Math. Z. 233, 365–398 (2000)

Mathematische Zeitschrift

Upper bounds on the length of the shortest closed geodesic on simply connected manifolds

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Received November 10, 1997; in final form June 23, 1998

Abstract. The subject of this paper is upper bounds on the length of the shortest closed geodesic on simply connected manifolds with non-trivial second homology group. We will give three estimates. The first estimate will explicitly depend on volume and the upper bound for the sectional curvature; the second estimate will depend on diameter, a positive lower bound for the volume, and on the (possibly negative) lower bound on sectional curvature; the third estimate will depend on diameter, on a (possibly negative) lower bound for the sectional curvature and on a lower bound for the simply-connectedness radius.

The technique that we develop in order to obtain the last result will also enable us to estimate the homotopy distance between any two closed curves on compact simply connected manifolds of sectional curvature bounded from below and diameter bounded from above. More precisely, let c be a constant such that any metric ball of radius $\leq c$ is simply connected. There exists a homotopy connecting any two closed curves such that the length of the trajectory of the points during this homotopy has an upper bound in terms of the lower bound of the curvature, the upper bound of diameter and c.

0. Introduction

In this paper we will prove three theorems relating the length of the shortest closed geodesic on a simply connected Riemannian manifold either to the

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Current address: Courant Institute of Mathematical Sciences, 251 Mercer street, New York University, New York, NY 10012, USA diameter or to the volume of a manifold. That work was motivated by the paper [G2] of Gromov, in which he asks whether it is always possible to find a constant c(n), such that the length of the shortest closed geodesic is bounded from above by c(n)$vol(M)^{1/n}$,where n is the dimension of a manifold. (See also [Br] and [G3] for the related topics.) Gromov himself had solved this problem for essential manifolds in [G2]. Recall that essential manifolds are those (nonsimply connected) compact manifolds that admit a map f:$M \to K(\Pi, 1)$, such that $f_*[M] \neq 0$, where [M] is the fundamental homology class of M, f_* is the induced homomorphism and $K(\Pi, 1)$ denotes the aspherical space with the fundamental group Π, i.e. the space K that has the following properties: $\pi_1(K) = \Pi$ and $\pi_n(K) = 0$, for $n \neq 1$. (In particular, all compact surfaces with the exception of a sphere are essential, and so are all the manifolds that admit Riemannian metric of nonpositive sectional curvature.) In 1988 C.B. Croke showed that the length of the shortest geodesic on a sphere is always $\leq 31\sqrt{A}$ and $\leq 9D$, where A is the area and D is the diameter of a sphere, (see [Cr2], [M]). That result finished the problem for the compact surfaces. The only known to us results for the simply connected manifolds of higher dimension are those of Ballmann, Thorbergsson and Ziller ([BalThZ]), who, in particular, have investigated the case of spheres endowed with a $\frac{1}{4}$-pinched metric of positive sectional curvature, and the results of Croke ([Cr2]) and Treibergs ([T]) for convex hypersurfaces.

In this paper we present three upper bounds on the length of the shortest closed geodesic on a simply connected Riemannian manifold with a non-trivial second homology group. Our first estimate will be in terms of the upper bound on sectional curvature and an upper bound on the volume of a manifold.

Theorem A. *Let M^n be a simply connected compact Riemannian manifold with a non-trivial second homology group, of sectional curvature $K \leq 1$ and a volume $\leq V$. Then the length of the shortest closed geodesic $\gamma(t)$ on the manifold M^n is bounded from above by*

$$g(V, n) = (c_1(n)(V + 1))^{c_2(n)(V+1)},$$

where $c_1(n) = 10^4(n!)^3, c_2(n) = 10^5(n!)^3$.

In the future the class of simply connected Riemannian manifolds with a non-trivial second homology group, of sectional curvature $K \leq 1$, and volume $\leq V$ will be denoted by Υ.

If, in addition, one assumes that the above manifold is 2-essential, one can somewhat improve the upper bound. Some of the examples of 2-essential manifolds are manifolds that are homotopically equivalent to Kähler manifolds, in particular $\mathbf{C}P^n$. In that case one obtains Theorem AA.

Theorem AA. *Let M^n be a 2-essential compact Riemannian manifold of sectional curvature $K \leq 1$ and a volume $\leq V$. Then the length of the shortest closed geodesic $\gamma(t)$ on M^n is bounded from above by*

$$\tilde{g}(V,n) = \tilde{c}_1(n) + (\tilde{c}_2(n)V)^{\tilde{c}_3(n)(V^{1/n}+1)},$$

where $\tilde{c}_1(n) = 10^3 n^3, \tilde{c}_2(n) = 10^3((n+1)!)^2, \tilde{c}_3(n) = 10^5 n^3$.

Let us now recall the definition of 2-essential manifolds:

Definition 0.1. *We will say that a compact and orientable manifold M is 2-essential if there exists $f : M \to \mathbf{C}P^\infty$ such that $f_*[M] \neq 0$,where [M] is the fundamental homology class of M.*

Our second estimate will be in terms of a lower bound for the sectional curvature, a positive lower bound for the volume and an upper bound for the diameter.

Theorem B. *Let M^n be a simply connected compact Riemannian manifold with a non-trivial second homology group, of sectional curvature $K \geq -1$, volume $\geq v > 0$ and diameter $d \leq D$. Then the length of the shortest closed geodesic $\gamma(t)$ on the manifold M^n is bounded from above by*

$$f(n,D,v) = \exp\left(\frac{e^{c_4(n)D}}{\min\{1,v\}^{c_5(n)}}\right),$$

where the constants c_4, c_5 can be explicitly calculated.

The estimate in our third result is also given in terms of a lower bound for the sectional curvature and an upper bound for the diameter but instead of a lower bound for the volume we use a lower bound for the simply-connectedness radius.

Theorem C. *Let M^n be a simply connected compact Riemannian manifold with a non-trivial second homology group, of sectional curvature $K \geq -1$, and diameter $d \leq D$. Assume that all metric balls of radius $\leq c$ in M^n are simply connected. Then the length of the shortest closed geodesic $\gamma(t)$ on the manifold M^n is bounded from above by a certain function $h(n,D,c)$ of n, D and c.*

If we assume, in addition, that every closed curve γ in a metric ball of radius less then c can be contracted to a point inside this ball by a homotopy which contains only closed curves of the length less than an explicitly given function of the length of γ then one is able to write down an explicit formula for $h(n,D,c)$.

In the future the class of simply connected compact Riemannian manifolds with a non-trivial second homology group, of sectional curvature $K \geq -1$ and diameter $d \leq D$ will be denoted by Ψ.

The proof of Theorem C that we will present is more involved then proofs of Theorems A and B and uses Gromov's ideas from his well-known paper on curvature, diameter and Betti numbers. Even though, Theorem C does not quite allow us to get rid of the dependence on v, it can be considered as a step in this direction.

We do not know how to get an explicit expression for the function h, existence of which is stated in the text of Theorem C (in contrast with Theorem B) without the assumption made in the last statement of Theorem C (or at least a weaker form of this assumption).

The reason that the proof of Theorem C is much more difficult than that of Theorems A and B is that in Theorem C we do not assume anything about how quickly closed curves in balls of radius $\leq c$ can be contracted to a point. On the other hand results of Grove and Petersen ([GrP]) yield not only an explicit contractibility function of manifolds satisfying $K \geq -1$, volume $\geq v > 0$ and diameter $\leq D$ but in fact provide enough information about how exactly closed curves contained in small metric balls can be contracted to a point inside larger metric balls.

Corollary 1. *Let M^n be a simply connected compact Riemannian manifold with a non-trivial second homology group, of sectional curvature $K \geq 0$, and diameter $d \leq D$. Assume that all metric balls of radius $\leq c$ in M^n are simply connected. Then the length of the shortest closed geodesic $\gamma(t)$ on the manifold M^n is bounded from above by $p(n, c/D)D$ for some function p.*

Proof. follows from Theorem C by a rescaling argument.

Recall that a geodesic in which its starting point and the end point coincide is called a geodesic loop. (That is, we do not require the resulting closed to curve to be smooth at this point.)

Corollary 2. *Let M^n be a simply connected Riemannian manifold with a non-trivial second homology group such that $K \geq -1$ and diameter $d \leq D$. Then the length of the shortest non-trivial geodesic loop on M^n is bounded from above by*

$$\lambda(n, D) = e^{e^{c(n)(D+1)}},$$

where $c(n) = 250n$.

If, moreover, $K \geq 0$ then the length of the shortest non-trivial geodesic loop is bounded from above by $e^{e^{2c(n)}} D$.

Proof. We will prove the existence of $\lambda(n, D)$. One derives the explicit formula for λ by following the proof of Theorem C. Let c be the same as in the text of Theorem C. If $c > 1$, then one has the upper bound $\lambda(n, D) = f(n, D, 1)$ even for the length of the shortest closed geodesic.

Otherwise there exists a closed curve in a ball of radius one which cannot be contracted to the center of this ball inside this ball.

It is not hard to see that if every closed curve formed by two minimizing geodesics between the center of the ball and a point of the closed curve is contractible within the ball then the curve itself will be contractible within the ball.

So, one of such curves will not be contractible. When we try to minimize its length in the class of curves which start and end at the center of the ball using a modification of the Birkhoff curve shortening process we arrive to a non-trivial geodesic loop of length less than twice that of the initial curve.

Last statement of Corollary 2 follows from a rescaling argument. QED.

Before we will give the outline of our proofs, let us recall the result of Lusternik and Fet, (see [B], [Ml] for more details).

Theorem. *Let M be compact simply connected Riemannian manifold. Then there exists at least one closed geodesic on M.*

In order to briefly state the proof of this theorem we will need a couple of facts:

1. Let $\Lambda M = \mathrm{Map}(S^1, M)$ denote the space of continuous maps from S^1 to M and let ΩM be the space of fixed point loops. Then

$$\pi_q(\Lambda M) = \pi_q(M) \oplus \pi_q(\Omega M);$$

and

$$\pi_{q+1}(M) \simeq \pi_q(\Omega M).$$

Therefore, there exists $i > 0$ such that $\pi_i(\Lambda M) \neq \{0\}$.

2. Given $c > 0$ let $\Lambda^c M$ denote the closed subset $E^{-1}([0, c])$, where E is the energy defined on piecewise differentiable curves. Let $P_N M$ be the set of all geodesic polygons consisting of N segments. For any fixed m there exists N_m such that $\pi_k(P_N M) = \pi_k(\Lambda^c M)$ for all $k \leq m, N \geq N_m$.

3. Let us also recall that closed geodesics are critical points of the energy function (or equivalently of the length function) on $P_N M$.

Proof. Consider the smallest i such that $\pi_i(\Lambda M) \neq \{0\}$, (or equivalently $\pi_{i+1}(M) \neq \{0\}$ and $\pi_i(M) = \{0\}$.) Let $\nu \in \pi_i(\Lambda M)$ and $\nu \neq 0$. It is easy to see that ν can be represented by a continuous map of S^i into the space $\Lambda^* M$ made of piecewise differentiable closed curves. Let $c = \sup_{x \in S^i}(E(\nu(x)))$. Consider $\Lambda^{2c} M$. For any fixed m there exists N_m such that for all $k \leq i, N \geq N_m, \pi_k(P_N^{2c} M) = \pi_k(\Lambda^{2c} M)$. Moreover, ν can be deformed into $P_N^{2c} M$ without the increase of energy in the process of homotopy. Thus, $\pi_i(P_N^{2c} M) \neq 0$.

Suppose $P_N^{2c} M$ does not contain closed geodesics. Then the energy function E on $P_N^{2c} M$ has no critical points on $P_N^{2c} M$ other than constant paths.

Let us define a vector field X on $P_N^{2c}M$ by the formula $- < X, Y >= dE(Y)$. The vector field X does not vanish on $P_N^{2c}M \backslash M$. Therefore we can deform $P_N^{2c}M$ into the tubular neighborhood of M, which can be retracted to M. But that would mean that $\pi_i(P_N^{2c}M) = \pi_i(M)$, which is a contradiction, since $\pi_i(M) = \{0\}$. QED.

Suppose now, that we want to estimate the length of a closed geodesic on a compact simply connected manifold. In light of the above proof, we can see that if we would actually construct a non-trivial element $h(q,t) \in \pi_i(\Lambda M)$ such that the length of each curve $h(q,*)$ is bounded from above by some constant, that would imply the existence of a closed geodesic, which length is going to be bounded by the same constant. That is exactly what we are going to do in our case. In our case $\pi_2(M) \neq \{0\}$ and $\pi_1(M) = \{0\}$. It follows that

$$\pi_1(\Lambda M) = \pi_1(M) \oplus \pi_1(\Omega M) \simeq \pi_1(\Omega M) \simeq \pi_2(M) \neq \{0\}.$$

Therefore, the problem can be reduced to constructing a non-trivial element $H_\tau(t)$ of $\pi_1(\Lambda M)$, such that for all τ length H_τ is bounded in terms of available geometric data. We will be using the notions of the width of a homotopy and the homotopy distance introduced in [SW] as follows:

Definition 0.2. *Width of a Homotopy.* Let $F_\tau(t)$ be a homotopy that connects two closed curves parametrized by $t \in [0,1]$ on a Riemannian manifold M. We say that W_F is the width of the homotopy $F_\tau(t)$ if $W_F = \max_{t\in[0,1]}$ length of the curve $F_\tau(t)$. That is W_F is the maximal length of the trajectory described by a point of one of the original closed curves during the homotopy. More generally if X, Y are metric spaces and $F : X \times [0,1] \longrightarrow Y$ is a homotopy then W_F is defined as $\sup_{x\in X}$length $F(x,*)$.

Definition 0.3. *Homotopy Distance.* Let $\alpha_1(t), \alpha_2(t)$ be two curves, then the homotopy distance $d_H(\alpha_1,\alpha_2)$=$\inf_H W_H$, where H is any homotopy between α_1 and α_2.

The constructions of $H_\tau(t)$ in the proofs of Theorems A, B and C are somewhat different. We will summarize the proof of Theorem C and then indicate the points where the proof of Theorems A and B will deviate from the proof of Theorem C.

There are several essential ideas in this proof that we want to emphasize.

1. In order to construct a non-trivial element $H_\tau(t)$ of $\pi_1(\Lambda M)$ such that length $H_\tau \leq f(n, D, c)$ we will have to learn how to construct a homotopy of any closed curve $\gamma(t)$ of length $\leq 3d$ to a point, and that homotopy has to have some special properties. What we have in mind is the following: there are two parameters of the homotopy that we need to control at the same time, i.e. the length of the curves in the homotopy and the homotopy width (by controlling we will mean providing an upper bound). We will, actually

be satisfied if we have only "partial" control over the width, that is at least two selected points on $\gamma(t)$ do not "travel for too long" until they reach p, i.e. $H_\tau(t_i)$ is bounded, where $i = 1, 2$. The attempt at using Birkhoff curve shortening process fails for the reason that even though we have absolute control over the length of the curves in a homotopy, each point on $\gamma(t)$ can travel a long distance till it gets to p.

2. Such a homotopy as in the previous paragraph can be constructed by first producing a different homotopy from which we demand only that its width should be bounded. The assumption that $l(M) > 3d$ will be used, where $l(M)$ denotes the length of the shortest closed geodesic. We can then construct a new homotopy based on the previous one that will satisfy the necessary conditions, (see Fig. 11.)

3. In order to accomplish (2) we will need to use notions very similar to those used by Gromov in [G1], i.e. rank and compressibility, that will be substituted by effective rank and effective compressibility. These notions will be defined in Sect. 3. We will be able to show that effective rank is bounded through curvature and diameter of a manifold.

4. The result of the various estimates will be that (assuming $l(M) > 3d$) for every curve $\gamma(t)$ of length $\leq 3d$ there exists a homotopy of this curve to a point, such that the length of the curves in a homotopy is bounded by a certain function of n, D and c and that we can insure that for some two points on the curve, the distance they travel is bounded by the same function. At that point we will be able to construct a non-trivial 2-cycle with some special properties. The argument that we will use in order to obtain it is the following: we will consider a map $f : S^2 \to M$, such that the image of S^2 under f represents a non-trivial element of $H_2(M)$. Next we will triangulate S^2 and consider the induced triangulation of $f(S^2)$. We will want the induced triangulation to be fine enough for the diameter of each simplex to be less than the injectivity radius of a manifold. Let us call $f(S^2)$ by σ. Then we will try to extend this map f to D^3, or, roughly speaking, we will attempt to "fill" σ using the following procedure. First we will pick any point $\tilde{p}$ in M. Then we will join the point $\tilde{p}$ with all the vertices of σ by minimal geodesics. After that we will consider all the closed curves composed of two geodesic segments that join $\tilde{p}$ and vertices $\tilde{v}_1$ and $\tilde{v}_2$ of σ and the edge of σ that joins $\tilde{v}_1$ and $\tilde{v}_2$. We will then use the "nice" homotopy to connect those closed curves with some points. We will extend f skeleton by skeleton by mapping the center of the disc D^3 to $\tilde{p}$, edges that connect the center of the disc with vertices of triangulation to minimal geodesics and 2-simplices to surfaces generated by the above homotopies. f cannot be extended to 3-skeleton, so it cannot be extended to at least one 3-dimensional simplex. Thus, we will obtain some cycles of M of a specific shape, at least one of which should

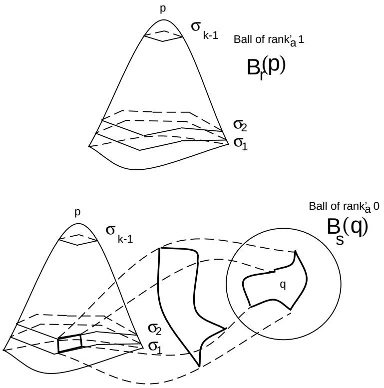

Fig. 1

be non-trivial, in order for σ not to bound. Finally, we will then construct a non-trivial element of $\pi_1(\Lambda M)$ with the desired properties.

Let us now somewhat extrapolate on (3). That work was mainly inspired by [G1]. We will establish the connection between effective rank of a ball and the homotopy distance between any closed curve inside that ball and some point. That is, we will find an upper bound on the homotopy distance between a curve and a point in terms of the effective rank. The proof of this uses the induction procedure on the effective rank of a ball containing the curve, that we will denote as $rank'_a$. For the curve $\gamma(t)$ that lies inside the ball, of radius less than the injectivity radius there exists an obvious homotopy of bounded width to the center of the ball. It is only slightly harder to construct a homotopy of bounded width for the curve inside the ball of $rank'_a = 0$, (but perhaps, with radius greater than the injectivity radius.) The above will be the base of induction. Now let us roughly describe how we can construct a homotopy of bounded width for the curve that lies inside a simply connected ball $B_r(p)$ of $rank'_a = 1$, since that will make the induction step clear.

We will begin by showing that there exists a finite sequence of closed piecewise geodesic curves that starts at our given curve and ends with some constant curve, such that two consecutive curves in the sequence are sufficiently close to each other. We will then construct a homotopy between two neighboring curves by reducing the problem to finding a homotopy of a closed curve that lies totally inside the ball $B_{r/10}(p')$, which is a subset of our original ball. That will be accomplished by first “bringing” the curve to a ball of $rank'_a = 0$ and then by homotoping it to a point, (see Fig. 1 below, and also Figs. 3–6.)

A technical difficulty which arises starting at the next step of induction is that we need to know in advance some information about geometry of the contraction of a given curve to a point inside $B_r(p)$. Fortunately, this information is of combinatorial nature and the existence of the desired upper bound follows from the contractibility of the considered curve inside $B_r(p)$ (see Lemma 3.5 for more details).

Proceeding by induction we can construct a homotopy of bounded width of a curve inside any simply connected ball in the manifold. Therefore if we have two parametrized curves such that the distance between their corresponding points is less then the simply connectedness radius divided by some large enough constant, then they can be connected by a homotopy of a bounded width. Now we can use the c/B-net (for large B) in the space of closed curves in M of length $\leq 3d$ in order to complete the construction of a contraction of bounded width for any closed curve of length $\leq 3d$.

Note that the procedure that we develop in Steps 2 and 3 in order to construct a homotopy of width bounded in terms of the lower bound on the sectional curvature and the diameter can also be used to prove the following theorem.

Theorem D. *Let M^n be a simply connected compact Riemannian manifold of sectional curvature $K \geq -1$ and diameter $d \leq D$. Suppose that all metric balls in M^n of radius $\leq c$ are simply connected. Then there exists a function $Q(n, c, D)$ such that for any closed curve $\omega(t)$ in M^n there exists a homotopy $H_\tau(t)$ of $\omega(t)$ to a point such that*

$$W_H \leq Q(n, c, D).$$

If we assume in addition that every closed curve γ in every metric ball of radius less than c can be contracted to a point inside this ball by a homotopy which passes only through closed curves of length bounded from above by an explicitly given function of the length of γ then we can write down an explicit formula for $Q(n, c, D)$.

To prove Theorem A we can assume that the injectivity radius of M^n is bounded from below. Otherwise Klingenberg's lemma together with Berger injectivity radius estimate will give an estimate on the length of the shortest closed geodesic and we would be done. But now, having the lower bound on the injectivity radius it will be much simpler to construct the required homotopy with special properties that we discussed above. We can then proceed as in the proof of Theorem C.

In the proof of Theorem B we will use Grove and Petersen results which imply that under assumptions of Theorem B there exist $r(n, v, D) > 0$ and $C(n, v, D)$ which can be written down explicitly such that any metric ball in M^n of radius $r < r(n, v, D)$ is contractible inside the concentric ball of radius $C(n, v, D)r$. Moreover, results of Grove and Petersen provide enough information about the geometry of this contraction to construct the required homotopy discussed above almost as easily as if we would have a lower bound for the injectivity radius.

In order to prove Theorem AA we will use some obstruction theory used by Gromov in [G2] to construct a homotopy non-trivial map of the boundary of 3-simplex into M^n with the following special properties: Denote the images of the vertices a, b, c, d. Then the images of 1-simplices will be geodesics joining a, b, c and d, and the images of faces will be formed by the surfaces generated by the homotopies with the "nice" properties described in (1). The lengths of contracted curves abc, bcd, acd and abd will be bounded in terms of the filling radius of M^n which does not exceeed $c(n)vol^{\frac{1}{n}}(M^n)$. Then we can proceed as in the proof of Theorem A.

Sections 1–4 will be dedicated to estimating the homotopy distance between closed piecewise geodesic curve and a point for class Ψ. In Sect. 1 we will define the notions of a-effective compressibility and a-effective rank. In Sect. 2 we will prove the lemma that will be essential in estimating a-effective rank. It will establish that the number of elements in the sequence of ϵ-almost critical points is finite for some ϵ under some conditions. In Sect. 3 we will establish the connection between the effective rank and the homotopy distance, and in Sect. 4 we will show that $rank'_a$ is bounded from above by $e^{3(n+1)(d+1)}$ for some a. Combining results of Sects. 3 and 4 we will obtain Theorem D.

In Sect. 5 we will construct the required homotopy for any $M^n \in \Psi$. We will deal with class Υ in Sect. 6. Finally in Sect. 7 we will prove Theorems A, B and C and use some ideas from the paper [G2] of Gromov to finish the proof of Theorem AA. In Sects. 1, 2 and 4 we will closely follow the proof of the main theorem from [G1] of Gromov as it was done in [Cg].

1. Basic definitions

Definition 1.1. *a-Effective Compressibility.* Let a be a positive number. We will say that $B_r(p)$ a-effectively compresses to $B_s(q)$ and write $B_r(p) \longmapsto_a B_s(q)$ if the following conditions are satisfied:

1. 5s+d(p,q)$\leq$5r;

2. There exists a homotopy F_τ of $B_r(p)$ into $B_s(q)$ with F_0 being the identity and $F_1 : B_r(p) \subset B_s(q)$;

3. $W_F \leq ar$.

(In particular, if there exists a homotopy F_τ of $B_r(p)$ to some point $p' \in B_r(p)$, such that F_0 is the identity and $F_1 = q'$, and the third property of the definition is satisfied, then we will say that $B_r(p)$ is a-effectively contractible.)

Definition 1.2. *a-Effective Rank.*

1. $rank_a(r,p)$:=0, if $B_r(p) \longmapsto_a B_s(q)$ with $B_s(q)$ a-effectively contractible.

2. $rank_a$ (r,p):=j if $rank_a$ (r,p) is not $\leq$ j-1 and if $B_r(p) \longmapsto_a B_s(q)$ such that for all $q' \in B_s(q)$ with $s' \leq$ s/10, we have $rank_a(s', q')$ $\leq$j-1.

Definition 1.3. *a-Effectively Incompressible Ball.* A ball $B_r(p)$ is called a-effectively incompressible if $B_r(p) \longmapsto_a B_s(q)$ implies that $s > r/2$.

Lemma 1.4. *Any ball $B_r(p)$ can be $3a$-effectively compressed either to an a-effectively contractible ball or to a ball that is incompressible a-effectively.*

Proof. Suppose $B_r(p) \longmapsto_a B_{s_1}(q_1)$. Then there are three possibilities:

1. $B_{s_1}(q_1)$ is a-contractible;

2. $B_{s_1}(q_1)$ is incompressible a-effectively;

3. $B_{s_1}(q_1)$ is compressible a-effectively, but not a-effectively contractible.

In the first and the second case we are done, since a-effective compressibility implies $3a$-effective compressibility. In the third case, $B_{s_1} \longmapsto_a B_{s_2}$ such that $s_2 \leq s_1/2$ by definition of a-effective compressibility. Once again we have three possibilities for $B_{s_2}(q_2)$. It can be either a-effectively contractible, a-effectively incompressible, or a-effectively compressible. Consider the last case and obtain $B_{s_2} \longmapsto_a B_{s_3}$ such that $s_3 \leq s_2/2$, and so on. The above process will have to terminate by our arriving either at a-effectively compressible ball, or the ball that is incompressible a-effectively. We, thus,obtain a sequence:$F^1, F^2, F^3, ..., F^n$ of homotopies such that $W_{F^1} \leq ar, W_{F^2} \leq as_1 \leq ar, W_{F^3} \leq as_2 \leq as_1/2 \leq ar/2$, ...,$W_{F^n} \leq (ar)/2^{(n-1)}$. Thus, we can get to a-effectively contractible or to a-effectively incompressible ball applying one homotopy after the other and the width W_F of the final homotopy will be $\leq ar + ar + ar/2 + ... + ar/2^{(n-1)} \leq 3ar$. QED.

2. Modified Gromov's lemma

In this section we will prove a slightly generalized version of the well-known Gromov lemma about the sequence of critical points, (see [Cg], [G1] for the proof of the original lemma).

Definition 2.1. *ϵ-Almost Critical Point.* We will say that a point q on a manifold M^n is ϵ-almost critical with respect to p, if for all vectors v in the tangent space M_q, there exists a minimal geodesic γ from q to p with the absolute value of the angle $\angle v, \gamma'(0) \leq \pi/2 + \epsilon$.

Lemma 2.2. *(Modified Gromov's Lemma.) Let q_1 be ϵ-almost critical point with respect to p and let q_2 satisfy $d(p, q_2) \geq \nu d(p, q_1)$ for some $\nu > 1$. Let γ_1, γ_2 be minimal geodesics from p to q_1, q_2 respectively, and let θ be the angle between $\gamma_1'(0)$ and $\gamma_2'(0)$. If sectional curvature K_M of the manifold*

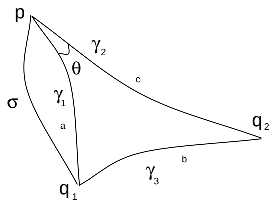

Fig. 2

M is bounded from below by -1 and $d(p, q_2) \leq d$ *then*

$$\cos\theta \leq \frac{\tanh\frac{d}{\nu}}{\tanh d}(\sin\epsilon + 1) + \sin\epsilon.$$

Proof. Let $a = d(p, q_1), b = d(q_1, q_2), c = d(p, q_2)$. Also let γ_3 be a minimal geodesic from q_1 to q_2. Since q_2 is ϵ-almost critical point to p, there exists a minimal geodesic $\sigma(t)$ from q_2 to p such that the angle $\angle\sigma'(0), \gamma_3'(0) \leq \pi/2 + \epsilon$, (see Fig. 2.) We will apply the Toponogov comparison theorem twice to the hinges $\sigma(t), \gamma_3(t)$ and $\gamma_1(t), \gamma_2(t)$, which in combination with the hyperbolic law of cosines will yield inequalities (1) and (2) respectively.

$$\cosh c \leq \cosh a \cosh b - \sinh a \sinh b \cos(\pi/2 + \epsilon) \tag{1}$$

$$\cosh b \leq \cosh a \cosh c - \sinh a \sinh c \cos\theta. \tag{2}$$

Let us substitute the inequality (2) into (1) to obtain:

$$\cosh c \leq \cosh a(\cosh a \cosh c - \sinh a \sinh c \cos\theta) + \sinh a \sinh b \sin\epsilon.$$

Now, let us use the triangle inequality to see that

$$\begin{aligned}\cosh c \leq \cosh^2 a \cosh c - \cosh a \sinh a \sinh c \cos\theta \\ + \sinh a \sinh(a + c) \sin\epsilon.\end{aligned}$$

Therefore, using the hyperbolic functions identities we obtain:

$$0 \leq \sinh^2 a(\sin\epsilon + 1) - \cosh a \sinh a \tanh c(\cos\theta - \sin\epsilon);$$

$$\cosh a \tanh c(\cos\theta - \sin\epsilon) \leq \sinh a(\sin\epsilon + 1)$$

$$\cos\theta \leq \frac{\tanh a}{\tanh c}(\sin\epsilon + 1) + \sin\epsilon \leq \frac{\tanh(d/\nu)}{\tanh d}(\sin\epsilon + 1) + \sin\epsilon.$$

QED.

It is clear that unless the expression on the right is strictly less than 1, our lemma will not provide any additional information. Thus, we need to find such an ϵ that

$$\frac{\tanh(d/\nu)}{\tanh d}(\sin\epsilon+1)+\sin\epsilon = x < 1.$$

We will use Lemma 2.2. in the situation when ν=5/4. In this case let

$$c_d = \frac{\tanh(4d/5)}{\tanh d},$$

and let

$$x = \frac{c_d+1}{2}.$$

It is clear that both c_d and x are strictly less than 1. Take ϵ such that

$$\sin\epsilon = \frac{1-c_d}{2(c_d+1)}.$$

After doing some calculation we will see that

$$\frac{1}{\sin\epsilon} = \frac{2(e^{9d/5}-e^{-9d/5})}{e^{d/5}-e^{-d/5}} \le 18e^{8d/5}.$$

Lemma 2.2 implies that $\cos\theta \le (c_d+1)/2$. Additional calculations imply that

$$(c_d+1)/2 \le 1 - \frac{e^{-8d/5}}{2}.$$

and hence

$$\theta \ge e^{-4d/5}.$$

Corollary 2.3. *Let $q_1, q_2, ..., q_N$ be a sequence of ϵ-almost critical points of p, where $\sin\epsilon$ satisfies the above condition. Suppose also that $d(p,q_{i+1}) \ge (5/4)d(p,q_i)$. Then N $\le (n-1)\pi^{n-1}e^{4(n-1)d/5}$.*

Proof. Consider minimal geodesics γ_i that join p and $q_i, i=1,...,N$. Next consider the set of the unit tangent vectors $\{\gamma_i'(0)\}$ that can be viewed as a subset of the unit sphere in the tangent space of M at p. Let θ_i be the angle between p, q_i. Then the balls of radius $\theta_i/2$ about the $\gamma_i'(0)$ are mutually disjoint. Thus the number of points in the sequence N $\le \frac{volS^{n-1}}{\min volB(p,\theta_i/2)}$, where $B(p,\theta_i/2)$ denote balls in S^{n-1} for $i=1,...,N$ and

$$\frac{volS^{n-1}}{\min volB(p,\theta_i/2)} = \frac{\int_0^{\pi}(\sin s)^{n-2}ds}{\int_0^{\theta/2}(\sin s)^{n-2}ds},$$

for $\theta=\min\theta_i$ Since $\sin s \ge 2s/\pi$, on the interval $(0,\pi/2)$ we estimate:

$$N \le \frac{\pi^{n-1}(n-1)}{\theta^{n-1}}.$$

Now substitute the lower bound for θ and obtain the result. QED.

3. Homotopy distance, nets in the space of closed curves and a-effective rank of a ball

In this section we will obtain upper bounds for the homotopy distance between two curves that are close to each other.

The constructions that we will perform are based on the following idea: if we know how to contract closed curves that lie in "small" metric balls to a point (i.e. closed curves that lie inside balls of radii $2r_0$) then we also know how to connect two curves that are close to each other (i.e. r_0-close) with a homotopy (see Lemma 3.3). Thus, in the situation when we are able to estimate homotopy distance between a closed curve in a sufficiently small (simply connected) metric ball and a point we are also able to estimate homotopy distance between any two closed curves which can be connected by homotopy that passes through closed curves of length $\leq L$. This estimate will depend on the number of points in the $r_0/80$-net in the space of closed curves of length $\leq L$.

Note that the idea does not work if the only information we have is that "small" balls are simply connected, but we do not know how to estimate the width of the homotopy that connects a closed curve that lies inside that ball and a point.

Thus, it is important to learn to construct contractions of closed curves inside balls of radius $2r_0$.

Three different cases will be considered:

In the situation, when we know the injectivity radius, we can take r_0 =injectivity radius, and the contractions are obvious.

The second case is that of a manifold with $K \geq -1$, $vol \geq v > 0$ and $diam \leq D$. In this case the work of Grove and Petersen gives to us the lower bound for the radius of contractibility, so we can take $r_0 = r_0(n, v, d)$ and $w(n, v, d)$ such that any closed curve in a ball of radius r_0 can be contracted to a point by homotopy of width $\leq w(n, v, d)$. Note that in this case "small" metric balls are not necessarily simply connected.

In the situation when a lower bound for the volume is not available our idea is to use the notion of a-effective rank which we introduce below. In that case we will examine how the homotopy distance between a curve $\alpha(t) \in B_r(p)$ of a-effective rank m and the center of the ball, depends on the rank m and the diameter d of a manifold M^n. Here we assume that M^n has sectional curvature ≥ -1 and that all balls of radius $\leq c$ are simply connected.

Definition 3.1. *Modified a-Effective Rank.*

1. $rank'_a(p, r) := 0$ if $B_r(p) \longmapsto_{3a} B_s(q)$ where $B_s(q)$ is a-contractible.

2. $rank'_a(p,r) := j$ if $rank'_a(p,r) \neq j-1$ and $B_r(p) \longmapsto_{3a} B_s(q)$ such that $B_s(q)$ is a-incompressible and for all $q' \in B_s(q)$ with $s' \leq s/10$, we have $rank'_a(q', s') \leq$ j-1.

Lemma 3.2. *Let $\gamma(\tau) \in B_r(p)$ with $rank'_a(p,r) = 0$ be a closed curve. There exists a homotopy F_τ of $\gamma(t)$ to a point with $W_{F_\tau} \leq 4ar \leq 4ad$.*

Proof. Since $rank'_a(p,r) = 0$ there exists an a-effectively contractible $B_s(q)$ such that $B_r(p) \longmapsto_a B_s(q)$, implying the existence of a homotopy F^2_τ with $W_{F^2_\tau} \leq 3ar$, such that $F^2_0(\gamma(t)) = \gamma(t)$ and $F^2_1(\gamma(t)) \subset B_s(q)$. But a-effective contractibility of $B_s(q)$ implies the existence of a homotopy F^1_τ such that

1. $W_{F^1_\tau} \leq ar$;
2. $F^1_0(F^2_1(\gamma(t))) = F^2_1(\gamma(t))$;
3. $F^1_1(F^2_1(\gamma(t))) = q'$,

where $q' \in B_s(q)$.

Take the composition of the above homotopies and obtain F_τ with $W_{F_\tau} \leq 4ar$. QED.

We are now ready to show that for every closed curve inside a ball B of a'-effective rank m the homotopy distance between the curve and any point on M^n is bounded by the function that depends on the a'-effective rank of the ball and the diameter of the manifold, where $a' = \frac{1}{\sin\epsilon}$, and where $\epsilon = \frac{1-c_d}{2(1+c_d)}$ as it was defined in Sect. 2, providing that B and all balls of smaller radius in M^n are simply connected. (Note that, $\frac{1}{\sin\epsilon} \leq 18e^{8d/5}$).

Our proof will be by induction on the effective rank of the ball and will be done in five steps.

Let us first note that the above statement is true, when $m = 0$. Since for any closed curve inside that ball $B_r(p)$ of a'-effective rank 0, there exists a homotopy F^0, such that $W_{F^0} \leq 4a'r$,(by Lemma 3.2.), thus, $W_{F^0} \leq \frac{4r}{\sin\epsilon} \leq e^{A(d+1)} \leq e^{A(d+1)(m+1)}$, where $A = 2 \cdot 10^5(n+1)$.

Let us assume now that the above statement is true for a curve lying in a ball of a'-effective rank m of radius smaller than the radius of simply connectedness, that is there exists a point q_m and homotopy F^m, such that $W_{F^m} \leq f(m,n,d)$. We want to show that for any closed curve lying inside that ball of a'-effective rank $m+1$ of radius r less than the simply connectedness radius there exists a homotopy F^{m+1} to a point q_{m+1}, such that $W_{F^{m+1}} \leq f(m+1,n,d)$. The homotopy F^{m+1} will be a product of several homotopies. We will proceed as follows.

Step 1: Given α_1 we will show that there exists a homotopy that we will call h^1 that connects our curve α_1 with the curve α_2 inside a ball $B_s(q)$, such that the width of h^1 will be bounded by $3a'r(\leq 3a'd)$ and for every $q' \in B_s(q)$ and $s' \leq s/10$ the ball $B_{s'}(q')$ has a'-effective rank m.

Step 2: We will use our induction assumption to show that for any two curves $\alpha_1(t), \alpha_2(t) \subset B_s(q)$ such that $d(\alpha_1(t), \alpha_2(t)) \leq s/20$ there exists a homotopy h^2 with the width $\leq 4f(m,n,d) + 4s/10 (\leq 4f(m,n,d) + 4d/10.)$ (see Figs. 3–6).

Step 3: We will then show that any closed curve in $B_s(q)$ of length bounded from above by $3s$ can be homotoped to a point q by a homotopy h^3 with the width bounded by a function of n, m, d.

Step 4: For any closed curve $\alpha_2 \subset B_s(q)$, regardless of its length, there exists a homotopy h^4 to a point q such that the width of the homotopy is bounded by $2s + 4W_{h^3} (\leq 2d + 4W_{h^3})$.

Step 5: Take the composition of h^1 and h^4 to get the required homotopy and estimate its width.

We will now proceed with the proofs.

Step 1: Immediately follows from the definition of the a-effective rank of the ball.

Step 2: will be the result of

Lemma 3.3. 1) *Let $\alpha_1(t), \alpha_2(t)$ be two closed curves in a ball $B_s(q)$, where s is smaller than simply connectedness radius of M^n with the distance $d(\alpha_1(t), \alpha_2(t)) \leq s/20$ for all t. Suppose also that $B_s(q)$ has a'-effective rank $\leq m$. Assume that every closed curve in every ball of a'-effective rank m and radius $\leq s$ can be connected to a point by a homotopy of width $\leq f(m,n,d)$. Then there exists a homotopy h^2 between those two curves with $W_{h^2} \leq 4f(m,n,d) + 4s/10$ ($\leq 4f(m,n,d) + 4d/10$).*

2) More generally, if $\alpha_1(t), \alpha_2(t)$ are two closed curves such that $d(\alpha_1(t), \alpha_2(t)) \leq s/20$ for all t in an arbitary Riemannian manifold M, and any closed curve in any ball of radius s in M can be contracted to a point by a homotopy of width $\leq W$ then there exists a homotopy H between α_1 and α_2 of width $\leq 4W + 4s/20$.

Proof. 1) Take $\alpha_1(t), \alpha_2(t)$, such that $d(\alpha_1(t), \alpha_2(t)) < s/20$. W.L.O.G. we can assume that α_1:[0,1] $\longrightarrow M^n$, α_2:[0,1] $\longrightarrow M^n$ are broken geodesics. We will partition the interval [0,1] into segments, such that each quadrangle with the vertices $\alpha_1(t_i), \alpha_1(t_{i+1}), \alpha_2(t_{i+1}), \alpha_2(t_i)$ and the edges $\alpha_j|_{[t_i,t_{i+1}]}$, $j = 1, 2, \sigma_j(s)$, where $\sigma_j(s)$ is a minimal geodesic joining $\alpha_1(t_i)$ and $\alpha_2(t_i)$, lies inside a metric ball of radius s/10. This can be done by requiring that the length of the curve $(\alpha_j|_{[i,i+1]}) \leq s/30$.

We will describe the homotopy, by providing the description of the images of the curve $\alpha_1(t)$ under the homotopy. Let $\alpha_1^i = \alpha_1|_{[t_i,t_{i+1}]}$ and $\alpha_2^i = \alpha_2|_{[t_i,t_{i+1}]}$. Then we claim that

1. $\alpha_1(t)$ is homotopic to the curve $\gamma_1 = \bigcup_{i=1} \alpha_1^i \cup \sigma_i \cup -\sigma_i$, (see Fig. 3.) Moreover, $W_{g^1} \leq s/10$, where g^1 is the homotopy between α_1 and γ_1.

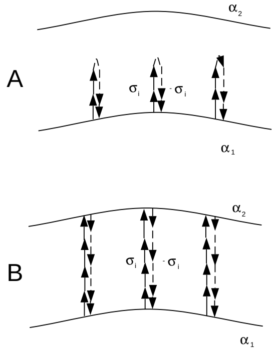

Fig. 3A,B.

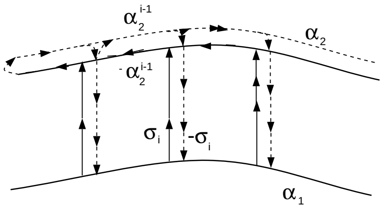

Fig. 4

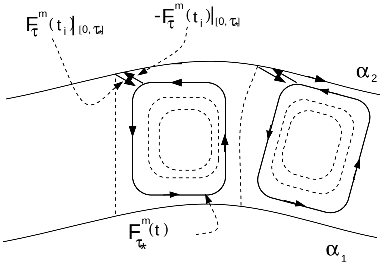

Fig. 5

2. The curve $\gamma_1(t)$ will be homotopic to the curve $\gamma_2(t) = \bigcup_{i=1} \alpha_1^i \cup \sigma_i \cup -\alpha_2^i \cup \alpha_2^i \cup -\sigma_i$ with $W_{g^2} \leq (3s)/10$, (see Fig. 4.)

3. The curve $\gamma_2(t)$ will be homotopic to the curve $\gamma_3(t)$, where $\gamma_3(t) = \bigcup_{i=1} \alpha_2^i \cup F_\tau^m(\alpha_2(t_{i+1}))|_{[0,\tau_*]} \cup F_\tau^m(t) \cup -F_\tau^m(\alpha_2(t_{i+1}))$ and $W_{g^3} \leq 2f(m)$, (see Fig. 5.)

4. The curve $\gamma_3(t)$ is homotopic to the curve $\gamma_4(t)$, where $\gamma_4(t) = \bigcup_{i=1} \alpha_2^i \cup F_\tau^m(\alpha_2(t_{i+1})) \cup -F_\tau^m(\alpha_2(t_{i+1}))$ and $W_{g^4} \leq 2f(m)$, (see Fig. 6).

Finally, we observe that $\gamma_4(t)$ is homotopic to $\alpha_2(t)$ and notice that $W_{h^2} \leq 4f(m) + 4s/10$ as required. The proof of 2) is quite similar. QED.

Step 3: will require proofs of several lemmas.

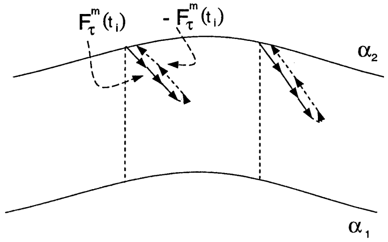

Fig. 6

Lemma 3.4. *Let $\Phi(B_r(p))$ be the space of piecewise differentiable closed curves of length $\leq L$ parametrized proportionally to their arclength in $B_r(p)$, a ball of radius r in a manifold M^n and let N be an upper bound on the number of elements in some cover $\{B_{\tilde{\epsilon}/24}(p_i)\}$ of the ball $B_r(p)$ for some $\tilde{\epsilon}$. There exists an $\tilde{\epsilon}/4$-net in the δ-neighborhood $N_\delta(\Phi(B_r(p)))$ with the number of elements*

$$Q \leq N^{24L/\tilde{\epsilon}+1},$$

where δ is some positive number and the length of every closed curve in this $\tilde{\epsilon}/4$-net is $\leq 3L$.

Proof. Given the cover $\{B_{\tilde{\epsilon}/24}(p_i)\}$ of $B_r(p)$ we will construct an $\tilde{\epsilon}/4$-net in the δ-neighborhood of $\Phi(B_r(p))$ as follows. We will consider the set I of $\gamma_i(t)$, where each $\gamma_i(t)$ will be a curve composed of the geodesic segments, that join points p_k and p_j if and only if $d(p_k, p_j) \leq \tilde{\epsilon}/8$ with the additional condition that the length of $\gamma_i \leq 3L$. The number of such curves will be $\leq N^{24L/\tilde{\epsilon}+1}$. If we will impose the additional condition of all curves being closed, then our set J of such curves will be a subset of I and the number of elements of J will also be $\leq N^{24L/\tilde{\epsilon}+1}$. We claim that J is our $\tilde{\epsilon}/4$-net.

For let $\gamma(t) \subset B_r(p)$ be any curve parametrized proportionally to its arclength, in particular, $\gamma : [0, l] \to B_r(p), l \leq L$. Let us partition [0,l] into segments $[t_j, t_{j+1}]$ such that $t_{j+1} - t_j = \tilde{\epsilon}/24$. For each t_j we will select p_j such that $\gamma(t_j) \in B_{\tilde{\epsilon}/24}(p_j)$. Note that $d(p_{j+1}, p_j) \leq \tilde{\epsilon}/8$ by triangle inequality. We will then construct a curve $\sigma(t)$ by joining centers of the balls by minimal geodesics. Note: $\sigma(t)$ will not be parametrized proportionally by its arclength, but $d(\sigma(t), \gamma(t)) < \tilde{\epsilon}/8$. QED.

Lemma 3.5. *Let $\alpha(t)$ be a closed piecewise differentiable curve in M^n parametrized proportionally to its arclength and contractible inside a ball $B_r(p)$ of radius r in M^n.*

1) Assume that $\alpha(t)$ can be contracted to a point inside $B_r(p)$ by a homotopy that passes only through piecewise differentiable curves of length $\leq L$. Then there exists a finite sequence $\{\sigma_i\}_{i=1}^k$ of closed broken geodesics, such that $d(\sigma_1(t), \alpha(t)) \leq \tilde{\epsilon}/4$ for all t; $\sigma_k = p$; length(σ_i) $\leq 3L$,

$d(\sigma_i(t), \sigma_{i+1}(t)) < \tilde{\epsilon}$ for all t and the number k of elements in this sequence is $\leq Q$, where Q is as in the Lemma 3.4.

2)Assume instead only that $B_r(p)$ is simply connected. Then there exists a sequence $\{\sigma_i\}_{i=1}^k$ of closed broken geodesics such that $d(\sigma_1(t), \alpha(t)) \leq \tilde{\epsilon}/4$, $d(\sigma_1(t), \sigma_2(t)) < \tilde{\epsilon}$ for all t, $\sigma_k = p$ and the number k of elements in this sequence does not exceed $f(r/\tilde{\epsilon}, N)$, where N is the same as in Lemma 3.4 and f is an increasing function.

Proof. 1) There exists a path P_τ in a space of all closed curves in $B_r(p)$ of length bounded from above by L and parametrized proportionally to the arclength connecting $\alpha(t)$ and the constant curve p. By the above lemma we can construct an $\tilde{\epsilon}/4$-net in the δ-neighborhood of $\Phi(B_r(p))$, that is the space of curves of length $\leq L$ and parametrized proportionally to their arclength. Now consider the sequence: $P_1, ..., P_k$ such that $d(P_i, P_{i+1}) < \tilde{\epsilon}/4, P_1 = \alpha_1, P_k = p$. We know that for all P_i there exists $\sigma_i \in N_\delta(\Phi(B_r(p)))$ such that $d(P_i, \sigma_i) < \tilde{\epsilon}/4$. Thus we obtain sequence σ_i such that $\sigma_k = p$ and $d(\sigma_i, \sigma_{i+1}) < \tilde{\epsilon}$. But we still have to estimate the number of elements in that sequence. So far there is a possibility that $\sigma_i = \sigma_j$ for different i and j. To avoid that we can delete all the subsequences between all the repeating elements from the sequence without changing the essential properties of the sequence, so our new sequence obtained in this way will be nonrepeating and the number of elements will be less than or equal to Q.

2) Consider α. Let us split α by geodesics from p to points of α into loops γ_i of length less then or equal to $3r$ similarly to what is on Figs. 7–9 and explained in details in the proof of Lemma 3.6 below. Loops γ_i are contractible inside $B_r(p)$ and if every of them can be contracted to p by a homotopy of width $\leq W$, then γ can be contracted to p by a homotopy of width $\leq 4W + 2r$, (see the proof of Lemma 3.6 below for details).

Consider a nerve of a covering of $B_r(p)$ by balls of radius $\tilde{\epsilon}/24$. Consider the natural map from the manifold to the nerve obtained using a partition of unity associated with the covering with derivatives bounded by a constant. Consider a simplicial approximation of γ_i. The number of simplices, l, in this simplicial curve Γ does not exceed a constant times the length of $\gamma_i/\tilde{\epsilon}$, and thus, does not exceed a constant times $r/\tilde{\epsilon}$. Γ must be contractible in the nerve since γ_i is contractible.

The crucial idea is that there exists a simplicial homotopy of Γ to a point and a function ω, such that its image consists of not more than $\omega(r/\tilde{\epsilon}, N)$ simplices (counting with multiplicities) in the nerve, where N is a number of vertices in the nerve.

Indeed, the number of simplices S in the nerve is bounded by a function of N. Now for every l and S there exists only finitely many simplicial complexes with not more than S simplices and contractible closed simplicial curves in the simplicial complex with not more than l simplices. Taking maximum over all such pairs of the number of 2-dimensional simplices in the optimal contracting homotopy we obtain an increasing function of l and S, which is bounded from above by an increasing function of $r/\tilde{\epsilon}$ and N.

Now we would like to turn this simplicial homotopy of Γ into a desired sequence of closed broken geodesics between γ_i and a point. To do this find a closed broken geodesic with vertices in the net used to construct the nerve closest to γ_i. This will be σ_1 (for the considered curve γ_i). We can think about simplicial homotopy of Γ to a point as a sequence of closed simplicial curves such that the difference between two consecutive curves is the boundary of exactly one triangle (i.e. two-dimensional simplex) of the nerve. Now we can assign to any closed simplicial curve in the nerve a broken geodesic passing through the centers of balls of the covering corresponding to the vertices of the nerve. Two broken geodesics in the manifold obtained from two consecutive closed curves in the simplicial homotopy of Γ will be $\tilde{\epsilon}/4$ close. It remains to recall that the number of 2-dimesional simplices in the homotopy of Γ (=the number of consecutive closed simplicial curves) is bounded from above by an increasing function of $r/\tilde{\epsilon}$ and N. Q.E.D.

Let us now apply those lemmas to the compact manifold M^n of sectional curvature $K \geq -1$ and where $L = 3r$. We estimate the number of points in the $\tilde{\epsilon}/24$ net on $B_r(p)$ using Bishop-Gromov volume comparison theorem, (see [GoHL]). Consider the maximal number of pairwise disjoint balls in $B_r(p)$ of radius $\tilde{\epsilon}/48$. Then the set of balls $\{B_{\tilde{\epsilon}/24}(p_i)\}_1^N$ will cover $B_r(p)$, where N can be estimated to be

$$N \leq \frac{vol(B_r(p))}{vol(B_{\tilde{\epsilon}/48}(p_i))} \leq \frac{vol(B'_r(p'))}{vol(B'_{\tilde{\epsilon}/48}(p'_i))},$$

where $B'_r(p')$, $B'_{\tilde{\epsilon}/48}(p'_i)$ are balls of radii $r, \tilde{\epsilon}/48$ respectively on a manifold of constant curvature -1. Calculation shows that

$$N \leq \frac{\int_0^r \sinh^{n-1} tdt}{\int_0^{\tilde{\epsilon}/48} \sinh^{n-1} tdt}.$$

Since the balls $\{B_{\tilde{\epsilon}/24}(p_i)\}$ cover $B_r(p)$, the set $\{p_i\}$ will be $\tilde{\epsilon}/24$-net. Let $\tilde{\epsilon} = r/20$. Then $Q \leq (960e^{d(n-1)})^{1441}$.

We are now ready to complete Step 3. Let $\gamma(t) \in B_r(p)$ be a curve of length $\leq 3r$ and let $\tilde{\epsilon} = r/20$. There exists a sequence of piecewise geodesics parametrized proportionally to their arclength of length $\leq 3r$ and having properties 1-3. Thus, we can obtain the required homotopy by taking the composition of homotopies between the consecutive curves. It is easy to see that the upper bound on the width of the final homotopy is what has been required.

Step 4: will require the lemma below.

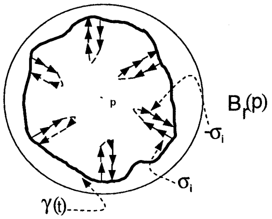

Fig. 7

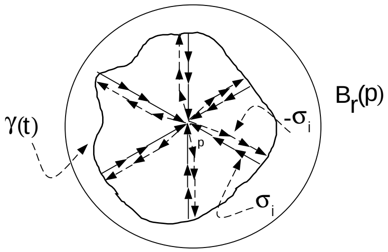

Fig. 8

Lemma 3.6. *Let $\gamma(t) \in B_r(p)$ be a piecewise geodesic curve of any length in $B_r(p)$ parametrized proportionally to its arclength. Then there is a homotopy $H_\tau(t)$ of $\gamma(t)$ to a point such that $W_{H_\tau} \leq 2d + 4W_{h^3}$, where $h^3_\tau(t)$ is the homotopy connecting a curve of length $\leq 3r$ to a point produced in Step 3.*

Proof. Let $\gamma(t)$ have length l. Partition the interval [0,l] into the subintervals $[t_i, t_{i+1}]$ such that $t_{i+1} - t_i \leq r$. Also let σ_i be minimal geodesic joining the points: $\gamma(t_i)$ and the center of the ball. Let $\gamma^i = \gamma|[t_i, t_{i+1}]$ then we claim that $\gamma(t)$ is homotopic to the curve $\gamma^1 = \bigcup \gamma^{i-1} \cup \sigma_i \cup -\sigma_i$ and $W_{h^1} \leq 2r (\leq 2d)$, where h^1 is the homotopy, (see Figs. 7 and 8). Let T^i be a geodesic triangle with vertices at $\gamma(t_i), \gamma(t_{i+1}), p$ and edges: $\sigma_i, \gamma^i, \sigma_{i+1}$. By Step 3, for each T_i there exists a homotopy that conects it to some point p_i. Let us call this homotopy $H^{m+1,i}$. Consider a curve $H^{m+1,i}_\tau(p)$ joining p and p_i, and denote $H^{m+1,i}_\tau|_{[0,t_*]}$ by $H^{m+1,i}_*$. Then we claim that γ^1 is homotopic to $\gamma^2 = \bigcup H^{m+1,i}_*(p) \cup H^{m+1,i}_{\tau_*}(T_i) \cup -H^{m+1,i}_*(p)$ with the width of the homotopy bounded by $2d + 2W_{H^{m+1,i}_\tau}$, (see Fig. 9). We can see that γ^2 is homotopic to $\gamma^3 = \bigcup H^i \cup -H^i$, which is homotopic to a point (see Fig. 10). QED

Step 5: We combine the results of Steps 1-4 and get the desired result:

$$W_{F^{m+1}} \leq \chi(n, d, m).$$

As it was stated at the beginning of the section, we can now prove the next lemma by following steps 1-5.

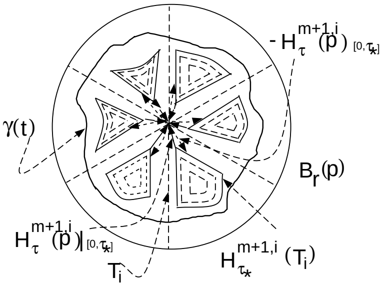

Fig. 9

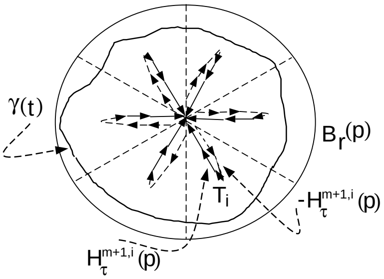

Fig. 10

Lemma 3.7. *Let M^n be a compact simply connected manifold of sectional curvature $K \geq -1$ and diameter d. Assume that every metric ball in M^n of radius $r \leq c$ is simply connected. Then every closed curve in M^n can be contracted to a point by a homotopy of width bounded by a function h of n, d, c and the modified a'-effective rank m of M^n. Here $a' = 1/\sin\epsilon$, where $\epsilon = \frac{\tanh d - \tanh(4d/5)}{2(\tanh d + \tanh(4d/5))}$. If we assume that every closed curve ω in any metric ball of radius $\leq c$ can be contracted to a point by a homotopy such that the length of curves in this homotopy is bounded by a function ϕ of the length of ω then one can write an explicit formula for h involving ϕ, n, d, c, m.*

Proof. The lemma basically follows from steps 1-5 above. It remains to observe that M^n can be regarded as a simply connected metric ball centered at any of its points and to use Lemmae 3.5 and 3.3 again. The first step is to cover a manifold M^n by simply connected metric balls and use Lemmae 3.5 and 3.3 to reduce the case of a general closed curve in M^n to the case of a closed curve that lies in a ball of radius less then the radius of simply connectedness. QED.

4. An upper bound on the effective rank

In this section we will establish an upper bound for the $\frac{1}{\sin\epsilon}$-effective rank of the ball.

Once again the proof of the following proposition is a modification of the similar proof in [G1]. We will need to use an effective version of the Isotopy Lemma ([Cg]) that was proved by Grove and Peterson in [GrP], (Theorem 1.6, pg.199).

Lemma 4.1. *Let $B_{r_1}(p)$ and $B_{r_2}(p)$ be two metric balls on a manifold M^n with $r_2 < r_1$. Suppose that there is no ϵ-almost critical points to p on the complement of $B_{r_2}(p)$ in the $B_{r_1}(p)$. Then there exists a homotopy that deforms $B_{r_1}(p)$, so that it lies inside $B_{r_2}(p)$ and the width of this homotopy is bounded from above by $\frac{r_1}{\sin\epsilon}$.*

Proposition 4.2. *Let M^n be a compact manifold of diameter d such that its curvature $K \geq -1$. Then modified a'-effective rank of any ball in M^n, where $a' = 1/\sin\epsilon$, $\epsilon = \frac{1-c_d}{2(1+c_d)}$ as in Sect. 2, will be bounded by $(n-1)\pi^{n-1}e^{4(n-1)d/5}$.*

In order to prove this proposition, we will first have to prove the following

Lemma 4.3. *Let $B_r(p)$ be a ball of radius r on a complete Riemannian manifold M^n. Assume $5s + d(p,y) \leq 5r; d(p,y) \leq 2r$. Then if $B_r(p)$ doesn't $\frac{1}{\sin\epsilon}$-effectively compress to $B_s(y)$ there exists an ϵ-almost critical point x of y with $s \leq d(x,y) \leq r + d(p,y)$.*

Proof. Let us assume that there are no ϵ-almost critical points in the complement of the $B_s(y)$ in $B_{d(p,y)+r}(y)$. Then the bigger ball can be $\frac{1}{\sin\epsilon}$-effectively deformed into a smaller one (this follows from the effective version of the Isotopy Lemma,) but $B_r(p) \subset B_{d(p,y)+r}(y) \subset B_{5r}(p)$,which is a contradiction. Therefore, there exists and ϵ-almost critical point x in the complement of $B_s(y)$ in $B_{d(p,y)+r}(y)$ QED.

Next Lemma will be a slight modification of the similar lemma in [G1], where the term “ $\frac{1}{\sin\epsilon}$-effectively” will be added in the appropriate places, (see also [Cg] for the proof of the original Lemma.)

Lemma 4.4. *Let M^n be a Riemannian manifold and let $rank'_{\frac{1}{\sin\epsilon}}(r,p) = j$. Then there exists $y \in B_{5r}(p)$ and $x_j, ...x_1 \in B_{5r}(p)$ such that for all $i \leq j$, x_i is ϵ-almost critical with respect to y and $d(x_i,y) \geq (5/4)d(x_{i-1},y)$.*

Proof. We begin by considering the metric ball $B_r(p)$ of $rank'_{\frac{1}{\sin\epsilon}}(r,p) = j$. By the definition of the $\frac{1}{\sin\epsilon}$-effective $rank'$ $B_r(p) \longmapsto_{\frac{3}{\sin\epsilon}} B_{r_j}(p_j)$, such that the following conditions are satisfied:
$B_{r_j}(p_j)$ is $\frac{1}{\sin\epsilon}$-effectively incompressible;

there exists $p'_{j-1} \in B_{r_j}(p_j)$ and $r'_{j-1} \leq r_j/10$ such that $rank'_{\frac{1}{\sin\epsilon}}(r'_{j-1}, p'_{j-1}) = j-1$.

Similarly, the ball $B_{r'_{j-1}}(p'_{j-1}) \longmapsto_{\frac{3}{\sin\epsilon}} B_{r_{j-1}}(p_{j-1})$ such that $B_{r_{j-1}}(p_{j-1})$ is $\frac{1}{\sin\epsilon}$-effectively incompressible;
there exists $p'_{j-2} \in B_{r_{j-1}}(p_{j-1})$ and $r'_{j-2} \leq r_{j-1}/10$, such that $rank'_{\frac{1}{\sin\epsilon}}(r'_{j-2}, p'_{j-2}) = j-2$, etc.

Note: $B_{3r_j/2}(p) \supset B_{5r_{j_1}}(p_{j_1})$.

By proceeding in the above fashion we obtain the sequence of balls $B_{r_i}(p_i), i = 0, 1, ..., j$. such that for $1 \leq i \leq j$ $B_{r_i}(p_i)$ is $\frac{1}{\sin\epsilon}$-incompressible and $B_{3r_i/2} \supset B_{5r_{i-1}}(p_{i-1}), r_{i-1} \leq r_i/10$. Let $y = p_0$. Then $y \in B_{3r_i/2}(p_i)$ for all $1 \leq i \leq j$. In particular, $d(p_i, y) + 5r_i/2 \leq 4r_i < 5r_i$ and $d(p_i, y) \leq 3r_i/2 < 2r_i$. Since $B_{r_i}(p_i)$ is $\frac{1}{\sin\epsilon}$-incompressible it doesn't $\frac{1}{\sin\epsilon}$-effectively compress to $B_{r_i/2}(y)$. Therefore, by Lemma 4.1 there exists an ϵ-almost critical point x_i with

$$r_i/2 \leq d(x_i, y) \leq r_i + 2 \cdot (3/2)r_i = 4r_i.$$

Then $d(x_i, y) \geq r_i/2 \geq 5r_{i-1} \geq (5/4) \cdot 4r_{i-1} \geq 5d(x_{i-1}, y)/4$. QED.

Corollary 4.5. $rank'_{\frac{1}{\sin\epsilon}}(r, p) \leq (n-1)\pi^{n-1}e^{4(n-1)d/5} \leq e^{3(d+1)(n-1)}$, *which proves Proposition 4.2.*

Proof of Theorem D. Combine Lemma 3.7 with Corollary 4.5 which provides an upper bound for m. QED.

We are now ready to construct a homotopy with the following properties:

1. the length of the curves in the homotopy is bounded;
2. there exists a point p for which the length of the curve $H_\tau(p)$ is bounded.

5. Construction of a homotopy with curves of bounded length

In this section we will prove the following proposition:

Proposition 5.1. *Let $M^n \subset \Psi$, and $\gamma(t)$ a closed curve in M^n. Assume that all metric balls in M^n of radius less than c are simply connected. There exists a homotopy $H^{new}_\tau(t)$ satisfying the following properties:*

1. $H^{new}_0(t) = \gamma(t)$;

2. $H^{new}_1(t) = p$, where $p \in M^n$;

3. $\sup_{\tau_*} lengthH^{new}_{\tau_*}(t) \leq length(\gamma) + 2Q(n, d, c)$

4. length $H^{new}_\tau(t_i) \leq Q(n, d, c)$, where $\gamma(t_i), i = 1, 2$ are two selected points on the curve $\gamma(t)$, and Q is the same function as in Theorem D.

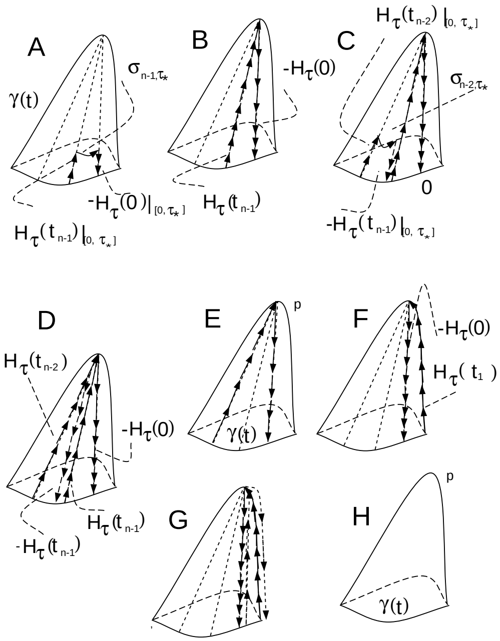

Fig. 11A–H.

Proof. Let us begin by observing that for any two curves $\alpha_1(t), \alpha_2(t)$ such that $d(\alpha_1(t), \alpha_2(t)) < inj/3$ for any t a function that for any $t \in (0,1)$ assigns the minimal geodesic that joins $\alpha_1(t)$ and $\alpha_2(t)$ is continuous.

Let $\gamma(t)$ be a closed curve as above, and let $H_\tau(t)$ be the homotopy of $\gamma(t)$ to p provided by Theorem D (so that we have an upper bound for the width of H). For any τ we will partition the unit interval into n subintervals $[t_i, t_{i+1}], t_0 = t_n$ so that $d(H_\tau(t_i), H_\tau(t_{i+1})) \leq inj/3$. That partition is possible to achieve because of the continuity of H_τ. Let $\sigma_{i,\tau}(s)$ be the minimal geodesic that joins points $H_\tau(t_i)$ and $H_\tau(t_{i+1})$. We are now ready to describe our new homotopy, (see also Fig. 11.) We claim:

1. $\gamma(t)$ is homotopic to the curve $\lambda_1 = \gamma|_{[0,t_{n-1}]} \cup H_\tau(t_{n-1})|_{[0,\tau_*]} \cup \sigma_{(n-1),\tau_*} \cup -H_\tau(0)|_{[0,\tau_*]}$. Moreover, the length of curves in the homotopy is bounded by $l(\gamma) + 4W_H$, (see Fig. 11A.)

2. λ_1 is homotopic to $\lambda_2 = \gamma|_{[0,t_{n-1}]} \cup H_\tau(t_{n-1}) \cup -H_\tau(0)$, (see Fig. 11B.)

3. λ_2 is homotopic to $\lambda_3 = \gamma|_{[0,t_{n-2}]} \cup H_\tau(t_{n-2}) \cup -H_\tau(t_{n-1}) \cup H_\tau(t_{n-1}) \cup -H_\tau(0)$. The length of curves in the homotopy is bounded by $l(\gamma) + 8W_{H_\tau}$, (see Figs. 11C and D.)

4. λ_3 is homotopic to $\lambda_4 = \gamma|[0, t_{n-2}] \cup H_\tau(t_{n-2}) \cup -H_\tau(0)$, (see Fig. 11E.)

5. λ_4 is homotopic to $\lambda_5 = \gamma|[0, t_1] \cup H_\tau(t_1) \cup -H_\tau(0)$, (see Fig. 11F.)

6. λ_5 is homotopic to $\lambda_6 = H_\tau(t_1) \cup -H_\tau(t_1) \cup -H_\tau(0) \cup H_\tau(0)$ which is homotopic to a point p, (see Figs. 11G and H.) Note also that for points $\{t_i\}, l(H_\tau^{new}(t_i)) \leq 4W_{H_\tau}$ and that we can partition the unit interval in such a way that some selected points on the curve γ are among $\gamma(t_i)$. QED.

Note that if we have an explicit upper bound for the width of the homotopy H_τ of $\gamma(t)$ to p provided by Theorem D then we can replace f_1 and f_2 in the right hand sides of 3. and 4. by explicit expressions.

In Sect. 6 we will establish a similar result for compact simply connected manifolds with a non-trivial second homology group, $K \leq 1$ and $vol(M^n) \leq V$, where K is sectional curvature.

6. Manifolds with bounded from above curvature

Our result will be based on the following two inequalities:

1. Croke's isoperimetric inequality: $vol(B) \geq const._n r^n$, where B(r,p) is any metric ball of radius r centered at $p \in M^n$ and $r < inj(M^n)/2$. We can take $const._n = \frac{2^{n-1}}{(n!)^2}$, (see [Cr1], [Ch]).

2. Berger's inequality: $vol(M^n) \geq c_n(inj(M^n)/\pi)^n$ where M^n is compact manifold of dimension n, and c_n can be estimated to be $\frac{\pi^n}{n!}$, (see [Ch]).

We will also need the following corollary to Klingenberg's lemma:

Lemma 6.1. *Let M^n be a compact manifold of sectional curvature bounded from above, i.e. $K \leq 1$. Then $inj(M^n) \geq min(\pi, l(M^n)/2)$, where $l(M^n)$ is the length of the shortest simple closed geodesic, (see [GoHL].)*

Suppose $l(M^n) = 2inj(M^n)$. Then we are done, because by Berger's inequality $l(M^n) = 2inj(M^n) \leq c_n' vol(M^n)^{1/n}$. Therefore, from now on we will assume that $l(M^n) > 2inj(M^n) > \pi$. Our approach will be similar to that of Sect. 3, but instead of first constructing a homotopy of bounded width we will right away construct a homotopy similar to the one of Sect. 5. We will show that it is possible to construct such a homotopy if the distance between two curves $d(\alpha_1(t), \alpha_2(t)) \leq \pi/9$. Then we will construct a sequence of curves $\{\sigma_i(t)\}_{i=1}^m$ such that

1. $\sigma_1(t) = \alpha(t)$;
2. $\sigma_m = p$, where p is a point on a manifold.
3. $d(\sigma_i, \sigma_{i+1}) \leq \pi/9$ and
4. $m \leq (\frac{(n!)^2 vol(M^n)}{20^{n-1}})^{216L/\pi+1}$

Lemma 6.2. *Let $\Phi_L(M^n)$ be the space of piecewise differentiable closed curves of length $\leq L$ parametrized proportionally to their arclength. There exists $\pi/36$-net on the δ-neighborhood $N_\delta(\Phi_L(M^n))$ of such curves and the number of elements in $\pi/36$-net will be $\leq (\frac{(n!)^2 vol(M^n)}{20^{n-1}})^{216L/\pi+1}$*

Proof. First, we will have to construct $\pi/216$-net on M^n and estimate the number of elements in it. It will be done using Croke's inequality. Let us consider the maximal number of pairwise disjoint balls in M^n of radius $\pi/432$. The number of such balls will be $\leq \frac{vol(M^n)}{\max vol(B_{\pi/432}(p_i))} \leq \frac{vol(M^n)}{\frac{2^{n-1}432^n}{(n!)^2\pi^n}} = const.'_n vol(M^n)$. The set $(B_{\pi/216}(p_i))_{i=1}^m$ is a cover of M^n, thus the set of points $\pi/216$-net on M^n. We will construct the required net on $N_\delta(\Phi_L(M^n))$ by joining p_i and p_j with minimal geodesics if and only if $d(p_i, p_j) \leq \pi/72$ and considering its subset consisting of closed curves. The number of elements in that set can be estimated to be

$$\begin{aligned} \leq [const.'_n vol(M^n)]^{216L/\pi+1} &= \left(\frac{(n!)^2\pi^n vol(M^n)}{2^{n-1}432^n}\right)^{216L/\pi+1} \\ &\leq \left(\frac{(n!)^2 vol(M^n)}{20^{n-1}}\right)^{216L/\pi+1}. \end{aligned}$$

Let us now apply Lemma 3.4 substituting $\pi/9$ for ϵ. QED.

Lemma 6.3. *Let M^n be a compact simply connected manifold with $K \geq 1$, $vol(M^n) \leq V$, and $l(M^n) \geq L > \pi$. Let $\gamma(t) \in \Phi_L(M^n)$. There exists a finite sequence $\{\sigma_i\}_{i=1}^k$ of broken geodesics such that $d(\sigma_1(t), \gamma(t)) \leq \pi/36$; $\sigma_k = p$; length(σ_i) $\leq 3L$; $d(\sigma_i, \sigma_{i+1}) < \pi/9$ and the number k of elements in this sequence is $\leq N$, where $N = (const.'_n vol(M^n))^{216L/\pi+1}$.*

Proof. as in Lemma 3.5, part 1.

We are now ready to construct a homotopy with the required properties.

Lemma 6.4. *Assume $l(M^n) > L$. Let M^n be as in Lemma 6.3, and let $\gamma(t) \subset M^n$ be any closed curve of length $\leq L$. Then there exists a homotopy $H_\tau(t)$ of a curve to a point p, such that:*

1. $\sup_\tau\{$*length* $H_\tau(t)/t \in [0,1]\} \leq 2L + \pi$*;*

2. length $H_\tau(t_i), \tau \in [0,1] \leq \frac{\pi(const.'_n vol(M^n))^{216L/\pi+1}}{9}$
$\leq (const.'_n vol(M^n))^{216L/\pi+1}$ *for at least two selected points* $t_i, i = 1, 2$.

Proof. First we will show that any two curves $\alpha_1(t), \alpha_2(t)$ such that $d(\alpha_1(t), \alpha_2(t)) \leq \pi/9$ can be connected by a homotopy $H'_\tau(t)$ for which

1. $\sup_\tau\{length H_\tau(t)/t \in [0,1]\} \leq 2L + \pi$.

2. $length H_\tau(t_i), t \in [0,1] \leq \pi/9$.

Then we will apply Lemma 6.3. to get a desired homotopy.

Let us subdivide the interval [0,1] into subintervals $[t_i, t_{i+1}]$, so that the figure with vertices: $\alpha_1(t_i)$, $\alpha_2(t_i)$, $\alpha_2(t_{i+1})$, and $\alpha_1(t_{i+1})$ and edges $\alpha_j|_{[t_i,t_{i+1}]}$ and $\sigma_i(s)$, where $\sigma_i(s)$ is a minimal geodesic joining $\alpha_1(t_i)$ and $\alpha_2(t_i)$ lies inside the ball of radius $\pi/3$. That is possible to achieve by

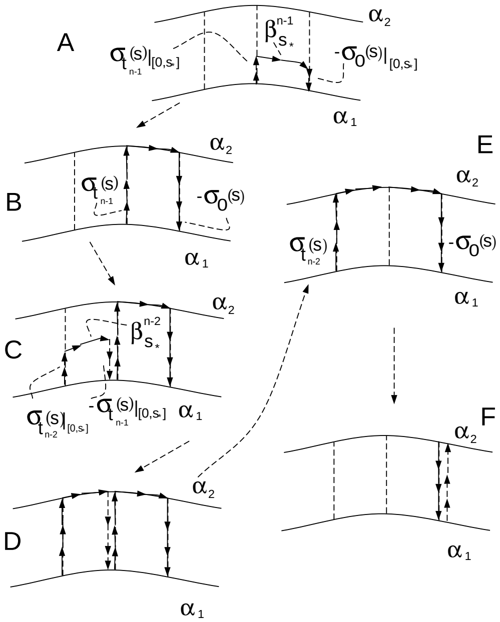

Fig. 12A–E.

demanding that max length $\alpha_j|_{[t_i,t_{i+1}]} \le \pi/9$. The function that for each s assigns the minimal geodesic $\beta^i_s(t)$ that joins points $\sigma_i(s)$ and $\sigma_{i+1}(s)$ is continuous. W.L.O.G. assume that $\alpha_j|_{[t_i,t_{i+1}]}$ is a geodesic. Let us call it α^i_j. Then we claim:

1. α_1 is homotopic to $\gamma_1 = \alpha_1|_{[0,t_{n-1}]} \cup \sigma_{t_{n-1}}(s)|_{[0,s_*]} \cup \beta^{n-1}_{s_*} \cup -\sigma_0(s)$. Moreover, the length of curves in the homotopy is bounded by $4\pi/9+l(\alpha_1)$.

2. γ_1 is homotopic to $\gamma_2 = \alpha_1|_{[0,t_{n-1}]} \cup \sigma_{t_{n-1}} \cup \alpha_2|_{[t_{n-1},0]} \cup -\sigma_0$. Length of curves in the homotopy bounded by $2\pi/9 + l(\alpha_1) + l(\alpha_2)$.

3. γ_2 is homotopic to $\gamma_3 = \alpha_1|_{[0,t_{n-2}]} \cup \sigma_{t_{n-2}}|_{[0,s_*]} \cup \beta^{n-2}_{s_*} \cup -\sigma_{t_{n-1}}|_{[0,s_*]} \cup \sigma_{t_{n-1}} \cup \alpha_2|_{[t_{n-1},0]} \cup -\sigma_0$. Length of curves in the homotopy is $\le 2\pi/3 + l(\alpha_1) + l(\alpha_2)$.

4. γ_3 is homotopic to $\gamma_4 = \alpha_1|_{[0,t_{n-2}]} \cup \sigma_{t_{n-2}} \cup \alpha_2|_{[t_{n-2},t_{n-1}]} \cup -\sigma_{t_{n-1}} \sigma_{t_{n-1}} \cup \alpha_2|_{[t_{n-1},0]} \cup -\sigma_0$.

5. γ_4 is homotopic to $\gamma_5 = \alpha_1|_{[0,t_{n-2}]} \cup \sigma_{t_{n-2}} \cup \alpha_2|_{[t_{n-2},0]} \cup -\sigma_0$.

6. γ_5 is homotopic to $\gamma_6 = \alpha_1|_{[0,t_1]} \cup \sigma_{t_1} \cup \alpha_2|_{[t_1,0]} \cup -\sigma_0$.

7. γ_6 is homotopic to $\gamma_7 = \sigma_0 \cup \alpha_2 \cup -\sigma_0$, which is homotopic to α_2. Note that the maximal length of curves in the resulting homotopy is bounded from above by $\pi + 2L$. Note also that for all t_i length $H(t_i) \le \pi/9$, (see Fig. 12.)

Now let us apply Lemma 6.3, i.e. take a sequence of broken geodesics $\{\sigma_i\}$. We know that σ_i is homotopic to σ_{i+1}, where the homotopy H^i has the desired properties. Take the composition of those homotopies to obtain $H_\tau(t)$. It is clear that it will satisfy 1 and 2 of the lemma. QED.

7. Construction of a closed path in the space of closed curves

Before we give the proofs of the main theorems let us state the following two definitions from [G2]:

Definition 7.1. *Filling Radius of n-dimensional Manifold M Topologically Imbedded into X.* Filling radius, denoted by Fill Rad(M $\subset X$), where X is an arbitrary metric space, is the infimum of $\epsilon > 0$, such that M bounds in the ϵ-neighborhood $N_\epsilon(M)$, i.e. homomorphism $H_n(M) \to H_n(N_\epsilon(M))$ induced by the inclusion map vanishes.

Definition 7.2. *Filling Radius of an Abstract Manifold.* Filling radius Fill Rad M of an abstract manifold M is Fill Rad ($M \subset X$), where $X = L^\infty(M)$, i.e. the Banach space of bounded Borel functions f on M, and the embedding of M into X that to each point p of M assigns a distance function $p \mapsto f_p = d(p,q)$.

In order to prove Theorem AA we will need the following result proved by Gromov in [G2].

Theorem 7.3. *Let M be a closed connected Riemannian manifold of dimension n. Then Fill Rad M $\leq (n+1)n\sqrt[n]{n!}(volM)^{1/n}$.*

We are now ready to prove the following proposition:

Proposition 7.4. *Let M^n be a compact 2-essential manifold of dimension n, with the property that for every closed curve $\gamma(t)$ of length $\leq \min\{6FillRadM^n, 3diam(M^n)\}$, there exists a homotopy $H_\tau(t)$ of that curve to a point p, such that the length of curves in the homotopy is bounded by L_1 and for two selected points on $\gamma(t) : \gamma(t_i), i = 1, 2$ the length $H_\tau(t_i) \leq L_2$. Then there exists a closed path in the space $\Lambda(M^n)$ of closed curves on a manifold with the property that the length of each curve is bounded by $h(L_1, L_2) = 3L_1 + 6L_2$ and such that this path represents a non-trivial element of $\pi_1(\Lambda M^n)$.*

Proof. The proof will be done in two steps: The idea of Step 1 is to obtain a non-trivial element of $H_2(M^n)$ with some special properties described below. This will be done in the following way: we will "fill" M^n with W, and then try to extend the map $f : M^n \to \mathbf{C}P^\infty$ to W. We will then obtain the required element as an obstruction. This part of the proof will be a modification of Lemma 1.2 B and Proposition (pg. 136) of Gromov in [G2]. In Step 2 we turn this element of $H_2(M^n)$ into the required loop in ΛM^n.

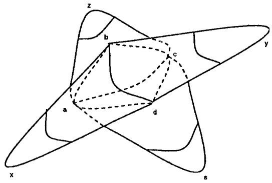

Fig. 13

Step 1. For any positive δ, M^n bounds in its Fill Rad $(M^n)+\delta$-neighborhood in $L^\infty(M^n)$. Let $M^n = \partial W$, where W is a compact $(n+1)$-dimensional polyhedron in $N_{FillRad(M^n)+\delta}(M^n)$. Since M^n is 2-essential there exists a function $f : M^n \to \mathbf{C}P^\infty$ such that $f_*[M] \neq 0$. We will try to extend f to W. Let us proceed as follows: first, extend f to 0-skeleton of W, then to 1-skeleton of W, etc. This process will have to be interrupted at the 4th stage since we know that f cannot be extended to W. (Here we somewhat oversimplify by assuming that W can be triangulated. In case it cannot, we can still approximate it by a simplicial complex.)

Extending to 0-skeleton: Subdivide W, so that all simplices have diam(σ) $\leq \delta$. Send vertices $w_i \in W$ to vertices of triangulation $m_i \in M^n$ for which $d(w_i, m_i) \leq d(w_i, M^n) + \delta <$ Fill Rad $M^n + \delta$. Suppose m_i, m_j come from the vertices w_i, w_j of some simplex in W. Then $d(m_i, m_j) \leq d(m_i, w_i) + d(w_i, m_j) \leq d(m_i, w_i) + d(w_i, w_j) + d(w_j, m_j) \leq$ 2Fill Rad $M^n + 3\delta$. Thus, m_i, m_j can be joined by geodesic of length $<$ 2Fill $RadM^n + 3\delta$.

Extending to 1-skeleton: send the 1-simplices $[w_i, w_j] \subset W/M^n$ to the above geodesics joining m_i and m_j. (In addition, we assume all 1-simplices in M^n to be already short.) So we can see that the boundary of each 2-simplex in W is sent to a curve of length $<$ 6Fill Rad $M^n + 9\delta$, (note, that it is also $\leq 3diam(M^n)$).

Extending to 2-skeleton: let σ be a 2-simplex of W. Consider its boundary $\partial\sigma$ and the image of $\partial\sigma$ under f. It will be a closed curve consisting of broken geodesics. Let us call it $\gamma(t)$. By our hypothesis we know that there exists a special homotopy $H_\tau(t)$ of that curve to a point. We will then map σ to the surface determined by this homotopy.

We have, thus, succeeded in extending our map to the 2-skeleton. Extending the map to 3-skeleton would have been equivalent to extending it to the whole of W, but that is impossible, since it would contradict $f_*[M^n] \neq 0$. Therefore, there exists a 3-simplex in W, such that the image of its boundary ω represents a non-trivial element of $H_2(M^n)$.

Step 2: Consider ω. Let's denote its vertices a,b,c,d. (Here, we will call vertices of ω the images of vertices a^*, b^*, c^*, d^* of Ω.) Let x,y,z,s be the images of the faces of Ω. (They are obtained by contracting closed curves [a,b,d], [b,d,c], [a,b,c], [d,a,c] in M^n by the homotopies from the hypothesis

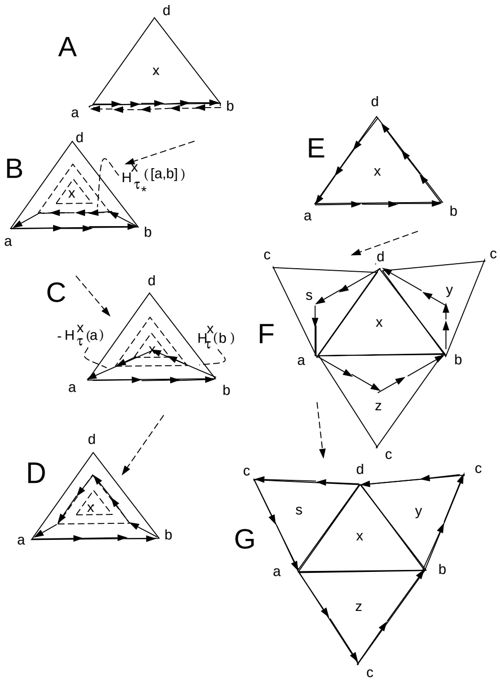

Fig. 14A–G.

of this proposition, which we will call H^x_τ, H^y_τ, H^z_τ, H^s_τ respectively.) Let us examine the face x of ω. We will claim that

1. a, regarded as a constant curve, is homotopic to $[a,b] \cup [b,a]$, (see Fig. 14.A.) and the length of curves in this homotopy $\leq L_1$.

2. $[a,b] \cup [b,a]$ is homotopic to $[a,b] \cup H^x_\tau(b) \cup -H^x_\tau(a)$ and the length of curves in the homotopy is $\leq 2L_1 + 2L_2$. Figure 14.B and C.

3. $[a,b] \cup H^x_\tau(b) \cup -H^x_\tau(a)$ is homotopic to $[a,b] \cup H^x_\tau|_{[0,\tau_*]}(b) \cup H^x_{\tau_*}([a,b]) \cup -H^x_\tau|_{[0,\tau_*]}(a)$. Figure 14.D. The length of curves in the homotopy is bounded from above by $2L_1 + 2L_2$.

4. $[a,b] \cup H^x_\tau|_{[0,\tau_*]}(b) \cup H^x_{\tau_*}(t) \cup -H^x_\tau|_{[0,\tau_*]}(a)$ is homotopic to $[a,b] \cup [b,d] \cup [d,a]$, and the length of the curves in the homotopy is $\leq 2L_1 + 2L_2$. Figure 14E.

By the same type of constructions we have

5. $[a,b] \cup [b,d] \cup [d,a]$ homotopic to $[a,z] \cup [z,b] \cup [b,y] \cup [y,d] \cup [d,s] \cup [s,a]$ homotopic to $[a,c] \cup [c,b] \cup [b,c] \cup [c,d] \cup [d,c] \cup [c,a]$ homotopic to $[a,c] \cup [c,a]$ homotopic to a, where the length of the curves in the homotopy is $\leq 3L_1 + 6L_2$. See Fig. 14F and G. QED.

Remark. We need Proposition 7.4 only to prove Theorem AA below. However, the construction of Step 2 producing a loop in ΛM^n out of a non-trivial cycle in $H_2(M^n)$ will be also used in the proofs of Theorems A, B and C. Note that 2-essentiality of M^n is not required for Step 2.

Now we can prove the main theorems. The proof of Theorem C will be a combination of application of the results of Sections 1-5 and the Step 2 of Proposition 7.4.

Proof of Theorem C. Consider a map $f : S^2 \to M^n$, such that the image of S^2 under f represents a non-trivial element of $H_2(M^n)$.

We will try to extend f to D^3, which should be impossible to do, therefore, as an obstruction to this extension, we will obtain a non-trivial element of $H_2(M^n)$ that has a special shape. As in the previous proposition we will do it skeleton by skeleton.

We will proceed as follows: first triangulate S^2 and consider an induced triangulation of the image of S^2. We will want the triangulation to be fine enough, so that the diameter of each simplex τ in the induced triangulation is less than the injectivity radius of a manifold, next we will attempt to construct a new map $f^* : D^3 \to M^n$, such that $f^*|_{S^2} = f$.

Let p be the center of D^3 and $\tilde{p}$ be any point on M^n. Define $f^*(p) = \tilde{p}$.

Connect p with the vertices of triangulation of S^2 by the straight lines, and also connect $\tilde{p}$ with the vertices of induced triangulation by minimal geodesics. Let f^* map the line segment $\bar{pv}$ to the minimal geodesic segment that joins $\tilde{p}$ and $f(v)$, for the vertex v of the triangulation of S^2.

Now consider all closed curves that are formed by three segments $[p, v_1]$, $[p, v_2]$, $[v_1, v_2]$, where $[p, v_1]$, $[p, v_2]$ are line segments that connect a point p with the vertices v_1, v_2 and $[v_1, v_2]$ is an edge of S^2. Consider also the corresponding closed curves that are formed by the three segments $[\tilde{p}, f(v_1)]$, $[\tilde{p}, f(v_2)]$, $[f(v_1), f(v_2)]$. The length of such closed curves is $\leq 3d$. We can contract each of those curves to a point using the homotopy of Proposition 5.1. Then each simplex $[p, v_1, v_2]$ can be mapped by f^* to the surface generated by this homotopy.

We now have succeeded at extending f to the 2-skeleton of D^3. We should not be able to extend it to the 3-skeleton. Therefore, among the 2-cycles of $H_2(M)$ that we obtained, at least one, should be non-trivial. This cycle has the shape of the one on Fig. 13, i.e. its vertices are $\tilde{p}, f(v_1), f(v_2), f(v_3)$, its edges are curves of length $\leq d$, and its faces are surfaces generated by the homotopies of Proposition 5.1 (except for the face that lies in $f(S^2)$, which is generated by the homotopy that is even nicer since it lies inside the ball of radius that is less than the injectivity radius of M^n.) We can now denote this cycle ω and follow Step 2 of Proposition 7.4. where $L_1 \leq Q(n, d, c)$ and $L_2 \leq 3d + 2Q(n, d, c)$, where L_1 and L_2 are as in Proposition 7.4. Therefore, by Step 2 of Proposition 7.4. there exists a closed non-trivial curve in the space of all closed curves such that the length of each curve is bounded by $3L_1 + 6L_2 \leq 18d + 15Q(n, c, d)$.

Proof of Theorem A. If $l(M^n) = 2\mathrm{inj}(M^n)$ then $l(M^n) \leq 2\sqrt[n]{n!}vol(M^n)^{1/n} \leq 2\sqrt[n]{n!}V^{1/n}$ and we are done. Otherwise, $inj(M^n) \geq \pi$ (since $K \leq 1$) and the proof is analogous to the proof of the Theorem C (but a much simpler argument explained in Sect. 6 replaces the argument of Sects. 2-4 using ideas from [G1]). The proof is an application of Proposition 6.4 and the argument of the proof of Proposition 7.4. As in Proposition 7.4 we first obtain a "special" non-trivial element of $H_2(M^n)$. Then we construct a corresponding closed curve in the space of closed curves out of it. One can just follow Step 2 of Proposition 7.4. In this case take $L_1 = 6d + \pi$ and $L_2 = (const.'vol(M^n))^{532(d+1)}$, and note that $d \leq (2n!)^3(vol(M^n))$. (d is bounded by the ratio of $vol(M^n)$ to the volume of the ball of radius $inj(M^n)/2$, which can be estimated with the help of Croke's inequality.) Obvious substitutions will imply the result. QED.

Proof of Theorem AA. Once again, let us assume that $l(M^n) > 2inj(M^n)$. The proof is again similar to the one of Theorem A and is an application of Proposition 6.4, Theorem 7.3 and Proposition 7.4. Note that in this case $L_1 = 12\text{Fill Rad } M^n + \pi$, $L_2 = (\frac{(n!)^2 vol(M^n)}{20^{n-1}})^{216 FillRad M^n + 1}$. Obvious substitutions and calculations imply the result. QED.

Proof of Theorem B. We know from the work of Grove and Petersen ([GrP]) how to find positive $\delta(n, v, D)$ and $C(n, v, D)$ such that any metric ball of radius $r \leq \delta(n, v, D)$ in a closed n-dimensional Riemannian manifold of diameter less than D, volume greater than v and $K \geq -1$ is contractible within the concentric ball of radius $C(n, v, D)r$. In fact, one can take $C(n, v, D) = \frac{\xi_1(n)e^{(n-1)D}}{v}$ and $\delta = \frac{\xi_2(n)v\min\{1,v\}}{De^{(n-1)D}}$. Moreover the work [GrP] implies an explicit upper bound for the width of the optimal homotopy contracting a closed curve in such small metric balls to a point, namely $C(n, v, D)r$. The standard Bishop-Gromov argument provides an upper bound for the number of balls of radius $\delta(n, v, D)/10$ covering any manifold with $K \geq -1$, volume greater than v and diameter less than D. Now the argument based on Lemma 3.3 (1) and used previously in the proofs of Theorems C,A and AA can be used to obtain an upper bound for the width of an optimal homotopy contracting a closed curve in M^n by reducing this to the case of closed curves in metric balls of radius $\leq \delta(n, v, D)$. The rest of the proof is the same as in the proofs of Theorems A and C. QED.

Acknowledgements. I would like to thank Prof. D. Gromoll for numerous discussions and helpful suggestions. I am grateful to A. Nabutovsky for valuable comments. I would also like to thank Prof. F. Tangerman and anonymous referee for their comments that helped to improve the exposition of the paper.

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