  $u : \mathbb{R} \to \mathbb{R}$  is upper semicontinuous if  $u(x) \ge \limsup_{y \to x} u(y)$  for every  $x \in \mathbb{R}$ . A lemma states that if  $u \in C_u(m, M)$ , then there is a sequence  $q_n$  of polynomials such that  $q_n(x)$  decreases monotonically to u(x) as  $n \to \infty$ , for every  $x \in [m, M]$ . The proof of this uses the (nontrivial) Stone-Weierstrass theorem from real analysis, as well as some elementary arguments. <sup>3</sup>

**3**. Given  $u \in C_u(m, M)$ , we can thus select a sequence of polynomials  $q_n$  such that  $q_n(x)$  decreases monotonically to u(x) for all  $x \in [m, M]$ . In view of the monotonicity property of the map  $A \to u(A)$  when u is a polynomial, already established at this point in the argument, it follows that  $Q_n = q_n(A)$  is a monotonically decreasing sequence of operators. We can then use Lemma 3 below to find a limit of this sequence of operators, and then to check that the limit is independent of the particular sequence of polynomials chosen. We define u(A) to be this limit.

4. Once u(A) is defined for all  $u \in C_u(m, M)$  by this limiting procedure, the last step is to check that the properties of homogeneity etc are still satisfied by this expanded definition. This is straightforward.

<sup>&</sup>lt;sup>3</sup>This step is the place in the proof where we use the assumption that u is upper semicontinuous. If we wished, we could replace this by the assumption of lower semicontinuity, and instead use monotonically increasing sequence of approximating polynomials.

5. The first continuity assertion (monotone convergence of functions implies strong convergence of operators) follows from the monotonicity property of the map  $u \mapsto u(A)$  and Lemma 3.

The second continuity assertion follows again from the monotonicity property of  $u \mapsto u(A)$ , which implies that

$$-(\max_{[m,M]} |u|) \ I \le u(A) \le (\max_{[m,M]} |u|) \ I$$

and Lemma 4 below. This completes the proof.

Here are the lemmas needed above. They are mostly quite fundamental and used very often.

**Lemma 3.** If  $\{A_n\}_{n=1}^{\infty}$  is a sequence of bounded, symmetric operators such that

$$mI \le A_n \le A_{n+1} \le MI$$

for all n, then there exists a symmetric operator A such that  $s-\lim_{n\to\infty}A_n = A$ .

Proof. We use the generalized Cauchy-Schwarz inequality

$$\langle A\psi,\phi
angle \leq \langle A\psi,\psi
angle^{1/2} \ \langle A\phi,\phi
angle^{1/2} \quad \forall\psi,\phi\in\mathcal{H}, \text{ when } A\geq 0.$$

The proof of this inequality exactly follows the usual proof of the Cauchy-Schwarz inequality, starting from the positivity of A, which implies that  $\langle A(\psi - \lambda \phi), \psi - \lambda \phi \rangle \geq 0$ . Using the above inequality, one easily checks that for every  $m \leq n$  say, and any  $\psi \in \mathcal{H}$ ,

$$\|(A_n - A_m)\psi\|^4 \le \langle (A_n - A_m)\psi,\psi\rangle \quad \langle (A_n - A_m)^2\psi, (A_m - A_n)\psi\rangle \\\le \langle (A_n - A_m)\psi,\psi\rangle \|A_m - A_n\|^3\|\psi\|^2.$$

Also,  $0 \le A_m - A_n \le (M - m)I$ , so Lemma 4 below implies that  $||A_m - A_n|| \le M - m$ . It follows that

$$\|(A_n - A_m)\psi\|^4 \le (\langle A_n\psi,\psi\rangle - \langle A_m\psi,\psi\rangle) \|A_m - A_n\|^3 \|\psi\|^2.$$

The assumptions imply that  $\{\langle A_n\psi,\psi\rangle\}$  is a bounded monotone sequence, and hence convergent, and the above inequality thus implies that  $\{A_n\psi\}$  is a Cauchy sequence in  $\mathcal{H}$ .  $\Box$ 

Lemma 4. If A is a bounded symmetric operator, then

$$\sup_{\|\psi\|=1} |\langle A\psi, \psi\rangle| = \sup_{\|\psi\|=1} \|A\psi\|$$

In particular, if  $mI \le A \le MI$ , then  $||A|| \le \max\{-m, M\} \le \max\{|m|, |M|\}$ .

*Proof.* It follows from the Cauchy-Schwarz inequality that  $\langle A\psi, \psi \rangle \leq ||A||$  whenever  $||\psi|| \leq 1$ . The other inequality in the above lemma is quite easy to obtain *once* one notices that

$$\|A\psi\|^{2} = \frac{1}{4} \left[ \left\langle A(\lambda\psi + \frac{A\psi}{\lambda}), \lambda\psi + \frac{A\psi}{\lambda} \right\rangle - \left\langle A(\lambda\psi - \frac{A\psi}{\lambda}), \lambda\psi - \frac{A\psi}{\lambda} \right\rangle \right]$$

for any  $\lambda \in \mathbb{R}$ . This is verified by expanding the right-hand side. If we let  $N_A := \sup_{\|\psi\|=1} |\langle A\psi, \psi \rangle|$ , then the right-hand side above is bounded by

$$\frac{1}{4}\left(N_A \|\lambda\psi + \frac{A\psi}{\lambda}\|^2 + N_A \|\lambda\psi - \frac{A\psi}{\lambda}\|^2\right) = \frac{1}{2}N_A\left(\lambda^2 \|\psi\|^2 + \frac{\|A\psi\|^2}{\lambda^2}\right).$$

In particular, if we set  $\lambda^2 = \|\psi\|^{-1} \|A\psi\|$ , we finally find that  $\|A\psi\|^2 \leq N_A \|\psi\| \|A\psi\|$  for all  $\psi$ , which proves the lemma.

Next, in the sketch of the proof of the proposition we also have used

**Lemma 5.** If A, B are bounded symmetric operators such that  $A \ge 0, B \ge 0$ , and AB = BA, then  $AB \ge 0$ .

*Proof.* The proof uses Lemma 6 below, which asserts the existence of a positive symmetric square root  $A^{1/2}$  of a positive symmetric operator A, which commutes with every operator that commutes with A. Then

$$\langle AB\psi,\psi\rangle = \langle A^{1/2}A^{1/2}B\psi,\psi\rangle = \langle A^{1/2}B\psi,A^{1/2}\psi\rangle = \langle BA^{1/2}\psi,A^{1/2}\psi\rangle \ge 0$$

using the positivity of B at the end.

**Lemma 6.** If A is a positive symmetric bounded operator, the A has a unique positive symmetric square root, i.e., an operator, denoted  $A^{1/2}$  or  $\sqrt{A}$ , such that  $A^{1/2}$  is positive and symmetric and satisfies  $(A^{1/2})^2 = A$ . This operator commutes with every operator that commutes with A.

*Proof.* existence: The idea is to construct  $A^{1/2}$  as a limit of a sequence of polynomials  $p_n(A)$ . We would like to construct this sequence such that  $p_n(A)$  can be shown to converge using Lemma 3, and so we have to be able to verify that  $p_{n+1}(A) - p_n(A) \leq 0$ . Note that we cannot argue that this follows from checking that  $p_{n+1}(x) - p_n(x) \leq 0$  for  $x \in [m, M]$ ; this is actually what we are trying to prove. (The fact in question is established in Step 1 of the proof of the Proposition 1. The argument there relies on Lemma 5, which in turn relies on the lemma that we are currently proving.)

However, it is easy to check directly that if  $A \ge 0$ , then  $A^k \ge 0$  for every k. So we can make sure that  $p_n(A) - p_{n+1}(A) \ge 0$  by arranging for  $p_n - p_{n+1}$  to be a polynomial with nonnegative coefficients.

It is convenient to modify the above plan slightly by arguing as follows: First, we may assume that  $0 \le A \le I$ . Second, to construct a solution  $X = A^{1/2}$  of the equation  $X^2 = A$ , let us look for Y = I - X, and let us write B = I - A. Then the equation we are trying to solve becomes

$$Y = \frac{1}{2}(Y^2 + B)$$

We define a sequence of operators  $\{Y_n\}$  by

$$Y_0 = 0,$$
  $Y_{n+1} = \frac{1}{2}(Y_n^2 + B).$ 

In other words,  $Y_n = q_n(B)$ , where  $q_0 = 0$  and  $q_{n+1}(x) = \frac{1}{2}q_n^2(x) + x$ . We will show that this is an increasing sequence of operators. (Clearly Lemma 3 implies to increasing sequences as well as decreasing.) Thus

$$q_{n+1} - q_n = \frac{1}{2}(q_n - q_{n-1})(q_n + q_{n-1})$$

and it follows easily by induction that  $q_{n+1} - q_n$ , and hence  $q_{n+1}$ , are polynomials with nonnegative coefficients for every n. It follows from our earlier discussion that  $Y_{n+1} \ge Y_n$  for every n. Also, since  $B \le I$ , it is easy to check that  $Y_n \le I$  for all n.

Thus lemma 3 implies that  $Y := s - \lim_{n \in \infty} Y_n$  exists. From the definition of  $Y_n$ , it follows that

$$Y = \frac{1}{2}(Y^2 + B)$$

and hence that  $\sqrt{A} = X = I - Y$  solves  $X^2 = A$ . Note that  $\sqrt{A}$  commutes with all operators that commute with A, as it is a limit of polynomials in A.

**uniqueness**: Suppose  $A \ge 0$ , and let X denote the positive square root constructed above. Let X' be another positive operator such that  $X'^2 = A$ . We must show that X = X'. Note that  $X'A = X'^3 = AX'$ , so that X' commutes with X.

Let Z and Z' denote the positive square roots constructed by the argument given above, so that  $Z^2 = A$  and  $Z'^2 = X'$ . For  $\psi \in \mathcal{H}$ , let  $\phi = (X - X')\psi$ . Then

$$||Z\phi||^{2} + ||Z'\phi||^{2} = \langle Z^{2}\phi,\phi\rangle + \langle Z'^{2}\phi,\phi\rangle = \langle X\phi,\phi\rangle + \langle X'\phi,\phi\rangle$$
$$= \langle (X+X')(X-X')\psi,\phi\rangle = \langle (X^{2}-X'^{2})\psi,\phi\rangle = 0.$$

Hence  $Z\phi = Z'\phi = 0$ . As a result,  $X\phi = X'\phi = 0$ . So

$$\|(X - X')\psi\|^2 = \langle (X - X')\psi, (X - X')\psi \rangle = \langle \phi, (X - X')\psi \rangle = \langle (X - X')\phi, \psi \rangle = 0.$$

We end this section with a remark: we assumed in Lemma 5 that A, B are bounded, positive, symmetric operators. In fact, the last assumption was redundant:

**Lemma 7.** If A is bounded and  $\langle A\psi, \psi \rangle$  is a real number for all  $\psi$ , then A is symmetric. In particular, a positive bounded operator is symmetric.

*Proof.* Suppose that  $\langle A\eta, \eta \rangle$  is a real number for all  $\eta$ . Then by expanding

$$\langle A(\psi + \lambda \phi), \psi + \lambda \phi \rangle = \langle A\psi, \psi \rangle + |\lambda|^2 \langle A\phi, \phi \rangle + \langle A\psi, \lambda \phi \rangle + \langle \lambda A\phi, \psi \rangle$$

we find that

$$\langle A\psi,\lambda\phi
angle+\langle\lambda A\phi,\psi
angle=\lambda\langle A\psi,\phi
angle+ar\lambda\overline{\langle\psi,A\phi
angle}$$

is a real number for all  $\lambda \in \mathbb{C}$ . This easily implies that  $\langle A\psi, \phi \rangle = \langle \psi, A\phi \rangle$  for all  $\psi, \phi$ . 

3.1.3. statement and proof in the bounded case. The statement below is almost exactly like in the unbounded case, except that the we assume the operator is bounded, and we strengthen the conclusions slightly.

**Theorem 2** (Spectral Theorem). Suppose that A is a bounded, self-adjoint operator on a Hilbert space  $\mathcal{H}$ , and assume that  $mI \leq A \leq MI$  for some  $m, M \in \mathbb{R}$ . Then there exists a unique family  $\{E_{\lambda}\}_{\lambda \in \mathbb{R}}$  of orthogonal projection operators satisfying the following:

(18) 
$$E_{\lambda}E_{\mu} = E_{\mu}E_{\lambda} = E_{\lambda} \quad for \ \lambda \le \mu$$

(19) 
$$E_{\lambda} = \operatorname{s-lim}_{\mu \searrow \lambda} E_{\mu}$$

(20) 
$$E_{\lambda} = 0 \text{ for } \lambda < m, \quad E_{\lambda} = I \text{ for } \lambda \ge M.$$

and finally, if  $u: [m^-, M] \to \mathbb{C}$  is any continuous function (i.e., u is continuous on  $(m-\delta, M]$ for some  $\delta > 0$ ), then

(21) 
$$u(A) = \int_{m^{-}}^{M} u(\lambda) dE_{\lambda}$$

where the Stieltjes integral denotes a limit in the operator norm of approximating sums, and the lower limit of integration  $m^-$  indicates that the approximating sums are based on partitions that begin at some number  $\lambda_0 < m$ . In particular (21) holds for  $u(\lambda) = \lambda$ .

*Proof.* Existence: For  $\lambda \in \mathbb{R}$ , define  $E_{\lambda} = e_{\lambda}(A)$ , where  $e_{\lambda}$  is defined in (15). The definition makes sense, as a consequence of Proposition 1. We must then verify that this family  $\{E_{\lambda}\}$  satisfies properties (18) - (21). The first conclusion follows directly from the fact that

$$e_{\lambda}e_{\mu} = e_{\mu}e_{\lambda} = e_{\min\{\mu,\lambda\}}$$

and the multiplicative property in Proposition 1. The second conclusion follows from the same proposition, after observing that

$$e_{\lambda} = \lim_{\mu \searrow \lambda} e_{\mu}.$$

To prove (20), note that if  $\lambda < m$ , then  $0 \le e_{\lambda} \le 0$  in [0, M], so that the monotonicity property of he map  $u \mapsto u(A)$  implies that  $0 \le e_{\lambda}(A) = E_{\lambda} \le 0$  for  $\lambda < m$ . Similarly one can check that if  $\lambda \ge M$ , the  $I \le E_{\lambda} \le I$ .

It only remains to prove (21). To do this, fix a continuous function  $\mu$ . Fix a partition P; that is, fix  $\lambda_0 < \lambda_1 < \ldots < \lambda_N$  with  $\lambda_0 < m \leq \lambda_1$  and  $\Lambda_N \geq M$ , and fix  $mu_i \in [\lambda_i, \lambda_{i+1}]$ . Define

$$u_P(\lambda) = \sum_{i=0}^M u(\mu_i) [e_{\lambda_{i+1}}(\lambda) - e_{\lambda_i}(\lambda)].$$

Then from the definition, one can check that the sum

$$u_P(A) = \sum_{i=0}^{M} u(\mu_i) [e_{\lambda_{i+1}}(A) - e_{\lambda_i}(A)]$$

is just an approximating sum for the Stieltjes integral  $\int_{m^{-}}^{M} u(\lambda) dE_{\lambda}$ . Moreover,

$$\begin{aligned} \|u_P(A) - u(A)\| &\leq \max_{\lambda \in [m,M]} |u_P(\lambda) - u(\lambda)| \\ &= \max_{\lambda \in [m,M]} \left| \sum_{i=0}^M u(\mu_i) - u(\lambda) [e_{\lambda_{i+1}}(\lambda) - e_{\lambda_i}(\lambda)] \right| \\ &= \max_{\lambda \in [m,M]} \sum_{i=0}^M |u(\mu_i) - u(\lambda)| \ [e_{\lambda_{i+1}}(\lambda) - e_{\lambda_i}(\lambda)] \\ &= \max_i \max_{\lambda_i \leq \lambda \leq \lambda_{i+1}} |u(\mu_i) - u(\lambda)|. \end{aligned}$$

This latter quantity tends to zero as the partition is refined, since a continuous function is uniformly continuous on any compact set. This proves (21).

**Uniqueness:** The uniqueness proof requires quite a lot of analysis to do in full detail, but here is the outline: Given two spectral families  $\{E_{\lambda}^{1}\}$  and  $\{E_{\lambda}^{2}\}$  for the same operator A, we fix  $\psi \in \mathcal{H}$ , and we let  $f^{i}(\lambda) = \langle E_{\lambda}^{i}\psi,\psi\rangle$  for i = 1, 2. Then for any continuous function uon [m, M], (12) implies that

$$\langle u(A)\psi,\psi\rangle = \int u(\lambda) \, df^i(\lambda)$$

for both i = 1, 2, and hence that

$$\int u(\lambda) \, df^1(\lambda) = \int u(\lambda) \, df^2(\lambda)$$

for all continuous u. (These are scalar-valued Stieltjes integrals, as discussed in Section 1). From this one can conclude that  $f^1 - f^2$  is constant; here a certain amount of work with the definition of Stieltjes integrals is required. Since (11) implies that  $f^1$  and  $f^2$  are equal when  $\lambda \leq m$ , it follows that  $f^1(\lambda) = f^2(\lambda)$  for all  $\lambda$ . And from this one can finally deduce that  $E^1_{\lambda} = E^2_{\lambda}$  for all  $\lambda$ .

For the record, the lemma we omitted in the proof of uniqueness of the spectral decomposition is:

**Lemma 8.** Suppose that  $f : \mathbb{R} \to \mathbb{R}$  is a function of bounded variation (that is, a difference  $f^1 - f^2$  of two monotone nondecreasing functions) that is right-continuous, so that  $f(\lambda) = \lim_{\mu \to \lambda} f(\mu)$  for all  $\mu$ . If

$$\int g(\lambda) \, df(\lambda) = 0$$

for all continuous g, then f is constant.

The proof is easy if f is  $C^1$ . The lemma can be proved in the generality stated here by approximating f by smooth functions, for example by mollification, a standard technique which can be found in any respectable analysis book.

3.2. the spectral theorem for unitary operators. The spectral theorem for unitary operators is this:

**Theorem 3** (Spectral Theorem for Unitary Operators). Suppose that U is a unitary operator on a Hilbert space  $\mathcal{H}$ . Then there exists a unique family  $\{E_{\varphi}\}_{0 \leq \varphi \leq 2\pi}$  of orthogonal projection operators satisfying the following:

(22) 
$$E_{\varphi_1} E_{\varphi_2} = E_{\varphi_2} E_{\varphi_1} = E_{\varphi_1} \quad \text{for } \varphi_1 \le \varphi_2$$

(23) 
$$E_{\varphi_0} = \operatorname{s-lim}_{\varphi \searrow \varphi_0} E_{\varphi}$$

(24) 
$$E_0 = 0, \quad E_{2\pi} = I$$

and finally

(25) 
$$U = \int_0^{2\pi} e^{i\varphi} dE_{\varphi}$$

The proof is very much like the proof of the spectral theorem in the bounded, selfadjoint case. A crucial point there was the monotonicity assertion of Proposition 1. The corresponding point here is supplied by the following

**Lemma 9.** Suppose that p is a polynomial such that  $p(e^{i\varphi}) \ge 0$  for all  $\varphi \in \mathbb{R}$ , and let U be a unitary operator on a Hilbert space. Then  $p(U) \ge 0$ .

The proof of this lemma hinges on the fact that if p is a polynomial satisfying the hypotheses of the theorem, then there exists a polynomial q such that  $p(e^{i\varphi}) = |q(e^{i\varphi})|^2$  for all real  $\varphi$ . From this one can check that  $p(U) = q(U)^*q(U)$ , which implies that

$$\langle P\psi,\psi\rangle = \langle q(U)^*q(U)\psi,\psi\rangle = \langle q(U)\psi,q(U)\psi\rangle = ||q(u)\psi||^2 \ge 0$$

for all  $\psi$ .

Once Lemma 9 is known, the proof of the theorem *exactly* follows the arguments in the bounded symmetric case.

3.3. the spectral theorem for unbounded self-adjoint operators. We will sketch a proof the spectral theorem for unbounded self-adjoint operators that deduces it from the corresponding result for unitary operators. One could also deduce it from the spectral theorem for bounded operators, for example by using the approximators  $A_{\lambda}$  used in Chapter 2 of GS in the proof of "existence of dynamics".

Our basic plan will be to construct a bijection between self-adjoint operators and unitary operators such that, if A is a self-adjoint operator and U is the unitary operator onto which A is mapped, then the spectral families of U and A are related in some natural way.

In the simplest case of a 1-dimensional Hilbert space, a unitary operator can be identified with a complex number u (or 1 by 1 matrix, if you prefer) such that  $\bar{u} = u^{-1}$  (so that  $|u|^2 = 1$ ) and a self-adjoint operator can be identified with a complex number a such that  $a = \bar{a}$  (so that a is in fact real.) The map  $v : \mathbb{C} \to \mathbb{C}$  defined by

(26) 
$$v(a) = \frac{a-a}{a+a}$$

has the property that it maps real numbers (i.e., self-adjoint operators) onto the unit circle (i.e., unitary operators) and in fact is a bijection of  $\mathbb{R}$  and  $\{e^{i\theta} : 0 < \theta < 2\pi\}$ ; the inverse is given by  $a(v) = i\frac{v+1}{v-1}$ .

Similarly, one can check by diagonalizing that if A is a self-adjoint operator on  $\mathcal{H} = \mathbb{C}^n$ , then  $V := (A-i)(A+i)^{-1}$  is a unitary operator on  $\mathcal{H}$ . Moreover, the map  $A \mapsto V$  is a bijection between the set of self-adjoint operators and the set {V unitary : I - V is invertible}, with inverse  $A(V) = i(I+V)(I-V)^{-1}$ . Finally, to understand the proof of the spectral theorem, it is useful to note that if  $\{\lambda_i\}_{i=1}^n$  are the eigenvalues of A, then  $\{v(\lambda_i)\}_{i=1}^n$  are the eigenvalues of V, for v as defined above in (26). The latter can also be written in the form  $\{e^{i\varphi(\lambda)}\}$  for  $\varphi(\lambda) = 2 \cot^{-1} \lambda$ .

The observations motivate the

proof of Theorem 1 (sketch). Let A be self-adjoint.

**1**. Define  $V = (A - i)(A + i)^{-1}$ . Check, using Lemma 10 below, that V is well-defined and in fact unitary.

**2.** Let  $\{F_{\varphi}\}_{0 \leq \varphi \leq 2\pi}$  be a spectral family for the unitary operator V. For  $\lambda \in \mathbb{R}$ , define  $\varphi : \mathbb{R} \to (0, 2\pi)$  by  $\varphi(\lambda) = 2 \cot^{-1} \lambda$ , so that  $v(\lambda) = e^{i\varphi(\lambda)}$  for all  $\lambda \in \mathbb{R}$ . Then define

$$E_{\lambda} = F_{\varphi(\lambda)}$$

3. The remainder of the proof consists in verifying that  $\{E_{\lambda}\}$  is a spectral family for A. It is clear from corresponding properties of  $\{F_{\varphi}\}$  that (9) and (10) hold, so it is only necessary to check the other two conditions. In checking (11), the main point is show that (I-V) is invertible, and that  $A = i(I+V)(I-V)^{-1}$ .

4. Use Lemma 11 below to verify (12). Thus, for  $\sigma > 0$  define

$$P_{\sigma} = E_{\sigma} - E_{-\sigma}, \qquad A_{\sigma} = \int_{(-\sigma,\sigma]} \lambda dE_{\lambda}$$

and verify that these satisfy the hypotheses of the lemma, and so determine a unique selfadjoint operator s- $\lim_{\sigma\to\infty} A_{\sigma}$ . It only remains to check that this operator in fact equals A, and to do this it suffices to show that (29) is satisfied. **Lemma 10.** Suppose that A is a symmetric operator on a Hilbert space  $\mathcal{H}$ . Then the following are equivalent:

- (1) A is self-adjoint
- (2) A is closed<sup>4</sup> and  $N(A^* + i) = N(A^* i) = \{0\}.$
- (3)  $Ran(A+i) = R(A-i) = \mathcal{H}.$

*Proof.* **Preliminaries**: For any operator A with dense domain (so that  $A^*$  is defined), if  $\phi \in D(A)$  and  $\psi \in N(A^* \pm i)$ , then

$$0 = \langle (A^* \pm i)\psi, \phi \rangle = \langle \psi, (A \mp i)\phi \rangle.$$

Thus  $N(A^* \pm i) \subset R(A - i)^{\perp}$ . By reversing the reasoning we obtain the opposite inclusion. Thus

(27) 
$$N(A^* \pm i) = R(A - i)^{\perp}$$

Next, if A is symmetric, then

(28) 
$$\|(A \pm i)\psi\|^2 = \|A\psi\|^2 \pm \langle A\psi, i\psi \rangle \pm \langle i\psi, A\psi \rangle + \|\psi\|^2 = \|A\psi\|^2 + \|\psi\|^2$$

for  $\psi \in D(A)$ .

(1)  $\Rightarrow$  (2): Now assume that A is self-adjoint. Since  $A = A^*$  and  $A^*$  is closed, it is clear that A is closed.

And if  $\psi \in N(A^* \pm i)$ , then (since A is self-adjoint)  $(A \pm i)\psi = 0$ , so that (28) implies that  $\psi = 0$ . Thus  $N(A^* + i) = N(A^* - i) = 0$ .

(2)  $\Rightarrow$  (3): Assume that A is symmetric and (2) holds. Then (27) implies that  $R(A \pm i)^{\perp} = \{0\}$ , which means that R(A+i) and R(A-i) are dense in  $\mathcal{H}$ . Thus, given any  $\psi \in \mathcal{H}$ , there exists a sequence  $\psi_n \in R(A+i)$  say, such that  $\psi_n \to \psi$  as  $n \to \infty$ . That is, there exists a sequence  $\phi_n \in D(A)$  such that  $(A+i)\phi_n = \psi_n \to \psi$ . Then (28) implies that

$$\|\psi_n - \psi_m\|^2 = \|(A+i)(\phi_n - \phi_m)\|^2 = \|A(\phi_n - \phi_m)\|^2 + \|\phi_n - \phi_m\|^2$$

Thus, because  $\{\psi_n\}$  is a Cauchy sequence, it follows that  $\{\phi_n\}$  and  $\{A\phi_n\}$  are both Cauchy sequences. So there exists  $\phi, \eta \in \mathcal{H}$  such that

$$\phi_n \to \phi, \qquad A\phi_n \to \eta$$

as  $n \to \infty$ . Since A is closed by assumption (2), we conclude that  $\eta = A\phi$ , and hence that

$$\psi = \lim \psi_n = \lim A\phi_n + i\phi_n = (A+i)\phi.$$

since  $\psi$  was arbitrary, this proves that  $R(A+i) = \mathcal{H}$ . The proof that  $R(A-i) = \mathcal{H}$  is exactly the same.

(3)  $\Rightarrow$  (1): Assume that A is symmetric and (3) holds. Note that  $A \subset A^*$ ; this is the definition of symmetric. Thus we only need to prove that  $A^* \subset A$ . To do this, fix  $\psi \in D(A^*)$ , and let  $\eta = (A^* + i)\psi$ . Since A + i is onto, there exists an element  $\tilde{\psi} \in D(A + i) = D(A)$  such that  $(A + i)\tilde{\psi} = \eta$ .

It suffices to prove that  $\tilde{\psi} = \psi$ .

<sup>4</sup>An operator A is closed if and only if

$$\left. \begin{array}{l} \psi_n \in D(A) \\ \psi_n \to \psi \\ A\psi_n \to \phi \end{array} \right\} \Rightarrow \psi \in D(A) \text{ and } A\psi = \phi.$$

To do this, note that, since  $A \subset A^*$ ,

$$(A^* + i)\psi = \eta = (A + i)\overline{\psi} = (A^* + i)\overline{\psi}.$$

Thus  $(A^* + i)(\psi - \tilde{\psi}) = 0$ . Moreover, since  $R(A - i) = \mathcal{H}$ , we deduce from (27) that  $N(A^* + i) = \{0\}$ , and it follows that  $\tilde{\psi} - \psi = 0$  as desired.

Finally, the proof of Theorem 1 also uses the following lemma:

**Lemma 11.** Assume that  $\{P_{\sigma}\}_{\sigma>0}$  is a family of orthogonal projection operators, and that  $\{A_{\sigma}\}_{\sigma>0}$  is a family of bounded, symmetric operators, and that the following conditions hold:

$$P_{\sigma}P_{\tau} = P_{\tau}P_{\sigma} = P_{\sigma} \quad \text{whenever } \sigma \le \tau;$$

s-lim<sub> $\sigma \to \infty$ </sub>  $P_{\sigma} = I;$ 

$$P_{\sigma}A_{\tau} \subset A_{\tau}P_{\sigma} = A_{\sigma}$$
 whenever  $\sigma \leq \tau$ .

(The notation  $B \subset C$  means that  $D(B) \subset D(C)$  and that B = C on D(B).) Then there exists a unique self-adjoint A such that

$$D(A) = \{ \psi \in \mathcal{H} : \lim_{\sigma \to \infty} \|A_{\sigma}\psi\| < \infty \}$$

and  $A\psi = \lim_{\sigma \to \infty} A_{\sigma}\psi$  for all  $\psi \in D(A)$ . In particular, A is the unique self-adjoint operator that satisfies

(29) 
$$P_{\sigma}A = AP_{\sigma} = A_{\sigma}$$

for all  $\sigma$ .

3.4. further remarks. The proof of Lemma 10 also shows that

**Lemma 12.** Suppose that A is a symmetric operator on a Hilbert space  $\mathcal{H}$ . Then the following are equivalent:

- (1) A is essentially self-adjoint (i.e.  $A^*$  is self-adjoint)
- (2)  $N(A^* + i) = N(A^* i) = \{0\}.$
- (3) Ran(A+i) and R(A-i) are dense in  $\mathcal{H}$ .

Also, it follows immediately from Lemma 10 that  $(A \pm i)$  are invertible if A is self-adjoint. In fact, it follows from (28) that  $||(A \pm i)^{-1}|| \leq 1$  for A self-adjoint.

(Actually, if A is only symmetric, it is still true, as a result of (28), that  $A \pm i$  are injective, and hence have inverses, and moreover that with  $||(A \pm i)^{-1} \leq || \leq ||\psi||$  for  $\psi \in D(A \pm i)$ . But if A is not self-adjoint, these domains are not dense, and thus  $A \pm i$  are not invertible in this case.)

One can easily deduce from the above considerations that

**Lemma 13.** If A is self-adjoint and  $\mu \in \mathbb{C}$  has nonzero imaginary part, then  $A - \mu$  is invertible, and

$$||(A - \mu)^{-1}|| \le |Im\mu|^{-1}.$$

### 4. THE SPECTRUM OF A SELF-ADJOINT OPERATOR

main results of this section: resolvent, point spectrum, continuous spectrum can be characterized in terms of spectral family  $\{E_{\lambda}\}$ .

Let A be a operator on a Hilbert space  $\mathcal{H}$ .

The resolvent set of A, denoted  $\rho(A)$ , is defined by

 $\rho(A) := \{ \lambda \in \mathbb{C} : A - \lambda I \text{ is invertible } \}.$ 

Recall that an operator is invertible if its inverse exists and is bounded, or equivalently, if it is one-to-one and onto.

The spectrum of A, denoted  $\sigma(A)$  is defined by

$$\sigma(A) := \mathbb{C} \setminus \rho(A) = \{\lambda : A - \lambda I \text{ is not invertible}\}.$$

#### Lemma 14.

First we show that the spectrum and the resolvent can be characterized in terms of the spectral family  $\{E_{\lambda}\}$ :

**Proposition 2.** Suppose that A is a self-adjoint operator on a Hilbert space  $\mathcal{H}$ , and let  $\{E_{\lambda}\}$  denote its spectral family.

- (1) If  $\lambda \mapsto E_{\lambda}$  is constant for all  $\lambda$  in an interval (a, b) then  $(a, b) \subset \rho(A)$ . In addition,  $\|(A - \lambda I)^{-1}\| \leq \max\{|a - \lambda|^{-1}, |b - \lambda|^{-1}\}.$
- (2) If  $\lambda_0$  is such that  $\lambda \mapsto E_{\lambda}$  is not constant in any interval containing  $\lambda_0$  (so that  $F_{\varepsilon} := E_{\lambda+\varepsilon} E_{\lambda-\varepsilon}$  is a nonzero projection for every  $\varepsilon > 0$ ) then  $\lambda_0 \in \sigma(A)$ .
- (3) For every  $\lambda$ ,  $E_{\lambda^{-}} := s \lim_{\mu \nearrow \lambda} E_{\mu}$  exists and is a projection operator.
- (4)  $\lambda$  is an eigenvalue if and only if  $E_{\lambda^{-}} \neq E_{\lambda}$

We further define the *point spectrum* of an operator A on a Hilbert space  $\mathcal{H}$ , denoted  $\sigma_p(A)$ , is defined by

 $\sigma_p(A) := \{\lambda \in \mathbb{C} : \lambda \text{ is an isolated eigenvalue of finite multiplicity}\}.$ 

Here "isolated" means that there exists  $\varepsilon > 0$  such that  $\sigma(A) \cap (\lambda - \varepsilon, \lambda + \varepsilon) = \{\lambda\}$ . Easy examples show that the inclusion  $\sigma_p(A) \subset \{\text{eigenvalues of } A\}$  can be strict.

If  $\lambda \in \mathbb{C}$  then a Weyl sequence for A and  $\lambda$  is a sequence  $\{\psi_n\} \subset D(A) \subset \mathcal{H}$  satisfying

$$\|\psi_n\| = 1 \text{ for all } n,$$
  
 $\|(A - \lambda)\psi_n\| \to 0 \quad \text{as } n \to \infty$ 

and

 $\psi_n \to 0$  weakly as  $n \to \infty$ .

The last condition means that for every  $\phi \in \mathcal{H}, \langle \psi_n, \phi \rangle \to 0$  as  $n \to \infty$ .

The continuous spectrum of A, denoted  $\sigma_c(A)$ , is defined by

 $\sigma_c(A) := \{\lambda \in \mathbb{C} : \text{ there exists a Weyl sequence for } A \text{ and } \lambda.\}.$ 

It is always true that  $\sigma_p(A) \cup \sigma_c(A) \subset \sigma(A)$ , regardless of whether or not A is self-adjoint. If A is self-adjoint, a stronger conclusion holds:

**Proposition 3.** If A is self-adjoint, then  $\sigma(A) = \sigma_c(A) \cup \sigma_p(A)$ , and  $\sigma_p(A) \cap \sigma_c(A) = \emptyset$ . Moreover

 $\sigma_p(A) = \{\lambda \in \sigma(A) : R(E_{\lambda+\varepsilon} - E_{\lambda-\varepsilon}) \text{ is finite-dimensional for some } \varepsilon > 0\}$ 

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 $\sigma_c(A) = \{ \lambda \in \mathbb{R} : R(E_{\lambda + \varepsilon} - E_{\lambda - \varepsilon}) \text{ is infinite-dimensional for all } \varepsilon > 0 \}.$ 

An operator can easily be diagonalized if we can find a complete set of eigenfunctions. Hence the following lemma is often useful.

**Lemma 15.** Let A be a self-adjoint operator on a Hilbert space  $\mathcal{H}$ . If  $\sigma(A)$  is countable, then there exists an orthonormal basis  $\mathcal{H}$  consisting of eigenfunctions of A.

*Proof.* The spectrum is

#### 5. Compact operators

Here we use the spectral theorem and related machinery to prove some facts about compact operators. It should be mentioned that these facts about compact operators can be proved without appealing to the spectral theorem. Indeed, it is much easier to prove them directly, than first to prove the spectral theorem and then deduce from things about the behavior of compact operators.

**definition** : If X and Y are Banach spaces, then an operators  $K : X \to Y$  is compact if, for every sequence  $\{x_n\} \subset X$  such that  $||x_n||_X \leq C$  for all n, the sequence  $\{Ax_n\} \subset Y$ has a convergent subsequence.

An operator is said to have finite rank if the dimension of the range is finite.

An operator is said to be rank-one if its range is 1-dimensional.

**Lemma 16.** If K is a compact operator and B is a bounded operator, then the following are true:

- (1) Every compact operator is bounded.
- (2) The space of compact operators is closed in the norm topology. In other words, if  $K_n$  and K are operators such that  $K_n$  is compact for every n and such that  $||K_n K|| \to 0$  as  $n \to \infty$ , then K is compact.
- (3) Every operator with finite rank is compact.

*Proof.* The first and third assertions are almost immediate. The second assertion can be proved using a diagonal argument for example.  $\Box$ 

**Proposition 4.** If  $\mathcal{H}$  is an infinite-dimensional Hilbert space and  $K : \mathcal{H} \to \mathcal{H}$  is a compact self-adjoint operator, then  $\sigma_c(A) = \{0\}$ .

*Proof.* (sketch) **1**. Take a Weyl sequence for  $A, \lambda$ , and prove (using the definition of compact) that the associated eigenvalue  $\lambda$  must equal zero. So  $\sigma_c(A) \subset \{0\}$ .

**2**. On the other hand, one can check (using the definition of compact) that every sequence  $\{\psi_n\}$  such that  $\|\psi_n\| = 1$  for all n and  $\psi_n \to 0$  weakly is a Weyl sequence for K, 0. In other words, every such sequence satisfies  $\|K\psi_n\| \to 0$ . Thus  $\{0\} \subset \sigma_c(A)$ .

**Proposition 5.** If K is compact and symmetric, K has a countable sequence of eigenvalues  $\lambda_k$  (repeated according to multiplicity) with  $\lambda_k \to 0$  as  $k \to \infty$  and a (finite or) countable orthonormal set  $\{\psi_k\}$  such that  $K\phi = \sum \lambda_k \langle \psi_k, \phi \rangle \psi_k$ .

For a general compact operator (not necessarily symmetric) K on  $\mathcal{H}$ , there exist a sequence of positive numbers  $\mu_k \to 0$  and orthonormal bases  $\{\psi_k\}$  and  $\{\eta_k\}$  such that  $K\phi = \sum \mu_k \langle \psi_k, \phi \rangle \eta_k$ .

*Proof.* **1**. In the symmetric case, this follows chiefly from Lemma 15 and Proposition 4. These imply that the spectrum is countable, and hence that there exists a (possibly uncountable) orthonormal basis, say  $\{\psi_{\alpha}\}_{\alpha \in A}$  of eigenvectors. From this it follows that

$$K\phi = \sum_{A} \langle \psi_{\alpha}, \phi \rangle \lambda_{\alpha} \psi_{\alpha}$$

where  $A\phi_{\alpha} = \lambda_{\alpha}\psi_{\alpha}$ . So to finish the proof, it remains to show that there are at most countably many eigenvectors with nonzero eigenvalues, and that they accumulate only at zero. Since the point spectrum is necessarily at most countable, and each eigenvalue in the point spectrum has only finite multipplicity, these follow from Proposition 4 and the that that the continuous spectrum by definition contains all accumulation points of the point spectrum.

2. If K is compact but not symmetric, then it follows from Lemma 16 that  $K^*K$  is compact, and it is clear that  $K^*K$  is symmetric. It is also clear that  $K^*K \ge 0$ , so that all eigenvalues of  $K^*K$  are nonnegative. Applying the proposition in the symmetric case, we obtain a sequence  $\{\psi_k\}$  of eigenvectors for  $K^*K$ , with positive eigenvalues  $\lambda_k$  tending to zero as  $k \to \infty$ , such that  $K^*K\phi = \sum \lambda_k \langle \psi_k, \phi \rangle \psi_k$ 

Let  $\eta_k = K\psi_k/||K\psi_k||$ , and let  $\mu_k = ||K\psi_k|| = \lambda_k^{1/2}$ . Then it is easy to check that

$$K\phi = \sum \lambda_k \langle \psi_k, \phi \rangle \eta_k.$$

Moreover,

$$\mu_k \mu_\ell \langle \eta_k, \eta_\ell \rangle = \langle K \psi_k, K \psi_\ell \rangle = \langle \psi_k, K^* K \psi_\ell \rangle = \lambda_k \langle \psi_k, \psi_\ell \rangle = \lambda_k \delta_{k\ell}$$

so that  $\{\eta_k\}$  form an orthonormal set.

**Corollary 1.** The space of compact operators on a Hilbert space  $\mathcal{H}$  is the closure, in the operator norm, of the space of finite rank operators.

*Proof.* It follows from Lemma 16 that the space of compact operators contains the closure of the finite-rank operators.

On the other hand, if K is a compact operator, then writing  $K\phi = \sum \mu_k \langle \psi_k, \phi \rangle \eta_k$  as in Lemma 5, we can define  $K_N \phi = \sum_{k=1}^N \mu_k \langle \psi_k, \phi \rangle \eta_k$ , and one can easily check, using the fact that  $\mu_k \to 0$  as  $k \to \infty$ , that  $||K_N - K|| \to 0$  as  $N \to \infty$ .

## 6. Dynamics

# 6.1. Stone's Theorem.

6.1.1. statement. In this section we discuss some elements of the proof of

**Theorem 4.** Suppose that A is a self-adjoint operator on a Hilbert space  $\mathcal{H}$ . Then there exists a family  $\{U_t\}_{t\in\mathbb{R}}$  of operators satisfying

(30) 
$$U_{s+t} = U_s U_t = U_t U_s, \quad U_t^* = U_{-t} \qquad for \ all \ s, t$$

(31)  $t \mapsto U_t$  is strongly continuous. In other words, s- $\lim_{\tau \to t} U_\tau = U_t$  for all t.

(32) 
$$if \ \psi \in D(A) \ then \ \lim_{h \to 0} \frac{1}{h} (U_{t+h} - U_t) \psi = i U_t A \psi = i A U_t \psi.$$

(In particular,  $U_t$  maps D(A) into D(A), for every t.)

(33) 
$$D(A) = \{ \psi \in \mathcal{H} : \lim_{h \to 0} \frac{1}{h} (U_{t+h} - U_t) \psi \text{ exists} \}.$$

Conversely, if  $\{U_t\}_{t\in\mathbb{R}}$  is a one-parameter family of operators satisfying (30), (31), then there exists a self-adjoint operator A such that (32), (33) hold. In fact, this is still true if instead of (31) we assume only the weaker condition

(34) 
$$t \mapsto U_t \text{ is weakly continuous, i.e., } \lim_{\tau \to t} \langle U_\tau \psi, \phi \rangle = \langle U_t \psi, \phi \rangle \text{ for all } t \in \mathbb{R}, \psi, \phi \in \mathcal{H}.$$

Note that (30) implies that  $U_t$  is unitary for every t. A family of operators  $\{U_t\}$  satisfying (30), (31) (respectively (34) is called a strongly continuous (resp., weakly continuous) one parameter unitary group.

We often write  $e^{itA}$  instead of  $U_t$ . Note that if A is bounded and symmetric, then one can define

$$U_t = e^{itA} := \sum_{n=0}^{\infty} \frac{1}{n!} (itA)^n.$$

Then the power series is easily seen to be convergent, and properties (30)–(33) can be verified with varying amounts of difficulty.

The statement that (30) and (34) imply the existence of a self-adjoint A satisfying (32) (33) is known as *Stone's Theorem*.

It is easy to see that if  $U_t$  is unitary for every t and (34) holds, then in fact (31) holds. Indeed, under these assumptions,

$$\begin{split} \lim_{\tau \to t} \| (U_{\tau} - U_t)\psi \|^2 &= \lim_{\tau \to t} \left( \| (U_{\tau}\psi \|^2 - 2\operatorname{Re}\langle U_{\tau}\psi, U_t\psi \rangle + \| U_t\psi \|^2 \right) \\ &= \lim_{\tau \to t} 2 \left( \| U_t\psi \|^2 - \operatorname{Re}\langle U_{\tau}\psi, U_t\psi \rangle \right) \\ &= 2 \left( \| U_t\psi \|^2 - \operatorname{Re}\langle U_t\psi, U_t\psi \rangle \right) \quad \text{by (34)} \\ &= 0. \end{split}$$

6.1.2. some ingredients in the proof. A main point in the proof of existence of the oneparameter unitary group  $\{U_t\}$ , given a self-adjoint operator A, is the following

**Lemma 17.** Let A be a self-adjoint operator, and suppose that  $f : \mathbb{R} \to \mathbb{C}$  is continuous and bounded. Then there is an operator f(A), characterized uniquely by the property

(35) 
$$\langle f(A)\psi,\phi\rangle := \int_{\mathbb{R}} f(\lambda)d\langle E_{\lambda}\psi,\phi\rangle$$

for  $\psi, \phi \in \mathcal{H}$ . Moreover,

(36) 
$$(af + bg)(A) = af(A) + bg(A)$$
 for  $a, b \in \mathbb{C}$  and  $f, g$  continuous;

(37) 
$$f_1(A)f_2(A) = (f_1f_2)(A)$$

(38) 
$$\bar{f}(A) = f(A)^*$$

(39) 
$$||f(A|| \le \sup_{\lambda \in \mathbb{R}} |f(\lambda)|$$

We have proved other results of a similar character, see in particular Proposition 1. In that earlier result, the operator A was assumed to be bounded and symmetric, so that polynomials p(A) automatically make sense, and expressions of the form u(A) for u uppersemicontinuous were eventually defined via approximating u by polynomials uniformly on a compact interval containing the spectrum of A.

In the present setting it would be much more difficult to carry out this sort of polynomial approximation argument. For example, if A is unbounded then polynomials p(A) are in general still more unbounded (ie, smaller domains). Also, continuous functions can be approximated only locally uniformly by polynomials.

sketch of the proof of Lemma 17. Step 1. We want to take (35) as the definition of f(A). We first need to check that f(A) is uniquely defined by this identity. Check, using the definition of the Stieltjes integral, that

$$\left| \int_{\mathbb{R}} f(\lambda) d\langle E_{\lambda} \psi, \phi \rangle \right| \le |f(A)| \|\psi\| \|\phi\|.$$

Conclude that for  $\psi$  fixed, the map

$$\phi \mapsto \int_{\mathbb{R}} f(\lambda) d\langle E_{\lambda} \psi, \phi \rangle$$

is a bounded linear functional, and thus there exists a unique element of  $\mathcal{H}$  — let us call it  $f(A)\psi$ , such that

$$\langle f(A)\psi,\phi\rangle = \int_{\mathbb{R}} f(\lambda)d\langle E_{\lambda}\psi,\phi\rangle$$

for all  $\phi \in \mathcal{H}$ . This shows that f(A) is well-defined.

**2**. It is easy to check that (36), (38), and (39) follows from (35). To verify (37), it suffices (by polarization) to check that

(40) 
$$\int_{\mathbb{R}} f_1(\lambda) d\langle E_\lambda f_2(A)\psi,\psi\rangle = \int_{\mathbb{R}} f_1(\lambda) f_2(\lambda) d\langle E_\lambda \psi,\psi\rangle$$

for arbitrary  $\psi \in \mathcal{H}$ . To do this, rewrite the left-hand side as follows:

$$\int_{\mathbb{R}} f_1(\lambda) d\langle E_\lambda f_2(A)\psi,\psi\rangle = \int_{\mathbb{R}} f_1(\lambda) d\langle f_2(A)\psi,E_\lambda\psi\rangle$$
$$= \int_{\mathbb{R}} f_1(\lambda) d\left(\int_{\mathbb{R}} f_2(\mu) d\langle E_\mu\psi,E_\lambda\psi\rangle\right).$$

Let  $g(\lambda) = \langle E_{\lambda}\psi, \psi \rangle$ . Using the continuity of  $f_2$ , the definition of Stieltjes integrals and the fact that  $\langle E_{\mu}\psi, E_{\lambda}\psi \rangle = g(\min\{\lambda, \mu\})$ , one checks that for every  $\lambda_1 < \lambda_2$ , there exists a  $\mu \in [\lambda_1, \lambda_2]$  such that

$$\int_{\mathbb{R}} f_2(\mu) d\langle E_{\mu}\psi, E_{\lambda_2}\psi\rangle - \int_{\mathbb{R}} f_2(\mu) d\langle E_{\mu}\psi, E_{\lambda_1}\psi\rangle = f_2(\mu)(g(\lambda_2) - g(\lambda_1)).$$

From this point it is not so hard to complete the verification of (40).

# sketch of the proof of Theorem 4. 1. self-adjoint operator generates 1-parameter unitary group:

Let A be self-adjoint. Define

$$U_t = e^{itA}$$

and verify that  $U_t$  has all the required properties. The easiest is (30), which is an immediate consequence of (37). We omit the proof of (31), which is similar to that of (32) but easier. For (32), we claim that

(41) 
$$\|(\frac{U_{t+h} - U_t}{h} - iU_t A)\psi\|^2 = \int_{\mathbb{R}} |\frac{e^{i(t+h)\lambda} - e^{it\lambda}}{h} - ie^{it\lambda}\lambda|^2 d\langle E_\lambda\psi,\psi\rangle$$
$$= \int_{\mathbb{R}} |\frac{e^{ih\lambda} - 1}{h} - i\lambda|^2 d\langle E_\lambda\psi,\psi\rangle.$$

This formally follows from (37) and (38), which, when they hold, imply that

$$||f(A)\psi||^2 = \langle \bar{f}(A)f(A)\psi,\psi\rangle = \langle |f|^2(A)\psi,\psi\rangle = \int_{\mathbb{R}} |f(\lambda)|^2 d\langle E_\lambda\psi,\psi\rangle$$

We have not verified that (37), (38) hold for unbounded functions such as the ones we consider here, so to conclude (41) we argue instead by breaking up the left-hand side into terms

$$\left\|\frac{U_{t+h} - U_t}{h}\psi\right\|^2, \left\|iA\psi\right\|^2, 2\operatorname{Re}\left\langle\frac{U_{t+h} - U_t}{h}\psi, iA\psi\right\rangle$$

then justifying for each term rewriting as a suitable Stieltjes integral, using the Spectral Theorem for  $||iA\psi||^2$  and Lemma 17 for the other two terms. When assembled, these yield (41).

To conclude (32), note that

$$\lim_{h \to 0} \left| \frac{e^{ih\lambda} - 1}{h} - i\lambda \right|^2 = 0 \qquad \qquad \left| \frac{e^{ih\lambda} - 1}{h} - i\lambda \right|^2 \le c\lambda^2$$

for every  $\lambda$ . Thus the Dominated Convergence Theorem implies that

$$\|(\frac{U_{t+h} - U_t}{h} - iU_t A)\psi\|^2 \to 0 \qquad \text{as } h \to 0$$

**Proof of** (33): · · ·

# 2. 1-parameter unitary group is generated by a self-adjoint operator:

idea: define  $A\psi = \frac{1}{i} \lim_{h \to 0} \frac{1}{h} (U_h - I) \psi$ , for suitable domain.

It is easier to define instead an operator  $\tilde{A}$  with domain  $D(\tilde{A})$  chosen to consist of a rather small subspace of  $\mathcal{H}$  on which we can easily justify all calculations that we wish to carry out, and then to verify that  $\tilde{A}$  is essentially self-adjoint.

Given  $\psi \in \mathcal{H}$  and  $f \in C_0^{\infty}(\mathbb{R}; \mathbb{C})$ , define

$$\psi_f := \int f(t) U_t \psi dt$$

Here the integral is understood as a strong limit of Riemann sums; it is easy to see, by essentially the proof from single-variable calculus, that this limit exists, since  $t \mapsto f(t)U_t\psi$  is continuous and compactly supported in t. Further define

$$D(\hat{A}) := \{ \psi_f : \psi \in \mathcal{H}, f \in C_0^\infty \}$$

and for  $\phi = \psi_f \in D(A)$ , define

$$\tilde{A}\phi = \frac{1}{i}\lim_{h\to 0}\frac{1}{h}(U_h - I)\phi.$$

Check that this limit exists, and in fact  $\tilde{A}\psi_f = \psi_{-f'} = -\psi_{f'}$ .

Check further that D(A) is dense, and check also that A is symmetric.

Verify that  $\tilde{A}$  is essentially self-adjoint by checking that  $N(\tilde{A}^* \pm i) = 0$ . To do this, suppose that  $(\tilde{A}^* + i)\psi = 0$ , so that  $\langle \psi, (\tilde{A} - i)\phi \rangle = 0$  for all  $\phi \in D(\tilde{A})$ . Fix  $\eta \in D(\tilde{A})$ , and let  $g(t) := \langle \psi, U_t \eta \rangle$ . We compute:

$$g'(t) = \frac{d}{dt} \langle \psi, U_t \eta \rangle = \langle \psi, i \tilde{A} U_t \eta \rangle = -\langle \psi, U_t \eta \rangle = -g(t).$$

It follows that  $g(t) = e^{-t}g(0)$  for all t. On the other hand, it is easy to see that  $g(t) \leq ||\psi|| ||\eta||$  for all t, from which we conclude that g(0) = 0. Thus  $\psi \perp D(\tilde{A})$ , so that  $N(A^* + i) = 0$ . Exactly the same argument shows that  $N(A^* - i) = 0$ .

Finally, let A be the unique self-adjoint extension of A, and verify that A has the required properties.  $\Box$ 

6.2. spectrum and dynamics. In this section we fix a self-adjoint operator A, and we use the notation

 $\mathcal{H}_e :=$  the closed span of the eigenvectors of A

= the smallest closed subspace containing all eigenvectors .

We write  $P_e$  to denote orthogonal projection onto  $\mathcal{H}_e$  and  $P_e^{\perp} := I - P_e$ .

**Theorem 5.** If K is a compact operator and  $\phi \perp \mathcal{H}_e$ , then

(42) 
$$\frac{1}{T} \int_0^T \|Ke^{itA}\phi\|^2 dt \to 0 \qquad \text{as } T \to \infty.$$

Moreover

(43) 
$$\left\|\frac{1}{T}\int_0^T e^{-itA}KP_e^{\perp}e^{itA} dt\right\| \to 0 \qquad \text{as } T \to \infty.$$

The integral in (43) is a s-lim of Riemann sums.

In fact, conclusions (42), (43) still hold if K is bounded and  $K(A+i)^{-1}$  is compact.

Proof.

#### 7. TROTTER PRODUCT FORMULA