APM 421/ MAT 1723 Assignment number 1, fall 2005, due October 6, 2005.

Please do **not** hand in problem 1, problem 2b, or any of problems 3-6 (which are just asking you to verify in detail some easy facts asserted in the lectures.)

Please hand in problems 2a and 2c and problem 7 (if you do it). Problem 7 is harder, and students who have not seen Hilbert spaces are encouraged to look at it, but can skip it.

1. **(Do not hand in!)** find a Langranian $L : \mathbb{R}^{3N} \times \mathbb{R}^{3N} \to \mathbb{R}$ for which the Euler-Lagrange equation is

$$\ddot{\mathbf{q}}_i = -\sum_{j \neq i} m_j \frac{\mathbf{q}_i - \mathbf{q}_j}{|\mathbf{q}_i - \mathbf{q}_j|^3}, \qquad \text{i} = 1, \dots, \text{N}$$

where $\mathbf{q}_i \in \mathbb{R}^3$ represents the position of the i member of a collection of N interacting particles. The Lagrangian L should be a function of $q = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ and $v = (\mathbf{v}_1, \dots \mathbf{v}_N)$.

hint: in some cases a problem becomes easier when it is stated in a more general form. Compare the specific ODE here (with a correspondingly specific Lagrangian that you must find) to the more general Lagrangian in problem 2.

2. Consider a pair of particles in $\mathbb{R}^3$ whose mechanical behavior is governed by the Lagrangian

$$L(q, v) = L(\mathbf{q}_1, \mathbf{q}_2, \mathbf{v}_1, \mathbf{v}_2) = \frac{1}{2}(m_1|\mathbf{v}_1|^2 + m_2|\mathbf{v}_2|^2) - F(|\mathbf{q}_1 - \mathbf{q}_2|)$$

for some smooth function $F : \mathbb{R} \to \mathbb{R}$.

The Lagrangian L has 6 "symmetries", corresponding to translational invariance and rotational invariance.

(a) using the translational symmetries $\Phi_i(\mathbf{q}_1, \mathbf{q}_2, \sigma) = (\mathbf{q}_1 + \sigma e_i, \mathbf{q}_2 + \sigma e_i)$ for $i = 1, 2, 3$, find three conserved quantites (which are interpreted as conserved momenta).

hint: it is best to write the answer as a single vector of conserved quantities rather than 3 separate conserved quantities. The answer will tell you that the center of mass of the system moves with constant velocity.

(b) The Lagrangian L and the Euler-Lagrange equations can be rewritten in terms of new variables $\mathbf{Q} = (m_1 + m_2)^{-1}(m_1\mathbf{q}_1 + m_2\mathbf{q}_2)$ and $\mathbf{r} := \mathbf{q}_1 - \mathbf{q}_2$, and in these variables they become

$$\ddot{\mathbf{Q}} = 0, \qquad \ddot{\mathbf{r}} = -\frac{m_1 m_2}{m_1 + m_2} \nabla F(|\mathbf{r}|) \frac{\mathbf{r}}{|\mathbf{r}|}.$$

with Lagrangian

$$L(\mathbf{Q}, \mathbf{r}, \dot{\mathbf{Q}}, \dot{\mathbf{r}}) = \frac{1}{2}((m_1 + m_2)|\dot{\mathbf{Q}}|^2 + \frac{m_1 m_2}{m_1 + m_2}|\dot{\mathbf{r}}|^2) - F(|\mathbf{r}|)$$

Verify these facts to your satisfaction.

(c) Since the equation for $\mathbf{Q}$ is trivially solved, we now consider the reduced equation $m'\ddot{\mathbf{r}} = -\nabla F(|\mathbf{r}|)\frac{\mathbf{r}}{|\mathbf{r}|}$ with Lagrangian $L = \frac{m'}{2}|\dot{\mathbf{r}}|^2 - F(|\mathbf{r}|)$, where we are writing $m' = m_1 m_2/(m_1 + m_2)$. Find three more symmetries and the associated conserved quantities.

hint: it is clear that L depends only on the length of the vector $\mathbf{r}$ and not on its direction, so L should be preserved by rotating $\mathbf{r}$ around any axis.

remark: the answer will tell you that angular momentum is conserved. Note that these symmetries and conserved quantites have counterparts for the original system as written in terms of $\mathbf{q}_1, \mathbf{q}_2$; it is just that the algebra is clearer for the reduced system.

In the following questions, $\mathcal{H}$ denotes a Hilbert space. Do **not** hand any of problems 3-6.

3. Verify the *parallelogram law*: $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ for all $x, y \in \mathcal{H}$.

4. Verify the *polarization identity*: $4(x,y) = \|x+y\|^2 - \|x-y\|^2 \pm i(\|x+iy\|^2 - \|x-iy\|^2)$.

5. Note that an immediate corollary of the polarization identity is the fact that $U : \mathcal{H}_1 \to \mathcal{H}_2$ is a linear operator such that $\|Ux\|_{\mathcal{H}_2} = \|x\|_{\mathcal{H}_1}$ for all $x \in \mathcal{H}_1$, then U is an isometry, that is $(Ux, Uy)_{\mathcal{H}_2} = (x,y)_{\mathcal{H}_1}$ for all $x, y \in \mathcal{H}_1$.

6. Recall that if $\mathcal{M}$ is a subspace of a Hilbert space $\mathcal{H}$ then the orthogonal complement $\mathcal{M}^\perp$ of $\mathcal{M}$ is defined to be
$$\mathcal{M}^\perp := \{x \in \mathcal{H} : (x,y) = 0 \text{ for all } y \in \mathcal{M}\}.$$
Recall also that a closed set in a Hilbert space is one that contains all of its limit points (that is, all limits of convergent sequences of elements of that set.) Also, the closure of a set is the set of all of its limit points.

   Prove that

   (a) for any subspace $\mathcal{M}$ of a Hilbert space, $\mathcal{M}^\perp$ is closed.

   (b) If $\mathcal{M}$ is any subspace of $\mathcal{H}$ and $\bar{\mathcal{M}}$ is the closure of $\mathcal{M}$, then $\mathcal{M}^\perp = \bar{\mathcal{M}}^\perp$.

   (c) $\bar{\mathcal{M}} = (\mathcal{M}^\perp)^\perp$

7. Prove that if a vector space $\mathcal{V}$ has a norm $\|\cdot\|$ that satisfies the parallelogram rule, then it is in fact an inner product space, in the sense that it can be given an inner product $(\cdot,\cdot)$ such that $\|x\|^2 = (x,x)$ for all $x \in \mathcal{V}$.

   Recall that a norm on a (complex) vector space $\mathcal{V}$ is a function $\mathcal{V} \to [0, \infty)$, typically written $\|\cdot\|$, such that
   a. $\|x\| = 0$ if and only if $x = 0$,
   b. $\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in \mathcal{V}$ and $\lambda \in \mathbb{C}$
   c. $\|x+y\| \le \|x\| + \|y\|$ for all $x, y \in \mathcal{V}$.