APM 421/ MAT 1723 Assignment number 2, fall 2005, due Tuesday October 25, 2005.

questions 1 through 4 of this assignment: do exercises 1, 2d, 3, and 4 in the handout "spectral theorem. part 1", posted on the course web site.

5. Let $\mathcal{H} = \ell^2$, here thought of as the set of doubly infinite square summable sequences $x = (x_n)_{n=-\infty}^{\infty}$, with the usual inner product $(x, y) = \sum_{-\infty}^{\infty} \bar{x}_n y_n$.

   Define
   $$D(A) = \{x \in \mathcal{H} \ : \exists N \text{ such that } x_n = 0 \text{ for all } n \geq N\}$$
   $$(Ax)_n = nx_n.$$

   Compute the adjoint A^*? Is A essentially self-adjoint?

   Note that $D(A)$ is dense. I do not want to see the proof of this fact.

6. For $\mathcal{H}$ as in the previous exercise, define A by
   $$D(A) = \{x \in \mathcal{H} \ : \sum n^2|x_n|^2 < \infty, \sum x_n = 0\}$$
   $$(Ax)_n = nx_n.$$

   Compute the adjoint A^*? Is A essentially self-adjoint?

   Note that $\sum_{n\neq 0} |x_n| = \sum_{n\neq 0} \frac{1}{n}(nx_n) \leq \left(\sum n^{-2} \sum n^2|x_n|^2\right)^{1/2} < \infty$. Thus the condition $\sum x_n = 0$ makes sense.

   One can also check that $D(A)$ is dense in $\mathcal{H}$. You need not write up the proof, which can easily be deduced from the elementary fact that, given any $z \in \mathbb{C}$ and any $\epsilon > 0$, one can find an element $x \in H$ such that $\sum x_n = z$, $\sum n^2|x_n|^2 < \infty$, and $\|x\| < \epsilon$.

7. Probe that if A is an invertible operator and if B is a bounded operator such that $\|B\| \, \|A\|^{-1} < 1$, then $A - B$ is invertible. (Here we define $D(A - B) = D(A)$, consistent with our standard conventions.)

   To do this, first note that $\sum \|(A^{-1}B)^n\| \leq \sum(\|A\|^{-1}\|B\|)^n < \infty$. It follows from an easy general result that the partial sums $\sum_{n=0}^{N}(A^{-1}B)^n$ converge in the operator norm as $N \to \infty$ to a limit, which is (naturally) denoted $\sum_{n=0}^{\infty}(A^{-1}B)^n$. We understand $(A^{-1}B)^0$ as the identity I

   (a) Prove that $\sum_{n=0}^{\infty}(A^{-1}B)^n$ provides the inverse of $I - A^{-1}B$, which is therefore an invertible operator.

   (b) Prove that if A and C are invertible, and C is bounded, then AC is invertible, with $(AC)^{-1} = C^{-1}A^{-1}$.

   recall that in general $D(A_1A_2) = \{x \in D(A_2) : A_2x \in D(A_1)\}$.

   The invertibility of $A - B$ then follows from writing $A - B = A(I - A^{-1}B)$