APM 421/ MAT 1723 Assignment number 4, fall 2005,

This will not be collected.

Here I assemble some exercises that were mentioned during the lectures.

1. Suppose that A is a self-adjoint operator with spectral family $\{E_\lambda\}$, and that ψ is an element of the Hilbert space such that $\langle \psi, \phi \rangle = 0$ for every eigenvector ϕ of E.

   (a) Prove that $\lambda \mapsto \langle E_\lambda \psi, \psi \rangle$ is continuous.

   (b) Prove that $\lambda \mapsto \langle E_\lambda \psi, \eta \rangle$ is continuous for every fixed $\eta \in \mathcal{H}$.

2. We have proved the following theorem in the lectures:

   Theorem 1. *If K is a rank-1 operator and $\phi \perp \mathcal{H}_e$, then*

   $$\frac{1}{T} \int_0^T \|K e^{itA} \phi\|^2 \, dt \to 0 \qquad \text{as } T \to \infty. \tag{1}$$

   Moreover

   $$\left\| \frac{1}{T} \int_0^T e^{-itA} K P_e^{\perp} e^{itA} \, dt \right\| \to 0 \qquad \text{as } T \to \infty. \tag{2}$$

   The integral in (2) *is understood to be a s-lim of Riemann sums.*

   Here we use the notation

   $$\begin{aligned} \mathcal{H}_e &:= \text{the closed span of the eigenvectors of } A \\ &= \text{the smallest closed subspace containing all eigenvectors .} \end{aligned}$$

   We write P_e to denote orthogonal projection onto $\mathcal{H}_e$ and $P_e^{\perp} := I - P_e$.

   Prove that the theorem is still valid if K is a compact operator. In doing this, you can take for granted the following

   Lemma 1. *Every compact operator is a norm limit of finite-rank operators. In other words, if K is compact and $\varepsilon > 0$, then there exists an operator K_ε which is a linear combination of finitely many rank-1 operators, and such that $\|K - K_\varepsilon\| < \varepsilon$.*

3. Suppose that A is an unbounded, self-adjoint operator on a Hilbert space $\mathcal{H}$. For $\psi, \phi \in D(A)$, define

   $$[\psi, \phi] := \langle A\psi, A\phi \rangle + \langle \psi, \phi \rangle.$$

   Verify that $[\cdot, \cdot]$ is a norm, and that with this norm, $D(A)$ is a Hilbert space.