APM 421/ MAT 1723 Assignment number 3, fall 2005, due Tuesday November 8, 2005.

Do at least three of the following 6 problems. You can use the conclusion of any of the exercises in solving the other exercises, except in cases where one problem is an easy corollary of another.

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function such that $f(\lambda) = 0$ for all $\lambda < m$ and $f(\lambda) = a$ for all $\lambda \geq M$. Show that if $g : \mathbb{R} \to \mathbb{R}$ is continuous, then

$$a \inf_{[m,M]} g(\lambda) \leq \int g(\lambda) df(\lambda) \leq a \sup_{[m,M]} g(\lambda).$$

In the following questions, $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ always denotes a spectral family of projection operators associated with a (possibly unbounded) self-adjoint operator A.Exercise 1 may be useful.

2. Suppose that $\lambda \mapsto E_\lambda$ is constant for all λ in an interval (a, b).
   (a) Show that if $\lambda \in (a, b)$ then $(A - \lambda I)^{-1}$ exists and is bounded. In other words, show that $(a, b) \subset \rho(A)$.
   (b) Estimate the operator norm of $(A - \lambda I)^{-1}$ in term of $\max\{\lambda - a, b - \lambda\}$.

3. Assume that λ_0 is such that $\lambda \mapsto E_\lambda$ is not constant in any interval containing λ_0, so that $F_\varepsilon := E_{\lambda+\varepsilon} - E_{\lambda-\varepsilon}$ is a *nonzero* projection for every $\varepsilon > 0$. Prove that $\lambda_0 \in \sigma(A)$.

4. (a) Prove that $E_{\lambda^-} := s - \lim_{\mu \nearrow \lambda} E_\mu$ is a projection operator. (This limit exists because any bounded monotone sequence of symmetric operators has a limit.)
   (b) Show that $E_{\lambda^-} \neq E_\lambda$ if and only if there exists some nonzero $\psi \in \mathcal{H}$ such that $A\psi = \lambda\psi$.
   (c) Deduce that $(A - \lambda I)^{-1}$ exists if and only if $E_{\lambda^-} = E_\lambda$.
   (Thus λ belongs to the point spectrum of A if and only if $\mu \mapsto E_\mu$ is (s-)discontinuous at $\mu = \lambda$.)

The above exercises show that for a self-adjoint operator, the resolvent set, the spectrum, and the eigenvalues can all be characterized in terms of the spectral decomposition $\{E_\lambda\}$.

Before stating the next problems we give some definitions.

definition 1: The *point spectrum* of an operator A on a Hilbert space $\mathcal{H}$, denoted $\sigma_p(A)$, is defined by

$$\sigma_p(A) := \{\lambda \in \mathbb{C} \ : \ \lambda \text{ is an isolated eigenvalue of finite multiplicity}\}.$$

Here "isolated" means that there exists $\varepsilon > 0$ such that $\sigma(A) \cap (\lambda - \varepsilon, \lambda + \varepsilon) = \{\lambda\}$. (In other words, λ is isolated, not merely from other eignevalues, but also from other parts of the spectrum as well.)

definition 2: If $\lambda \in \mathbb{C}$ then a *Weyl sequence* for A and λ is a sequence $\{\psi_n\} \subset D(A) \subset \mathcal{H}$ satisfying

$$\|\psi_n\| = 1 \text{ for all } n,$$

$$\|(A-\lambda)\psi_n\| \to 0 \qquad \text{as } n \to \infty$$

and

$$\psi_n \to 0 \text{ weakly as } n \to \infty.$$

The last condition means that for every $\phi \in \mathcal{H}$, $\langle \psi_n, \phi \rangle \to 0$ as $n \to \infty$.

definition 3: The *continuous spectrum* of A, denoted $\sigma_c(A)$, is defined by

$$\sigma_c(A) := \{\lambda \in \mathbb{C} \ : \ \text{there exists a Weyl sequence for } A \text{ and } \lambda.\}.$$

It is always true that $\sigma_p(A) \sup \sigma_c(A) \subset \sigma(A)$, regardless of whether or not A is self-adjoint. If A is self-adjoint, a stronger conclusion holds:

5. If A is self-adjoint, then $\sigma(A) = \sigma_c(A) \cup \sigma_p(A)$, and $\sigma_p(A) \cap \sigma_c(A) = \emptyset$.

In fact, if A is self-adjoint then

$$\sigma_p(A) = \{\lambda \in \sigma(A) \ : \ R(E_{\lambda+\varepsilon} - E_{\lambda-\varepsilon}) \text{ is finite-dimensional for some } \varepsilon > 0\}$$

and

$$\sigma_c(A) = \{\lambda \in \mathbb{R} \ : \ R(E_{\lambda+\varepsilon} - E_{\lambda-\varepsilon}) \text{ is infinite-dimensional for all } \varepsilon > 0\}.$$

One way to do exercise 4 is by proving these statements. There may be easier ways to do it.

We state one more exercise after giving yet another definition:

definition 4: If X and Y are Banach spaces, then an operators $K : X \to Y$ is compact if, for every sequence $\{x_n\} \subset X$ such that $\|x_n\|_X \le C$ for all n, the sequence $\{Ax_n\} \subset Y$ has a convergent subsequence.

6. Show that if $\mathcal{H}$ is an infinite-dimensional Hilbert space and $K : \mathcal{H} \to \mathcal{H}$ is a compact self-adjoint operator, then $\sigma_c(A) = \{0\}$.