

LENGTHS OF THREE SIMPLE PERIODIC GEODESICS ON A RIEMANNIAN 2-SPHERE

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ABSTRACT. Let M be a Riemannian 2-sphere. A classical theorem of Lyusternik and Shnirelman asserts the existence of three distinct simple non-trivial periodic geodesics on M . In this paper we prove that there exist three simple periodic geodesics with lengths that do not exceed $20d$, where d is the diameter of M .

We also present an upper bound that depends only on the area and diameter for the lengths of the three simple periodic geodesics with positive indices that appear as minimax critical values in the classical proofs of the Lyusternik-Shnirelman theorem.

Finally, we present better bounds for these three lengths for “thin” spheres, when the area A is much less than d^2 , where the bounds for the lengths of the first two simple periodic geodesics are asymptotically optimal, when $\frac{A}{d^2} \rightarrow 0$.

1. MAIN RESULTS.

Theorem 1.1. *Let M be a Riemannian manifold of diameter d diffeomorphic to S^2 . There exist three distinct non-trivial simple periodic geodesics of length not exceeding $20d$ on M .*

Recall that the term “simple” means that the geodesics do not have self-intersections.

The upper bounds $5d$ and $10d$ for lengths of two shortest simple geodesics were found in [NR], but there we were not able to estimate the length of the third simple periodic geodesic.

Note that one can get a better upper bound $4d$ for the length of the shortest non-trivial but not necessarily simple periodic geodesic ([S], [NR0]). Yet the example of a “three-legged starfish” glued out of three long thin tentacles illustrates the difference between majorization of the length of the shortest periodic geodesic and the length of the shortest simple periodic geodesic: The shortest periodic geodesic will be a figure eight shaped self-intersecting curve hugging two of three tentacles. On the other hand it seems that the shortest simple periodic geodesic will be a curve that goes all the way along one of the tentacles and passes through its top and the saddle point where two other tentacles intersect. This example suggests that the approach developed in [NR] and [Sa] is not applicable for majorization of lengths of simple periodic geodesics. It also suggests that

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apparently there is no upper bound for the length of the shortest simple periodic geodesic in terms of the area of M alone. On the other hand the examples of ellipsoids that are very close to the standard sphere demonstrate that 1) there can be only three simple periodic geodesics; and that 2) the length of the fourth periodic geodesic (simple or not) cannot be majorized in terms of the diameter, the area and even an upper bound for the absolute value of the curvature of the sphere. (This is classical result of M. Morse, who proved that the fourth periodic geodesic becomes uncontrollably large for ellipsoids with distinct but very close semiaxes. Also, a very nice extension of this result of Morse for Riemannian 2-spheres of almost constant curvature was proven by W. Ballmann in [B1].) So, in this respect our result is the optimal possible. Finally note that techniques from [LNR] lead to upper bound for the lengths of three simple periodic geodesics in terms of the diameter *and* the area of M , that are, however, much better than the estimates in [NR] and the present paper, in the case when the area is much smaller than the square of the diameter (see Theorem 1.3 below).

The main idea of the proof of Theorem 1.1 is to express three homology classes of the space of non-parametrized curves that are used in classical proofs of the Lyusternik-Shnirelman theorem by cycles that consist of simple closed curves “mainly made” of curves in a meridian-like family that connects two fixed points of M . Following ideas of [Cr] and [NR] and using some observation of [M] we attempt to construct such a family where the lengths of all meridians are bounded by *const* d for an appropriate *const*. Our repeated attempts can be blocked only by appearance of different “short” simple periodic geodesics of index 0. So, we either get three short simple periodic geodesics of index 0, or our third attempt to construct a “meridional slicing” succeeds. Once one of our attempts succeeds, and we get a slicing of M into short meridians, the original proof of the Lyusternik-Shnirelman theorem yields the desired upper bounds.

Also note that the standard proofs of the Lyusternik-Shnirelman theorem produce three simple geodesics that have positive indices (cf. [T]). In fact, these geodesics appear as minimax critical values corresponding to certain families of 1-, 2-, and 3-dimensional cycles with \mathbb{Z}_2 coefficients in the space of non-parametrized simple curves on the Riemannian 2-sphere that correspond to the generators of the respective homology groups of this space. However, our quantitative version of Lyusternik-Shnirelman produces three geodesics that can be local minima of the length functional. Can one find an effective version of the Lyusternik-Shnirelman theorem, where one majorizes the lengths of three simple geodesics with positive indices that appear in the original proof and are minimaxes for certain specific families of simple closed curves? We believe that there are no such estimates solely in terms of the diameter of M , and that Riemannian metrics constructed in [L] using ideas from [FK] provide a basis for counterexamples.

On the other hand we will use the results from our paper [LNR] to prove that the lengths of three simple periodic geodesics that appear in the classical proofs of the Lyusternik-Shnirelman theorem can be majorized in terms of the volume and the area. In particular:

Theorem 1.2. *Let M be a Riemannian manifold diffeomorphic to the 2-sphere of area A and diameter d . There exist three distinct simple periodic geodesics on M of length less than $800d \max\{1, \ln \frac{\sqrt{A}}{d}\}$ such that none of these geodesics has index zero. (Moreover, these geodesics are exactly the minimax geodesics that appear in the classical proofs of the Lyusternik-Shnirelman theorem).*

Finally, we consider the special case of “thin” 2-spheres. More precisely, we provide upper bounds for the lengths of three simple periodic geodesics in terms of d and the area, which are much better than the estimates in terms of d , when $\frac{A}{d^2}$ is very small. As an extra bonus, our estimates are for the lengths of three geodesics that appear in classical proofs of Lyusternik-Shnirelman and correspond to the generators of the first three homology groups of the space of simple non-parametrized curves.

Theorem 1.3. *Let M be a Riemannian manifold diffeomorphic to the 2-sphere of area A and diameter d . There exist three distinct simple periodic geodesics on M such that none of these geodesics has index zero. and with lengths that do not exceed, correspondingly, $d + 700\sqrt{A}$, $2d + 1400\sqrt{A}$ and $4d + 2800\sqrt{A}$.*

We believe that the coefficient at d in the first of these estimates is optimal as evidenced by the example of a thin three-legged starfish with tentacles of equal size. The example of a thin long ellipsoid of revolution can be used to easily *prove* that 2 is the optimal coefficient at d in the second estimate. Also, one can try to use the ideas of F. Balacheff, C. Croke and M. Katz from [BCK] , where they constructed 2-spheres where all non-trivial periodic geodesics were longer than $2d$ by introducing a mild asymmetry to the Riemannian metric. It seems very plausible that an application of the same idea to long thin ellipsoids of revolution will result in Riemannian metrics on the 2-sphere, where the length of the second shortest simple periodic geodesic is strictly greater than $2d$, thus, justifying the appearance of the $O(\sqrt{A})$ term. We believe that an $O(\sqrt{A})$ term is also required in the first estimate. On the other hand, the coefficient 4 in the estimate for the length of the third shortest simple periodic geodesic on a “thin” 2-sphere is, most probably, not optimal.

2. GEOMETRIC REALIZATION OF CYCLES FROM THE PROOF OF LYUSTERNIK AND SHNIREMAN.

2.1. A brief review of a classical proof of the existence of three simple periodic geodesics The existence of three simple periodic geodesics on every Riemannian 2-sphere

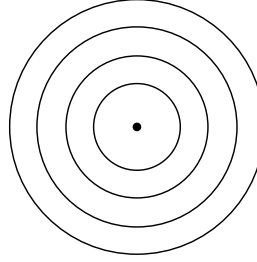
Cycle z_1

FIGURE 1.

was first proven by L.A. Lyusternik and L.G. Shnirelman ([LS], [Ly]). They considered the space ΠM of non-parametrized curves on a 2-dimensional Riemannian manifold M diffeomorphic to S^2 , as well as its subset $\Pi_0 M$ that consists of all constant curves (and, thus, can be identified with M). Following [T] we consider the three relative homology classes of the pair $(\Pi S^2, \Pi_0 S^2)$ with coefficients in \mathbb{Z}_2 , where we regard S^2 as the unit round sphere in \mathbb{R}^3 : 1) The 1-dimensional class represented by the relative 1-cycle z_1 formed by all circles on S^2 in planes parallel to the XZ -plane in the ambient \mathbb{R}^3 (see Fig. 1); 2) The 2-dimensional class represented by the relative cycle z_2 formed by all circles in planes parallel to the Z -axis; and 3) The 3-dimensional class represented by the relative 3-cycle z_3 formed by all round circles on the sphere (including points regarded as “degenerate” circles). Lyusternik and Shnirelman described a curve shortening flow in ΠM .

They observed that this flow “gets stuck” on critical points representing simple periodic geodesics, when applied to z_1, z_2, z_3 , and proved that if two of these three geodesics coincide, then there is a whole critical level with a 1-parametric set of distinct simple periodic geodesics. Some errors in the construction of their curve shortening flow had been later corrected by W. Ballman ([B]), J. Jost ([J]) and I. Taimanov ([T0], [T]). Alternatively, one can prove the existence of three simple periodic geodesics using either the curvature flow and Grayson’s theorem ([Gr]) or an especially simple curve shortening flow constructed by J. Hass and P. Scott ([HS]).

An immediate corollary of any of those versions of the proof of Lyusternik-Shnirelman theorem is the following assertion:

Proposition 2.1. *Let $u_i, i = 1, 2, 3$ be relative singular cycles of $(\Pi M, \Pi_0 M)$ with coefficients in \mathbb{Z}_2 such that u_i is homologous to z_i for each i . Assume that the lengths of simple closed curves on M in the image of each simplex of u_i are bounded by L_i . Then there exist three distinct simple geodesics on M with lengths not exceeding L_1, L_2 and L_3 , correspondingly. Moreover, here we can also assume that these three simple geodesics*

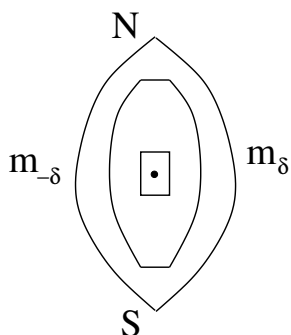


FIGURE 2.

have positive indices, i.e. none of them is a local minimum of the length functional on the space of simple closed curves on M .

Now assume that $f : S^2 \rightarrow M$ is a diffeomorphism from the standard round sphere S^2 of radius 1 in \mathbb{R}^3 onto M that sends each meridian connecting the South pole $(0, 0, -1)$ and the North pole $(0, 0, 1)$ of S^2 into a curve of length $\leq L$. Fix a (small) positive δ . We would like to describe 1-, 2-, and 3-dimensional relative cycles u_1, u_2, u_3 in $(\Pi M, \Pi M_0)$ with coefficients in \mathbb{Z}_2 such that the lengths of all closed curves on M that appear as images of points of singular simplices of u_i do not exceed, correspondingly, $2L + \delta$, $2L + \delta$ and $4L + \delta$. Moreover, if L_0 denotes the minimal length of the image of a meridian of S^2 under f , then the lengths of closed curves in the images of simplices of u_1 will not exceed $L_0 + L + \delta$. Here is the description of the first two u_1 and u_2 :

2.2. The cycles u_1 and u_2 . The cycles u_i , $i = 1, 2, 3$ will be the images under the map induced by f of the following cycles v_1, v_2, v_3 in $(\Pi S^2, \Pi S^2_0)$: Assume that m_0 is a meridian such that the length of $f(m_0)$ is the minimal length of the image under f of a meridian of S^2 . Then v_1 is constructed as follows. Denote a meridian that forms an angle α with m_0 by m_α . Consider a 1-dimensional family of simple curves formed by m_α and $m_{-\alpha}$, where α runs through the interval $(\delta, \pi - \delta)$, where we will make a positive δ very small. Now contract $m_\delta \cup m_{-\delta}$ through simple closed curves that go along $m_{t\delta}$ and $m_{-t\delta}$ from the parallel with latitude $(\pi/2)t$ in the Southern hemisphere to the parallel with the latitude $(\pi/2)t$ in the Northern hemisphere and connects between $m_{t\delta}$ and $m_{-t\delta}$ along these two parallels (see Fig. 2). Here t varies between 0 and 1. When $t \rightarrow 0$ these curves converge to a point corresponding to the value $t = 0$. Similarly, we can contract $m_{-(\pi-\delta)} \cup m_{\pi-\delta}$. The resulting relative 1-cycle will be v_1 .

Note that we obtain a homologous cycle if instead of $m_\alpha \cup m_{-\alpha}$, $\alpha \in [-\pi, \pi]$ we consider $m_0 \cup m_\alpha$, $\alpha \in [0, 2\pi]$. If m_0 is the shortest meridian, then the maximal length of curves in these family can be less than in the family $m_\alpha \cup m_{-\alpha}$.

The 2-dimensional relative cycle v_2 is defined by rotating v_1 by the angle π : For each $\theta \in [0, \pi]$ we take m_θ instead of m_0 and define a similar path in ΠS^2 starting and ending

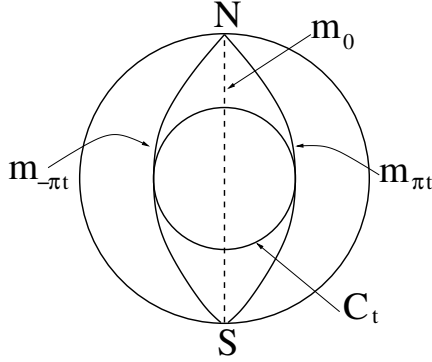


FIGURE 3.

in $\Pi_0 S^2$. Denote the resulting relative 1-cycle by v_{1t} . Note that $m_\pi = m_{-\pi}$. Therefore, $v_{1\pi} = v_{10} = v_1$. (Here it is important that the curves in ΠM are non-parametrized, and the ring of coefficients is \mathbb{Z}_2 . Therefore, when we change the orientation of the singular 1-simplex in $(\Pi S^2, \Pi S_0^2)$ we obtain the same 1-cycle.)

It had been observed in [NR] that v_1 is homologous to z_1 , and v_2 is homologous to z_2 , where z_1 and z_2 were defined in section 2.1. Indeed, in order to see that v_1 is homologous to z_1 , observe that the maps of $[0, 1]$ to ΠS^2 representing these 1-cycles are homotopic to each other relatively to $\{0, 1\}$. The homotopy can be described as follows: After a reparametrization of these paths and assuming that m_0 passes through $(0, -1, 0)$ we can assume that the circle C_t corresponding to the image of each point $t \in (0, 1)$ under the map representing z_1 is tangent to $m_{t\pi}$ and $m_{-t\pi}$, and is contained inside the digon D_t formed by these two meridians and containing m_0 inside (see Fig. 4).

For each t the homotopy is merely a homotopy between D_t to C_t via simple curves that obviously can be made continuous with respect to t (see Fig. 4). Rotating this homotopy, we will obtain homotopy between the maps representing 2-cycles v_2 and z_2 .

2.3. Cycles v_3 and u_3 . Again, u_3 will be the image under the homomorphism induced by f of a cycle $v_3 \in Z_3(\Pi S^2, \Pi_0 S^2; \mathbb{Z}_2)$ that we are going to describe below.

Let P denote all planes in \mathbb{R}^3 at a distance ≤ 1 from the origin. Then $z_3 = \{p \cap S^2 | p \in P\}$. In other words, $z_3 \subset \Pi S^2$ is a family of all circles and points on S^2 . Let $[z_3]$ denote the homology class of z_3 in $H_3(\Pi S^2, \Pi_0 S^2, \mathbb{Z}_2)$.

Suppose that a diffeomorphism $f : S^2 \rightarrow (S^2, g)$ maps each meridian to a curve of length $\leq L$ on (S^2, g) . We will construct a family $X \subset \Pi S^2$ of closed curves, such that 1) X will constitute a 3-dimensional relative cycle in $(\Pi S^2, \Pi_0 S^2; \mathbb{Z}_2)$; 2) $X \in [z_3]$ and the length of $f(\gamma)$ does not exceed $\leq 4L + \varepsilon$ for every $\gamma \in X$. Then we define v_3 as the relative cycle corresponding to X . In fact, below we sometimes will slightly abuse this notation by writing X for v_3 and considering X as an element of $Z_3(\Pi S^2, \Pi_0 S^2; \mathbb{Z}_2)$.

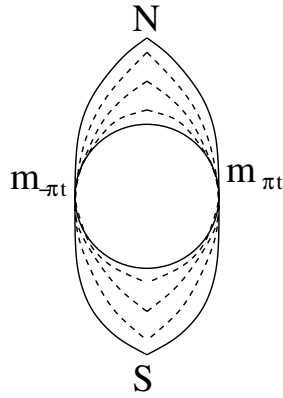


FIGURE 4.

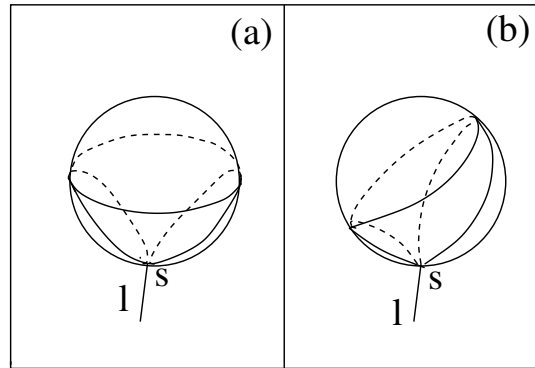


FIGURE 5.

First, we will give a brief informal description of the construction of X . Consider a 1-parametric family of planes in \mathbb{R}^3 that pass through the South pole S and contain some fixed line l in the tangent space of S^2 at S . They intersect S^2 in a family of circles. Note that these circles form a 1-cycle homologous to v_1 . Notice that this family of circles can be isotoped to a family of curves each consisting of two meridians on S^2 . On the other hand, consider a 2-parametric family of circles on S^2 that lie on a plane that is parallel to l . Each circle C in this family can be sandwiched between two planes containing l and intersecting C tangentially at its highest and lowest points. (If the circle is in a horizontal plane, then the two points of intersection are at the same height. However, even in this case the pair of points of intersection is uniquely defined.) This defines a bijective correspondence between all circles in planes that are parallel to l and all pairs of circles on S^2 that pass through S and are tangent to l at S (see Fig. 5).

These pairs of circles in turn can be homotoped to a family of curves made out of four arcs of meridians (similarly to how it was done in section 2.2). As we rotate l by

π , circles parallel to l will yield the 3-parametric family of all circles on S^2 . We have a surjective continuous map from the set of all pairs of circles passing through S and having a common tangent at S to the family of all circles on S^2 . This map fails to be injective only for horizontal circles on S^2 , each of which will have a 1-parametric family of inverse images. Therefore, the cycle formed by all considered pairs of circles will be homologous to v_3 . Furthermore, the cycle formed by all closed curves that go along four meridians that form two pairs with common bisectoral planes is also homologous to v_3 . Finally, one can slightly perturb curves in this last cycle to make them nonself-intersecting without significantly changing their length.

Cycle X of short curves. Now we are going to give a more detailed description of X . We will construct a cycle $X (= v_3)$, such that each closed curve in X consists (up to a small perturbation) of at most four subarcs of the meridians on S^2 . This implies the above estimate.

We will describe the construction of X in two steps. First, let us define a chain X' , which has a non-empty boundary and curves with self-intersections where the curve can touch itself but cannot cross itself. Then we perturb the family so that it only consists of simple curves and connect the boundary through simple curves of bounded length to a 2-cycle in $\Pi_0 S^2$. The image of this homotopy will be attached to the perturbed X' , and the result will be X that can now be regarded as a relative 2-cycle in $(\Pi S^2, \Pi_0 S^2)$.

Step 1. Defining X'

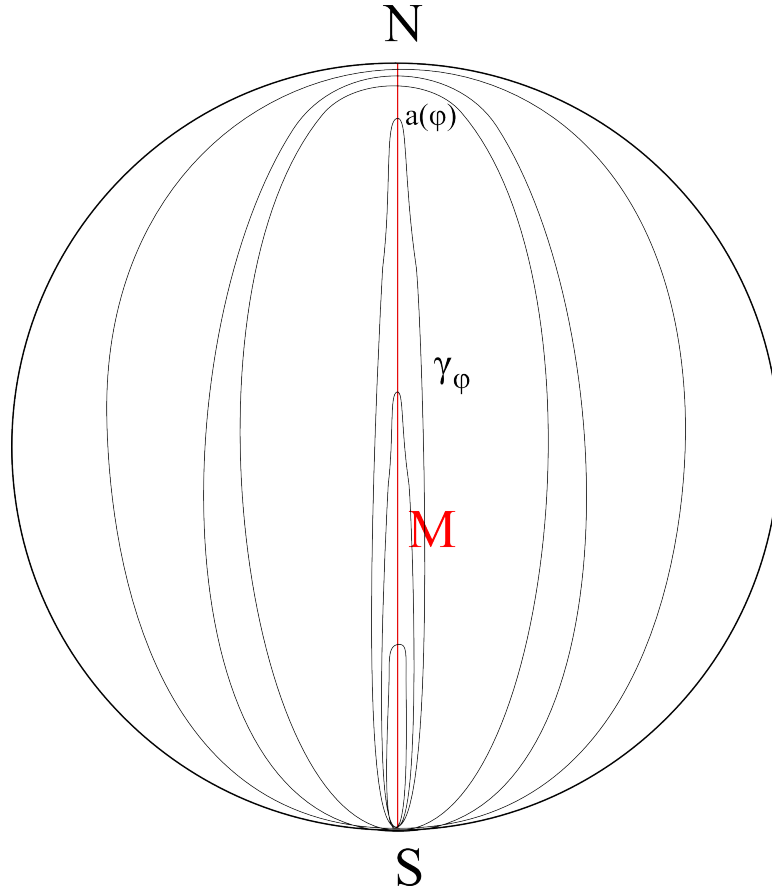
Fix a meridian m parametrized by the arclength. Let m^ϕ denote a meridian making an angle ϕ with $m = m^0$, $\phi \in [-\pi, \pi]$. For $t \in [0, \pi]$ define a family of closed curves

$$\gamma_t^m = \begin{cases} m^0|_{[0,3t]} \cup -m^0|_{[0,3t]} & \text{for } t \in [0, \frac{\pi}{3}] \\ m^{3t-\pi} \cup -m^{\pi-3t} & \text{for } t \in [\frac{\pi}{3}, \frac{2\pi}{3}] \\ m^\pi|_{[0,-3(t-\pi)]} \cup -m^\pi|_{[0,-3(t-\pi)]} & \text{for } t \in [\frac{2\pi}{3}, \pi] \end{cases}$$

We identify the point curves γ_0^m and γ_π^m . We perform a small perturbation that makes curves γ_t simple and disjoint everywhere except at the South Pole. The perturbation is illustrated in Figure 6. After the perturbation, $\gamma_{\pi/2}^m$ is the big circle that intersects South and North pole and is perpendicular to the meridian m . For each value of the z coordinate for $-1 < z < 1$ there are exactly two curves γ_ϕ^m and $\gamma_{\pi-\phi}^m$ that are touching the $z = \text{const}$ plane at their highest point, which we will denote by $a(\phi)$. Moreover, curves γ_ϕ^m and $\gamma_{\pi-\phi}^m$ are mirror images of each other reflected along $\gamma_{\pi/2}$.

Consider triangle $T = \{(\phi_1, \phi_2) | \phi_1 \geq \phi_2\} \subset [0, \pi]^2$. For each $(\phi_1, \phi_2) \in T$ let $\gamma_{(\phi_1, \phi_2)} = \gamma_{\phi_1}^m \cup \gamma_{\phi_2}^m$. We connect the endpoints of γ_{ϕ_1} and γ_{ϕ_2} in such a way that after a small perturbation $\gamma_{(\phi_1, \phi_2)}$ can be made simple (see Fig. 8).

Two sides of T , $a = \{(x, 0) | 0 \leq x \leq \pi\}$ and $b = \{(\pi, y) | 0 \leq y \leq \pi\}$, parametrize the same family of curves, so we glue them together accordingly. Note that $(0, 0)$, $(0, \pi)$ and (π, π) are all identified into one point. As a result we obtain a family of curves

FIGURE 6. One-parametric family γ_t

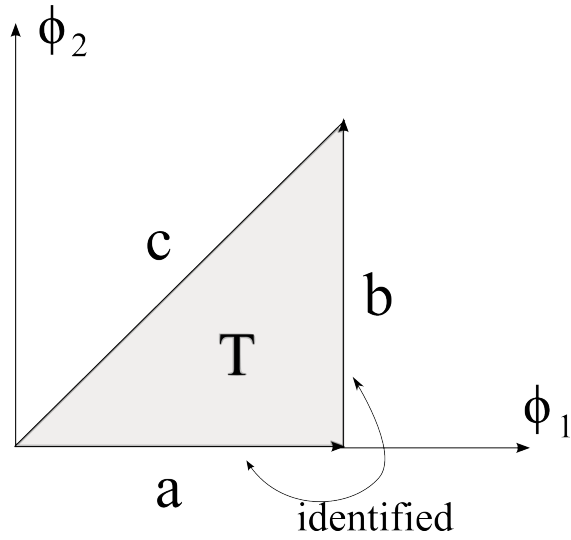
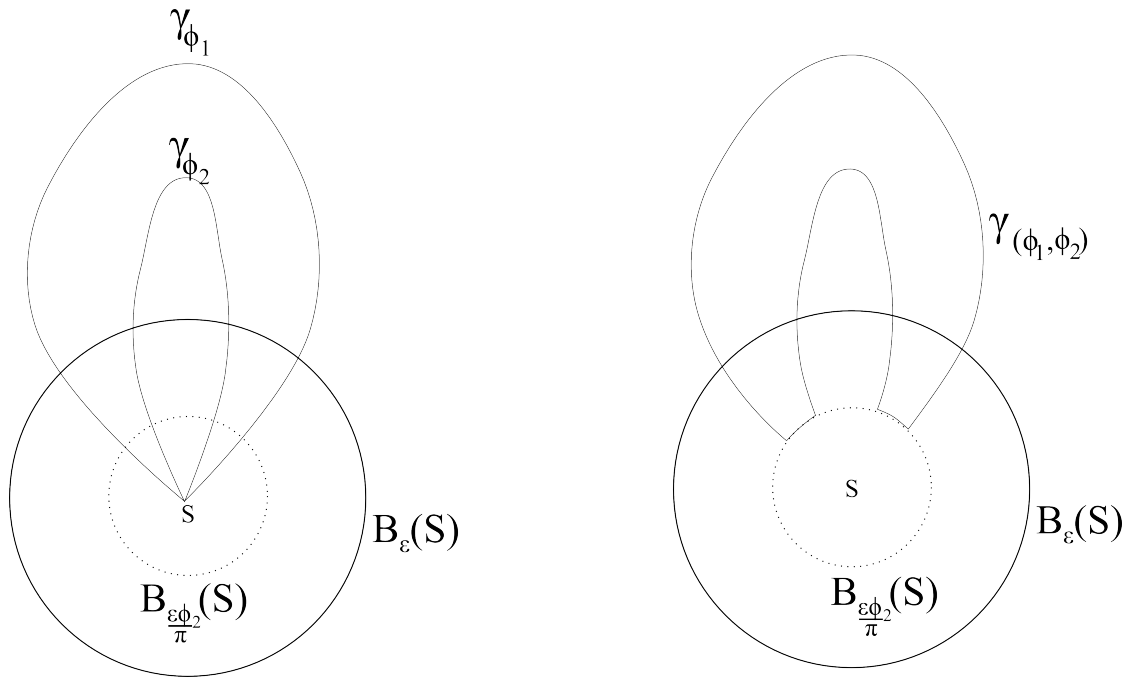
parametrized by a Möbius band (in the space $\mathbb{P}S^2$), where the boundary is formed by all $c = \{(\phi, \phi) | \phi \in [0, \pi]\}$, consisting of curves γ_ϕ^m traversed twice. See Figure 7.

Denote by X_m the family of curves parametrized by $T/a \sim b$. Rotating m by an angle π and identifying the curves of X_m and $X_{m\pi}$ we obtain a 3-parametric family X' .

Step 2. Contracting the boundary of X' and removing self-intersections

Closed curves in X' do not have transverse self-intersections, but some of them have arcs traced multiple times and non-transverse intersections at the South Pole.

First we define a perturbation that gets rid of self-intersections at the South Pole. Choose a small ε_0 -ball $B_{\varepsilon_0}(S)$ around the South Pole. Consider $\gamma_{(\phi_1, \phi_2)} \in X_m$ ($\phi_1 \geq \phi_2$) and let $\varepsilon(\phi_2) = \frac{(z(a(\phi_2))+1)\varepsilon_0}{2}$. We replace the intersection of $\gamma_{(\phi_1, \phi_2)}$ with $B_{\varepsilon(\phi_2)}(S) \subset B_{\varepsilon_0}(S)$ by two arcs on the boundary circle of $B_{\varepsilon(\phi_2)}(S)$ as on Figure 8. One can check that this defines a continuous mapping on X' .

FIGURE 7. Triangle T parametrizing family X_m FIGURE 8. Perturbation of $\gamma_{(\phi_1, \phi_2)}$ at the South Pole

Now we get rid of double traced arcs and contract the boundary of chain X' . Consider a one-parametric family of curves $L_m = \{\gamma_{(\phi_1, \phi_2)} | \phi_1 - \phi_2 = \varepsilon\} \subset X_m$ for some small

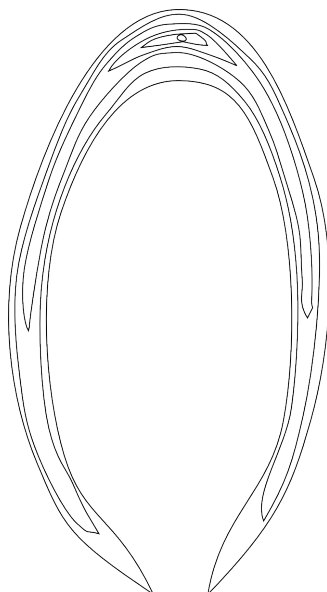


FIGURE 9. Contracting a curve near the boundary of X_m

$\varepsilon > 0$. Each curve $\gamma_{(\phi_1, \phi_2)} \in L_m$ bounds a narrow disc around γ_{θ_1} for $\theta_1 = \phi_1 + \varepsilon/2$. We can contract each $\gamma_{(\phi_1, \phi_2)}$ to a point $a(\theta_1)$ (the point of maximal z -coordinate on γ_{θ_1}) in this disc through simple closed curves as on Figure 9. This is continuous for all families L_m as we vary the meridian m . We replace the part of X' formed by all curves $\{\gamma_{(\phi_1, \phi_2)} \mid \phi_1 - \phi_2 < \varepsilon\}$ by the image of this contracting homotopy. All points $\gamma_{(\phi, \phi)}$ on the boundary of the Möbius band will be mapped to the trivial curves $\{a(\phi)\}$. This completes our construction of X .

2.3.2 Proof that $X \in [z_3]$ We claim that X represents the same homology class in $H_3((\Pi S^2, \Pi_0 S^2), \mathbb{Z}_2)$ as the family z_3 of all circles on S^2 . (In this subsection z_3 will sometimes be regarded as a subset of ΠS^2 and sometimes as the corresponding relative 3-cycle.) To prove this we will construct a homotopy $\sigma_t : X \rightarrow \Pi S^2$ with σ_0 the identity map and σ_1 coinciding with z_3 on a dense open subset. Fixing a CW structure on cycles σ_1 and z_3 we will observe that the two maps coincide on the interior of their 3-cells. In particular, it will follow that they represent the same homology class.

Description of $\sigma_1(X)$ (See Fig. 10). First, we describe σ_1 . If $\gamma \in X$ is a point curve we set $\sigma_1(\gamma) = \gamma$. Otherwise, by construction, for each curve $\gamma \in X$ there is unique big circle C that passes through the South pole and the point, where γ attains its maximal z -value. Note that γ is symmetric under the reflection along C . The curve γ intersects C at two points v_1 and v_2 . We set $\sigma(\gamma)$ to be the circle on S^2 that passes through v_1 and v_2 and whose center lies on C . Clearly σ is continuous and maps X onto z_3 . It is not injective. Note that each longitudinal circle has a one-parametric family of pre-images under σ_1 .

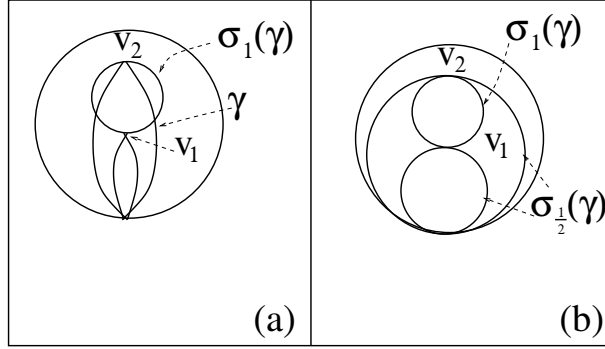


FIGURE 10.

The homotopy between the identity map on X and σ_1 will be constructed in two stages. First we homotop X to a family $Y = \sigma_{1/2}(X)$ of curves obtained from two circles that pass through the South pole and whose centres lie on the same meridional big circle. We described this family at the beginning of section 2.3 as a family of curves obtained from a pair of circles that have a common tangent line at the South pole. Then we will homotop Y to $\sigma_1(X)$.

Homotopy from X to Y Given a curve γ_θ^m define $\bar{\gamma}_\theta^m$ to be the circle that passes through the South Pole and $a(\theta)$ (the point where γ_θ^m attains its maximal z value) and whose center lies on $m \cup m^\pi$. For each $\gamma_{(\phi_1, \phi_2)} \in X_m$ we set $\bar{\gamma}_{(\phi_1, \phi_2)} = \bar{\gamma}_{\phi_1}^m \cup \bar{\gamma}_{\phi_2}^m$.

We perform perturbations that remove self-intersections at the South Pole and doubly traced curves in exactly the same way as we described above for the family X . This correspondence is a homeomorphism and it is easy to check that there exists an isotopy from X to $\sigma_{1/2}(X) = Y$ through simple curves that gradually “fattens” each curve γ_ϕ^m into the corresponding circle $\bar{\gamma}_\phi^m$.

Homotopy from Y to $\sigma_1(X)$

Let $\gamma \in Y$. We will define a homotopy γ_t , $t \in [-1, 1]$, that starts on γ and ends on $\bar{\gamma} = \sigma_1(\sigma_{1/2}^{-1}(\gamma))$. It will be clear from the definition that the homotopy depends continuously on $\gamma \in Y$.

As before let S be the big circle of symmetry of γ and v_1 and v_2 be points of intersection of γ with S . Let D_1 be the disc of $S^2 \setminus \gamma$ that contains $\bar{\gamma}$. Let D_2 denote the disc of $S^2 \setminus \bar{\gamma}$ that does not contain γ . Note that $D_2 \subset D_1$.

Let a_1 and a_2 be the z -coordinates of points v_1 and v_2 correspondingly and assume without any loss of generality that $a_1 \leq a_2$.

For t so small that the plane $\{z = t\}$ does not intersect γ we set $\gamma_t = \gamma$. Suppose $\{z = t\}$ intersects γ , but $t \leq a_1$. Let $\gamma^{\geq t}$ denote the intersection of the curve γ with the halfspace $\{z \geq t\}$. Let α_t denote the intersection of the disc D_1 with the plane $\{z = t\}$. Note that α_t consists of two arcs connecting the endpoints of $\gamma^{\geq t}$. We define $\gamma_t = \gamma^{\geq t} \cup \alpha_t$.

Suppose $a_1 \leq t \leq a_2$. Define β_t to be the intersection of $D_1 \setminus D_2$ with the plane $\{z = t\}$. Let $c^{\leq t}$ denote the intersection of $\bar{\gamma}$ with $\{z \leq t\}$. We define $\gamma_t = \gamma^{\geq t} \cup \beta_t \cup c^{\leq t}$.

For $t \geq a_2$ we set $\gamma_t = \bar{\gamma}$.

As we vary the initial curve $\gamma \in Y$, the 1-parametric family obtained in this way changes continuously. This completes our definition of homotopy σ . And, finally, we observe that:

Lemma 2.1. $[\sigma_1(X)] = [z_3] \in H_3((\Pi S^2, \Pi_0 S^2), \mathbb{Z}_2)$.

Proof. Let $L = \{l_s\} \subset z_3$, $-1 \leq s \leq 1$, denote all longitudinal circles $z = s$. We fix a CW structure on z_3 so that L is contained in the 1-skeleton and a CW structure on X so that $\sigma_1^{-1}(L)$ is contained in the 2-skeleton. Note that σ maps $X \setminus \sigma_1^{-1}(L)$ homeomorphically onto $z_3 - L$. It follows that the simplicial 3-chain corresponding to z_3 coincides with the image of a 3-dimensional chain with \mathbb{Z}_2 -coefficients representing the fundamental class of X under the homomorphism induced by σ . \square

2.4. Meridional slicing of M and the lengths of three simple periodic geodesics.

The discussion in section 2.1 and the description of cycles u_1 , u_2 and u_3 in sections 3.2 and 3.3 imply the following proposition:

Proposition 2.2. *Assume that p, q are (not necessarily distinct) points of M , and ϕ is a continuous map of S^1 into the space of piecewise smooth curves connecting p and q in M such that for each point $r \in M$ different from p and q exists unique $t \in S^1$ such that $\phi(t)$ passes through r . Assume also that for each $t \in S^1$ the length of $\phi(t)$ does not exceed L . (In other words, ϕ is a slicing of M into piecewise smooth curves of length $\leq L$ connecting p and q .) Assume also, that $\min_{t \in S^1} \text{length}(\phi(t)) \leq L_0$ for some $L_0 \leq L$. Then there exist three distinct simple periodic geodesics on M of non-zero index with lengths not exceeding, respectively, $L_0 + L$, $2L$ and $4L$.*

Proof of Theorem 1.2. Combining this proposition with Theorem 1.3 B from [LNR] we immediately obtain a proof of Theorem 1.2. (Theorem 1.3B from [LNR] asserts that for any Riemannian 2-sphere M of area A and diameter d and any point $p \in M$ there exists a slicing of M into nonself-intersecting loops based at p of length $\leq 200d \max\{1, \ln(\frac{\sqrt{A}}{d})\}$ so that these loops intersect each other only at p .) \square

Now we are going to prove Theorem 1.3.

Proof of Theorem 1.3.

We prove Theorem 1.3 by combining Proposition 2.2 with results of [LNR].

We consider two cases.

Case 1. Suppose the diameter d of the 2-sphere M satisfies $d \leq \sqrt{3}\sqrt{A}$. By Theorem 1.3 from [LNR] there exists a slicing of M into nonself-intersecting based loops of length bounded by $664\sqrt{A} + 2d < 700\sqrt{A}$. It follows from the proof that these loops do not intersect except at their base point.

Case 2. Suppose $d > \sqrt{3}\sqrt{A}$. Let p and q be two points on M that are at the maximal distance d from each other. Let $S_p(r) = \{x | \text{dist}(x, p) = r\}$ be the geodesic sphere around p of radius r . For almost every r the set $S_r(p)$ is a finite collection of simple closed curves on M . By coarea inequality we can find r , $|r - d/2| \leq \frac{\sqrt{3}}{2}\sqrt{A}$, such that $S_r(p)$ is a collection of simple closed curves of total length at most $\frac{\sqrt{3}}{3}\sqrt{A}$.

Let γ be a connected component of $S_r(p)$ that separates p and q . In other words, $M \setminus \gamma = D_1 \sqcup D_2$ with $p \in D_1$ and $q \in D_2$.

Next we will construct a slicing in each of the discs D_i using Theorem 1.6 C from [LNR]. For a Riemannian 2-disc D define $d_D = \max\{\text{dist}(p, x) | p \in \partial D, x \in D\}$. Note that d_D is bounded from above by the diameter of D . Theorem 1.6 C from [LNR] asserts that for any two points x and y on the boundary of D there exists a path homotopy between two arcs of ∂D with endpoints x and y , such that the lengths of all curves in the homotopy are less than $2L + 686\sqrt{A} + 2d_D$, where L denotes the length of ∂D . Moreover, it follows from the proof that the curves have no self-intersections and intersect each other only at the endpoints.

We claim that $d_{D_i} \leq d/2 + \frac{3\sqrt{3}+3}{2}\sqrt{A}$. Indeed, let l be a minimizing geodesic from p to q . Since γ is a connected component of a geodesic sphere, l will have a unique intersection point with γ . Let a denote this intersection point, l_1 be the arc of l from p to a and l_2 be the arc of l from a to q .

Let $p_1 \in D_1$ and $a_1 \in \gamma$ be such that $\text{dist}(p_1, a_1) = d_{D_1}$. Using the triangle inequality we obtain the following inequalities:

$$d \geq \text{dist}(p_1, q) \geq \text{dist}(p_1, \gamma) + \text{dist}(\gamma, q) \geq d_{D_1} - \frac{\text{length}(\gamma)}{2} + \text{length}(l_2).$$

By definition of γ we have $\text{length}(\gamma) \leq \frac{\sqrt{3}}{3}\sqrt{A}$ and $\text{length}(l_2) \geq d/2 - \frac{\sqrt{3}}{2}\sqrt{A}$. We obtain the desired bound $d_{D_1} \leq d/2 + \frac{3\sqrt{3}+3}{2}\sqrt{A}$. For disc D_2 the proof is identical. Choosing any two points x and y on γ and applying Theorem 1.6 C from [LNR] to D_1 and D_2 we obtain a meridional slicing of M by curves connecting x and y with length bounded by $d + 700\sqrt{A}$.

By Proposition 2.2 we obtain three distinct simple closed curves of positive index with the length of the first curve bounded by $d + 700\sqrt{A}$, the length of the second curve bounded by $2d + 1400\sqrt{A}$, and the length of the third curve bounded by $4d + 2800\sqrt{A}$. □

3. SLICING OF M INTO SHORT CURVES IN THE ABSENCE OF SHORT SIMPLE PERIODIC GEODESICS.

3.1. Geodesic segments between a pair of the most distant points of M . In this subsection we review some ideas of C.B. Croke from [Cr]. Let p, q be two points on M such that the distance between them is the maximal possible (and is equal to d). In [Cr]

Croke observed that either there exists a periodic geodesic on M of length $\leq 2d$, or there exists a sweep-out of M by a continuous 1-parametric family of curves connecting p and q of length $\leq 8d$, leading to an upper bound $9d$ for the length of the shortest non-trivial periodic geodesic.

The argument of Croke starts from the observation that p and q can be connected by finitely many minimizing geodesics that divide M into geodesic digons with angles at p and q that do not exceed π . This fact easily follows from Berger's lemma. Indeed, if this is not so, then one would be able to increase the distance between p and q by moving one of these two points inside a geodesic digon that has the angle adjacent to this point greater than π and is free of minimizing geodesics.

Now we would like to interrupt our exposition to review several definitions and facts. A simple closed curve on M is called *convex* with respect to one of two discs D bounded by this curve if each sufficiently short geodesic in M connecting two points of γ is contained in D . Sometimes we will say that a closed curve is convex without mentioning the relevant disc, if it is clear from the context which of the two discs we have in mind. A homotopy between γ and another closed curve β in a 2-disc D is called *monotone* if all curves γ_t , $t \in [0, 1]$, $\gamma_0 = \gamma$, $\gamma_1 = \beta$, in this homotopy are simple, and for each t_1, t_2 such that $t_1 < t_2$ the closed disc bounded by γ_{t_2} in D is contained in the closed disc bounded by γ_{t_1} in D . In other words, "monotone" means here that the disc inside the digon bounded by a curve at a later moment of time is contained in the disc bounded by a curve at an earlier moment of time. A *path homotopy* between two curves connecting a pair of points u and v is a homotopy that passes through curves connecting u and v . Further, we will be using the Birkhoff curve-shortening process which is explained in [Cr] and has the following properties (also explained in [Cr]): For each closed curve γ the Birkhoff curve-shortening process produces a length non-increasing homotopy that connects γ with either a point or a periodic geodesic. If the geodesic to which Birkhoff curve-shortening process converges has index > 0 , we can perturb it decreasing its length, and continue using the Birkhoff curve-shortening process. Therefore, without any loss of generality we can assume that the process converges either to a point or to a non-trivial periodic geodesic of index 0.

Birkhoff curve-shortening process depends on a small parameter $\varepsilon > 0$. If the initial curve is simple and convex to a disc D , then for all sufficiently small positive values of this parameter all intermediate curves will be simple, convex and contained in D ; the resulting homotopy will be monotone. Other features of this well-known process are not important for us here, and, in fact, this process can be replaced by other curve-shortening processes with similar properties such as Grayson's curvature flow ([Gr], or a process devised by J. Hass and P. Scott ([HS])).

The next observation of Croke is that these geodesic digons with the vertices at p and q and angles not exceeding π are convex, and, therefore, the Birkhoff curve-shortening process will shrink the boundary of each digon inside the digon in a monotone way (for a sufficiently small value of the parameter). The process will terminate either at a point

or at a non-trivial periodic geodesic of length not exceeding $2d$ inside the digon. If there are no such periodic geodesics, the boundary of the digon contracts to a point by a length non-increasing monotone homotopy.

Then Croke observes that this homotopy contracting the boundary of the digon can be transformed into a path homotopy between two geodesics forming the boundary that passes through paths of length $\leq 8d$ containing in the digon. All those homotopies for various digons can be combined into a sweep-out of M into paths between p and q of length $\leq 8d$. (See Lemma 3.1 below for an explicit statement in this direction that also yields a better bound than $8d$.) Some of these paths are the original minimizing geodesics between p and q of length $\leq d$. Choosing one of these minimizing geodesics and attaching it to all the paths one obtains a 1-parametric family of loops of length $\leq 9d$ based at p and representing a non-trivial homology class of the space of free loops of M . Now the standard Morse-theoretic proof of the existence of periodic geodesics implies the existence of a non-trivial periodic geodesic on M of length $\leq 9d$.

3.2. Some observations of M. Maeda. M. Maeda improved this argument in two directions ([M]). Let D be one of the convex digons formed by two minimizing geodesics between p and q . Let $x \in D$ be a point where $f(x) = \text{dist}(x, \partial D)$ attains its maximum. Denote this maximal value by r . First, he observed that r can exceed $\frac{d}{2}$ for at most one of the digons (and, of course, in this case it still does not exceed d). The reason is simple: if there are two digons D_1, D_2 and points $x_1 \in D_1, x_2 \in D_2$ such that $\text{dist}(x_i, \partial D_i) > \frac{d}{2}$ for $i = 1, 2$, then $\text{dist}(x_1, x_2) > d$ which is impossible.

To explain the second observation of Maeda let $\Delta_t, t \in [0, 1]$, denote (convex) curves obtained from $\Delta_0 = \partial D$ during an application of the Birkhoff curve-shortening flow ending at a point u (so that $\Delta_1 = u$). Let γ denote a minimal geodesic connecting a closest to u point z on Δ_0 with u , then for each t there is exactly one point of intersection of γ with Δ_t , and this point of intersection moves continuously and monotonically from z to u .

Indeed, otherwise, γ might exit at a second point of intersection and will need to intersect Δ_t again to return to the disc bounded by Δ_t and containing q . As the result, γ will touch from inside one of the piecewise geodesic curves Δ_t which is impossible because of their convexity. Indeed, assume that γ touches a curve Δ_s from inside at a point w . (When we say that γ touches Δ_s from inside at w , we mean that an open arc of γ containing w stays inside the closed disc bounded by Δ_s and containing q .) Take two points $w_1, w_2 \in \Delta_s$ that lie on opposite sides of w on Δ_s and such that the length of the arc of Δ_s that connects them is very small. Then the convexity of Δ_s implies that the minimal geodesic connecting w_1 and w_2 must intersect γ at at least two distinct points which contradicts the minimality of γ .

The second option is γ reaching Δ_t at a moment of time s_1 , then going along Δ_t until a moment of time $s_2 > s_1$, then entering the open disc bounded by Δ_t (and not returning to Δ_t anymore). To exclude this possibility consider $w^* = \gamma(s_2)$. (In other words, w^* is

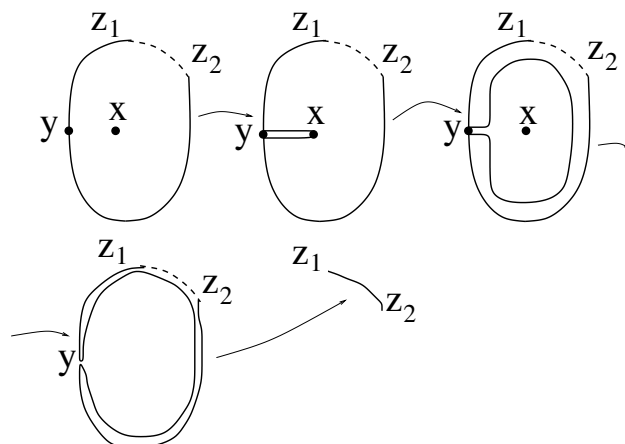


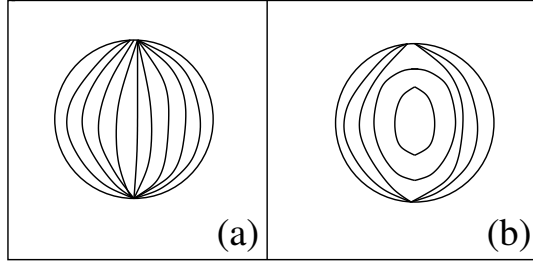
FIGURE 11.

the “last” point of γ on Δ_t .) Take two points w_1^* and w_2^* on Δ_t that are very close to w^* and lie on the opposite sides of w^* . (So, one of them is in the image of τ , and the other is not. Now, the convexity of Δ_t implies that the minimal geodesic between w_1^* and w_2^* must intersect γ at a point different from w_1^* and w_2^* , and we again obtain a contradiction with the minimality of γ .)

Now we know that γ intersect Δ_t at exactly one point. An obvious argument shows that this point of intersection moves continuously and monotonously along γ .

Denote the segment of γ from z to the point of intersection of γ with Δ_t by γ_t . Maeda considers loops based at z formed by first following γ_t , then Δ_t , then γ_t travelled in the opposite direction, and concludes that they provide a way to contract Δ_0 to a point through loops based at z of length less than or equal to $2r + 2d$, where r denotes $\max_{x \in D_t} \text{dist}(x, \Delta_t)$. Here D_t denote the disc bounded by Δ_t containing u . (One also needs to cancel the the loop obtained by following γ from z to u , and then back to z along itself at the end of this homotopy.) One can use the homotopy contracting Δ_0 as a loop based at z to construct a path homotopy between one side of Δ_0 to the other (keeping the endpoints fixed during the homotopy) (see Fig. 11). The length of curves in this homotopy is bounded by $2r + 3d \leq 4d$, if $r \leq \frac{d}{2}$, and by $5d$ in the general case (as r cannot exceed d).

Now, one can start from the only digon, where $r > \frac{d}{2}$, if such a digon exists, create its boundary out of a point by performing the Birkhoff curve-shortening in reverse, and then use the constructed path homotopies to move sides of the digon through adjacent digons. We can continue in this way getting rid of all digons until only one of digons, Δ , will be left. We could arrange for Δ to be any digon other than the digon we started from (which is the digon with $r \geq \frac{d}{2}$, if such a digon exists, and an arbitrary digon otherwise). Then it



Two ways to construct a generator of $H(\Pi M, \Pi_0 M, \mathbb{Z}_2)$:

(a) Meridional slicing

(b) The boundary of one of the geodesic digons is contracted to a point using the Birkhoff curve-shortening flow

FIGURE 12.

remains only to construct a path homotopy contracting the boundary of Δ - for example, using the Birkhoff curve-shortening process.

3.3. Homotopies that pass via simple curves. It had been observed in [NR], that one can perturb this homotopy so that the resulting homotopy would pass only through simple curves (as well as the points at the beginning and the end of the homotopy) (see Lemma 3.1 below for a formal statement of such nature that will be required for our purposes below). As the result, one obtains a segment I in the space of closed curves on M such that its endpoints are trivial curves and all intermediate curves are simple. This segment can be used to obtain via Morse theory (as above) one simple geodesic of controlled length ($\leq 5d$, if there are no geodesics of length $\leq 2d$ of index zero).

Alternatively, we could just start from a side of the initial digon, to connect it with the other side by a path homotopy that passes through nonself-intersecting curves of length $\leq 2r + 3d + \varepsilon$ for an arbitrary small ε using a homotopy constructed in Lemma 3.1 below (See Fig. 12).

Different curves in this path homotopy intersect each other only at their common endpoints. Then one can consider a similar path homotopy through the adjacent digon, then through the next one, and so on, and continue in this manner, until we return to the initial geodesic segment slicing out M into nonself-intersecting segments of length $\leq 2r + 3d + \varepsilon \leq 5d + \varepsilon$ passing through p and q and not intersecting at any other points. Proposition 2.2 implies the existence of three distinct simple periodic geodesics with lengths $\leq 6d$, $10d$ and $20d$, correspondingly. However, the first bound can be improved to $5d$, as the resulting 1-cycle in the space of non-parametrized simple curves will be homologous to the 1-cycle constructed in the previous paragraph. Therefore, the first

of these simple periodic geodesics will coincide with the periodic geodesic obtained in the previous paragraph, and its length will not exceed $5d$.

3.4. Path homotopies between arcs of convex curves. Here we prove the following lemma (some weaker versions of which can be found in [Cr], [M] and [NR1]). We will present a self-contained proof that incorporates some observations of Croke ([Cr]) and Maeda ([M]). We have already used this lemma in section 3.3 to conclude that if M has no simple closed geodesics of length $\leq 2d$, then there are three distinct simple periodic geodesics on M with lengths $\leq 5d$, $10d$ and $20d$, correspondingly.

Lemma 3.1. *Let $D \subset M$ be a 2-disc, and γ_t , $t \in [0, 1]$, be a monotone homotopy between its boundary γ_0 and a point $x = \gamma_1$ in D . Assume that for each t the length of γ_t does not exceed L , all curves γ_t are convex with respect to the discs contained in D , and that $y \in \gamma_0$ is a point such that $\text{dist}(x, y) = \text{dist}(x, \gamma_0) = r$. Then for each positive ε there exists a monotone homotopy l_t , $t \in [0, 1]$, between γ_0 and the constant loop y that passes through simple loops based at y and contained in D such that the lengths of each loop l_t does not exceed $L + 2r + \varepsilon$. Moreover, if z_1, z_2 are any two points on γ_0 , then there exists a path homotopy A_t between two arcs A_0 and A_1 of γ_0 between z_1 and z_2 such that for each t the length of A_t does not exceed $\max\{\text{length}(A_1), \text{length}(A_0)\} + L + 2r + \varepsilon$, all arcs A_t in this path homotopy do not self-intersect, and different arcs intersect each other only at their common endpoints.*

Proof. Let τ be a minimal geodesic from y to x . As y is a closest to x point of γ_0 , τ is contained in D . Now note that τ can intersect each curve γ_t only at a point or along an interval. Further, note that τ needs to intersect γ_t at least once to reach x . We are going to prove by contradiction that once τ enters the open disc D_t containing x bounded by γ_t it will not be able to exit this disc. Indeed, if it exits D_t it will need to return to γ_t later. This observation implies that for some $s < t$ τ touches γ_s at some point w but does not exit the closed disc bounded by γ_s . Now consider the minimizing geodesic in M connecting two points $w_1, w_2 \in \gamma_0$ that lie on the opposite sides of w and are very close to each other. The convexity of γ_s implies that this geodesic must intersect τ at two different points, which contradicts the minimality of τ (see Fig. 13).

In fact, we can continue this argument and demonstrate that $\tau \cup \gamma_t$ is a point and not a closed interval as it was done in section 3.2, but we do not need this.

Now we can construct a homotopy \tilde{l}_t between γ_0 and γ_1 as follows. For each t we follow τ from $\tau(0)$ to $\tau(t)$, then traverse γ_{t^*} such that $\tau(t) \in \gamma_{t^*}$ counterclockwise until we return back to $\tau(t)$, then return back along τ from $\tau(t)$ to $\tau(0)$. These curves will not be simple, as we use an arc of τ between $\tau(0)$ and $\tau(t)$ twice, but an obvious small perturbation makes these loops simple and pairwise intersecting only at y : We just move two segments of τ between $\tau(0)$ and $\tau(t)$ apart in different directions. For each t the perturbation is very small, but its magnitude increases with t . The point $\tau(t)$ is replaced by two very close

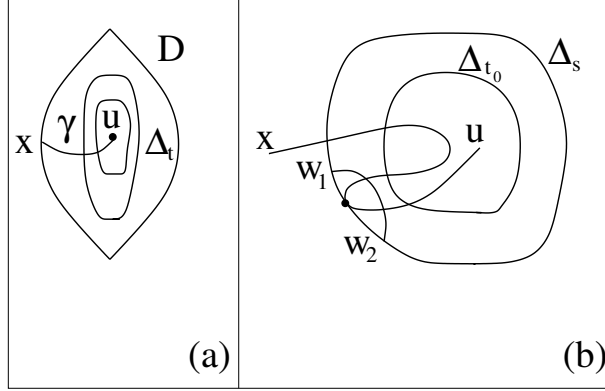


FIGURE 13.

points on γ_{t^*} on both sides of $\tau(t)$ that are used as the endpoints of the perturbed segments of τ . The resulting modified homotopy will be the desired homotopy l_t .

In order to prove the last assertion about the existence of the path homotopy connecting two arcs between z_1 and z_2 we can proceed as follows (see Fig. 11). Denote the arc between z_1 and z_2 that contains y by A_0 . Let A_0^1 and A_0^2 denote the arcs of A_0 from z_1 to y and from y to z_2 , respectively. Start from A_0 . If we ran the homotopy of based loops in reverse direction, we can gradually grow loop γ_0 from point y . At the end of this homotopy we obtain the curve $A_0^1 * \gamma_0 * A_0^2$ connecting z_1 and z_2 . Now we can cancel arcs between z_1 and y as well as z_2 and y (as each of these arcs will be appearing twice with opposite orientations, and we can cancel these arcs by means of obvious length non-increasing homotopies.) \square

4. SLICING OF M INTO SHORT CURVES IN THE PRESENCE OF ONE SHORT SIMPLE PERIODIC GEODESIC OF INDEX ZERO

Now assume that there exists one non-trivial periodic geodesic γ of length $\leq 2d$ of index 0 which is simple and is the only simple periodic geodesic of index 0 and length $\leq 4d$. We are going to construct a “meridional slicing” of M into nonself-intersecting curves starting at p and ending at q and not intersecting at any other points of length $\leq 10d + \varepsilon$ for an arbitrarily small ε . In this case Proposition 2.2 will imply the existence of two distinct simple periodic geodesics of length $\leq 11d$ and $20d$ with positive indices, so that they cannot coincide with the geodesic of index zero.

Then we will obtain the first of these two geodesics of a positive index using a homologous relative 1-cycle in the space of nonparametrized 1-curves that consists of curves of length $\leq 8d + \varepsilon$. As the result, we will be able to conclude that the first of these two periodic geodesics has length $\leq 8d$ (and not only $\leq 11d$).

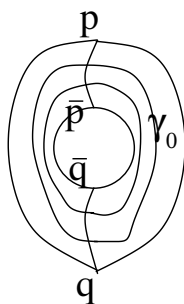


FIGURE 14.

Indeed, assume first that γ does not arise during the Birkhoff-curve shortening process applied to the boundary of one of the geodesic digons considered in the previous section obstructing its length non-increasing contraction to a point. Then we can proceed exactly as in the previous section. So, without any loss of generality we can assume that γ arises during the Birkhoff curve-shortening process applied to the boundary of one of the digons and denote this digon by D . (Recall that the considered digons are formed by pairs of minimizing geodesics connecting a pair of points $p, q \in M$ such that $\text{dist}(p, q) = d$. The angles of each digon at both p and q do not exceed π .) But γ can obstruct the process of contracting the boundary of only one of these digons. For the remaining digons we will be able to construct “short” path homotopies from one side to the other. After we will combine all these homotopies, we are going to have two options: First, we can complete this homotopy by constructing a path homotopy of one side of ∂D to the other. Second, we can contract the boundary of D through D to a point as a free loop (see Fig. 12). The second option will lead to shorter closed curves but will yield only one “short” simple geodesic of index 1. (In fact, Proposition 4.1 below will immediately imply that the boundary of D can be contracted through simple closed curves of length $\leq 8d$.)

So, assume that when one applies the Birkhoff curve-shortening process to ∂D (which is assumed to be convex to D), the process stops at a simple periodic geodesic $\gamma_0 \subset D$. Of course, the length of γ_0 does not exceed $2d$. Denote the piecewise geodesic convex closed curves arising during the Birkhoff curve-shortening process by h_t , where $h_0 = \partial D$ and $h_1 = \gamma_0$.

In the remainder of this section our goal will be to prove that there exists a path homotopy between the two minimal geodesics forming ∂D that passes through nonself-intersecting curves of length $\leq 10d$ that intersect each other only at their common endpoints.

Let \bar{p} and \bar{q} denote points of γ_0 that are the closest to, correspondingly, p and q (see Fig. 14). Connect p and \bar{p} by a minimizing geodesic α . It intersects each convex piecewise geodesic curve h_t at a point. (It should enter the domain bounded by h_t , but if it exists this domain, then it cannot return back because of the convexity of all curves h_s -see section

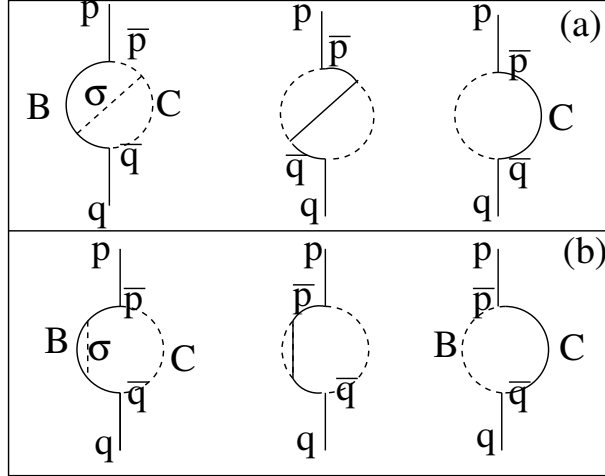


FIGURE 15.

3.2 for the details of this argument.) Also, it is easy to see that the point of intersection of α and h_t continuously depends on t . We can similarly connect q and \bar{q} by a minimizing geodesic β . It will also intersect each h_t at exactly one point that continuously depends on t .

Now each of two sides of the digon ∂D can be homotoped through curves that go along a segment of α to its point of intersection with h_t , then go along one of two arcs of h_t to its point of intersection with β , and then follow β to q .

In order to complete these two segments of a path homotopy to a desired path homotopy connecting both sides of ∂D we need to connect two arcs of γ_0 between \bar{p} and \bar{q} by a path homotopy passing through paths of length $\leq 8d + \varepsilon$ (and to attach α and β to the ends of each of these paths) (see Fig. 15).

To achieve this last goal we first consider the case, when there exists a pair of distinct points r and s on γ_0 that can be connected by a minimizing geodesic segment σ such that its interior lies entirely in the open subdisc of D bounded by γ_0 . Denote arcs of γ_0 connecting r and s by B and C . Then digons $B \cup \sigma$ and $\sigma \cup C$ are convex. These two digons can be contracted by means of a length nonincreasing homotopy using the Birkhoff curve-shortening procedure, as our assumptions imply that there are no “short” non-trivial periodic geodesics inside of these two digons. Therefore there exists a path homotopy between B and C that passes through σ and consist of paths of length $\leq 6d + \varepsilon$ connecting r and s . (Here we are using Lemma 3.1.) Now we need to consider cases when both points \bar{p} and \bar{q} are on the same arc, say, B , and another case when one of two points (say, \bar{q}) is on B and another on C . In the first case we first attach longer and longer segments of B traversed in both directions at the beginning and at the end of the arc connecting \bar{p} and \bar{q} inside B . At the end we obtain the arc that goes from \bar{p} to one of the ends of B , then

follows B to its other end and then returns to \bar{q} . Now we homotope the middle part of the path, B , to C via paths of length $\leq 6d + \varepsilon$ keeping its endpoints fixed. We end up with the complementary arc of γ_0 connecting \bar{p} and \bar{q} . The resulting homotopy passes through paths of length not exceeding $8d + \varepsilon$. In the second case we start from an arc connecting \bar{p} and \bar{q} , then elongate its subarc in B by inserting longer and longer complementary subarcs of B until the middle part of the curve will be B . Again, we homotop it to C keeping the endpoints fixed, and remove the unnecessary arc of C which will be traversed twice in different directions.

Now we consider the case, when no such $r, s \in \gamma_0$ exist. In this case for each point $x \in \gamma_0$ any minimizing geodesic between p and x is in the annulus formed by ∂D and γ_0 . (Of course, the same is true for q and x .) Indeed, this minimizing geodesic cannot cross two minimal geodesics forming ∂D , and also cannot cross γ_0 because of our assumption. Now note that in the argument above we do not need to assume anymore that \bar{p} is the closest to p point on γ_0 , and \bar{q} is the closest to q point on γ_0 . Instead we can choose \bar{p} and \bar{q} to be arbitrary points on γ_0 such that minimal geodesics between p and \bar{p} as well as q and \bar{q} do not intersect. But before explaining how to choose \bar{p} and \bar{q} we are going to state the following proposition and review its proof given in [NR]:

Proposition 4.1. *Let γ_0 be a simple closed curve of length L in a Riemannian surface of diameter d that bounds a disc D_0 . Denote $\max_{x \in D_0} \text{dist}(x, \gamma_0)$ by r . Curve γ_0 might be a periodic geodesic of index 0, but assume that there are no other non-trivial periodic geodesics of index 0 of length $\leq L + 2r$ in D_0 . Then for each $\varepsilon > 0$ γ_0 can be contracted to a point via simple closed curves of length $\leq \max\{\frac{L}{2}, 2r\} + L + 4r + \varepsilon$. Moreover, these curves are pairwise non-intersecting and are contained in D_0 .*

Proof. Let z be a point in D_0 at the maximal distance from $\gamma_0 = \partial D$. Then there exist points $x_1, x_2, \dots, x_k \in \gamma_0$ such that for each i $\text{dist}(z, x_i) = \text{dist}(z, \gamma_0)$, $k \geq 2$, and tangent vectors at z to minimizing geodesics from z to x_i subdivide the tangent space $T_z M$ into angles $\leq \pi$ (see Fig. 16).

(Otherwise, one can move z further away from γ_0 as in the proof of Berger's Lemma.) As the geodesics zx_i are perpendicular to γ_0 at x_i , they divide D_0 into convex triangles $T_i = zx_i x_{(i \bmod k)+1}$. Birkhoff's curve-shortening process contracts the boundary of each triangle T_i to a point inside this triangle; the resulting homotopy H_i is monotone and length non-increasing. Lemma 3.1 implies that each homotopy H_i can be easily converted into a path homotopy connecting one side (or two sides) of T_i with the curve formed by the two other sides (respectively, the other side) such that this path homotopy passes through T_i and "eliminates" T_i . Lemma 3.1 also provides an upper bound for the lengths of curves in this path homotopy. We can assume that the triangle corresponding to the longest arc $x_i x_{(i \bmod k)+1}$ of γ_0 will be the last one being contracted. (This helps to make an upper bound for the lengths of curves during the homotopy somewhat smaller.) An

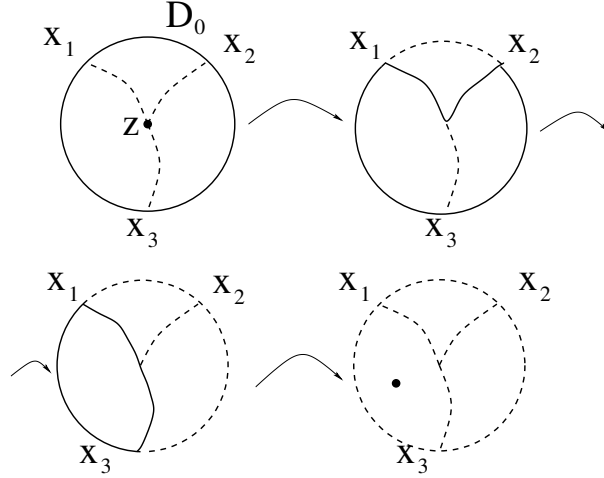


FIGURE 16.

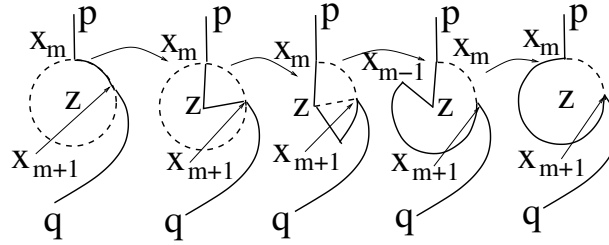


FIGURE 17.

easy calculation proves the claimed upper bound for the lengths of curves in the homotopy contracting γ_0 to a point. \square

Consider the triangles T_i defined in the previous proof. Among them choose a triangle with the maximal value of $\max_{y \in T_i} \text{dist}(y, \partial T_i)$. Denote this triangle by T_m , and its vertices on γ by x_m and x_{m+1} . We choose the pair of points x_m, x_{m+1} as the pair \bar{p}, \bar{q} .

More precisely, if $\text{dist}(p, x_m) + \text{dist}(q, x_{m+1}) \leq \text{dist}(p, x_{m+1}) + \text{dist}(q, x_m)$, then $\bar{p} = x_m, \bar{q} = x_{m+1}$, and, otherwise, $\bar{p} = x_{m+1}, \bar{q} = x_m$. This ensures that minimal geodesics between p and \bar{p} and between q and \bar{q} do not intersect.

Note that there is at most one triangle T_i such that $\max_{y \in T_j} \text{dist}(y, \partial T_j) > \frac{d}{2}$. Indeed, otherwise there would be two triangles T_{j_1}, T_{j_2} and points $z_1 \in T_{j_1}, z_2 \in T_{j_2}$ such that $\text{dist}(z_1, z_2) > d$, which is impossible.

Now it remains to find a path homotopy between the side $(x_m x_{m+1})$ of T_m and its complementary arc in γ that passes through simple curves of length $\leq 8d + \varepsilon$ contained in D_0 (see Fig. 17). This path homotopy will be constructed in stages. On the first stage

we construct a path homotopy in T_m between $x_m x_{m+1}$ and the broken geodesic formed by two other sides $x_m z x_{m+1}$ of the triangle T_m . We apply Lemma 3.1 to construct this path homotopy, and this lemma implies that the length of the paths during the homotopy will not exceed $8d + \varepsilon$. (Indeed, using the notations of Lemma 3.1, $L \leq 4d$, $2r \leq 2d$ and $\max\{\text{length}(A_1), \text{length}(A_2)\} \leq 2d$.) On each of the next stages we take either a triangle T_j adjacent to one of the triangles T_i that were already covered by the image of the constructed path homotopy that is the furthest in the clockwise direction (typically, $j = i + 1$), and construct a path homotopy in T_j between the common side of T_i and T_j and two other sides of T_j , or do the same for the triangle adjacent to the “processed” triangle that is the furthest in the counterclockwise direction. One of these two other sides of T_j will be a minimal geodesic from z to a new vertex of T_j , the other will be an arc of γ that will be added to already constructed part of $\gamma \setminus (x_m x_{m+1})$. The common side $z x_i$ of T_i and T_j as well as an arc of γ constructed on previous stages remain attached at the beginning of all paths obtained during this path homotopy. Note that we have a choice of which of the remaining triangles (other than the first one) to eliminate at the last step of the procedure. We choose the last triangle to be the triangle such that its side on γ is the longest among the remaining triangles. This automatically means that the lengths of the side on γ of each of the intermediate triangles will not exceed d . In this way we ensure that when we apply Lemma 3.1 on each of the intermediate steps $\max\{\text{length}(A_1), \text{length}(A_2)\} \leq 2d$. On the last step we eliminate the last triangle that we denote T_s by constructing the path homotopy between two sides of T_s adjacent to z and the third side which is an arc of γ . The already constructed arc of γ complementary to the side of T_s on γ is attached to all these paths. Again, an application of Lemma 3.1 implies that all these paths have the length not exceeding $8d + \varepsilon$. (Indeed, as before, $\max\{\text{length}(A_1), \text{length}(A_2)\}$ does not exceed $2d$. Also note that on each step the total length of all arcs of γ that already were constructed as well as the new one that enters d does not exceed $2d$. On all of the steps starting from the second one $2r \leq d$, but on each of the intermediate steps (between the first step and the last step) we also keep $z x_i$ fixed during the homotopy, and the length of this segment adds at most d to our upper bound for the lengths of curves during the constructed homotopy. Finally, the lengths of two geodesics between z and two other vertices of T_s also enter L on the last step. Their total length also does not exceed $2d$. Summarizing all these observations we see that the upper estimate provided by Lemma 3.1 on each of the steps of our construction is $8d + \varepsilon$, as desired.

5. SLICING OF M INTO SHORT CURVES IN THE PRESENCE OF MORE THAN ONE SHORT SIMPLE PERIODIC GEODESIC OF INDEX ZERO

Now we are going to consider the case, when there exists a simple periodic geodesic of length $\leq 2d$ and index 0 as well as another simple periodic geodesic of length $\leq 4d$ and index 0. If there are three simple periodic geodesics of length $\leq 6d$, then we are done.

Therefore, without any loss of generality we can assume that there is no third simple periodic geodesic of length $\leq 6d$ of index 0.

Our goal will be to construct a relative 1-cycle of $(\Pi M, \Pi_0 M)$ representing the non-trivial homology class with \mathbb{Z}_2 coefficients. This relative 1-cycle will be represented by a map of an interval so that the endpoints of the interval are mapped into constant curves, and all intermediate points are mapped into simple closed curves of length $\leq 20d$. As we saw in section 2, this will be sufficient to complete the proof of the theorem.

As in the previous section we will consider the digons formed by minimal geodesics p and q , where $d(p, q) = d$ and all angles in all digons do not exceed π . We will be trying to connect the sides of each digon by a path homotopy passing through curves of length $\leq 8d + \varepsilon$. This will be obviously sufficient for our purposes. If the boundary of each digon can be contracted by means of the Birkhoff curve-shortening process to a point, then Lemma 3.1 implies that this is, indeed, possible, and we are done. But, in principle, an application of the Birkhoff curve-shortening process can end at a simple closed geodesic of length $\leq 2d$. This can happen with one or two digons.

Case 1. The application of the Birkhoff process to two distinct digons results in simple closed geodesics. In this case we can be sure that 1) the lengths of these geodesics do not exceed $2d$; 2) These geodesics are simple and distinct; 3) They have index 0. Now our assumption implies that there are no other simple geodesics of index 0 and length $\leq 6d$. This means that each of these two periodic geodesics can be contracted to a point through simple pairwise non-intersecting closed curves of length $\leq 8d$ as in Lemma 4.1. Moreover, the images of these two homotopies are contained in the discs bounded by corresponding geodesics that are contained in the interiors of the corresponding digons. Now we can use one of these homotopies to go from a point to one of these two geodesics. Then we use the Birkhoff curve-shortening homotopy in reverse to go from the geodesic to the boundary of the corresponding digon. Then we continue our homotopy by moving one of two minimizing geodesics through neighboring digons using the fact that their boundaries can be contracted by means of the Birkhoff curve-shortening process and Lemma 3.1. We arrange this stage so that the last remaining digon will be the second digon that cannot be contracted by means of the Birkhoff curve-shortening process to a point. Once we obtain this digon we apply the Birkhoff curve-shortening process that ends at the second simple periodic geodesic. Finally, we contract this geodesic inside the disc contained in the second digon via simple closed curves of length $\leq 8d + \varepsilon$ as in Lemma 4.1. As the result, we obtain a desired relative 1-cycle in $(\Pi M, \Pi_0 M)$, where the lengths of all curves do not exceed $8d + \varepsilon$.

Case 2. The application of the Birkhoff curve-shortening process to all digons but one contracts them to points. The application of the Birkhoff curve shortening process to the last digon ends at a simple periodic geodesic γ_1 . The length of γ_1 does not exceed $2d$. Denote the disc bounded by γ_1 inside the digon by D , and its complement by E . One can contract γ_1 through “short” simple pairwise non-intersecting curves to a point inside E as

follows: First, connect γ_1 and the boundary of the corresponding digon by means of the Birkhoff curve-shortening process, where time goes in the opposite direction. Then keep one of two sides of this digon fixed and move the other side through all digons one by one using Lemma 3.1. When there will be only one last digon left, apply the Birkhoff curve-shortening process to contract it. The length of the closed curves during this homotopy does not exceed $6d + \varepsilon$.

Now we would like to contract γ_1 to a point inside D via simple pairwise non-intersecting closed curves of length $\leq 10d + \varepsilon$. The combination of these two homotopies will yield the desired relative 1-cycle.

As in the proof of Lemma 4.1, choose a point $x \in D$ that maximizes $\text{dist}(x, \gamma_1)$. Minimal geodesic segments from x to γ_1 subdivide D into a finite number of convex triangles. The application of the Birkhoff curve-shortening process to all of these triangles but possibly one contract them to points. As in the proof of Lemma 4.1 we can move arcs of γ_1 through all these triangles but possibly one eliminating these triangles in the process. The lengths of all curves during these homotopies will not exceed $9d + \varepsilon$. At the end of this stage we obtain the boundary of the last triangle. If the application of the Birkhoff curve-shortening process to this curve ends at a point, then we are done. Otherwise, we end at the second simple periodic geodesic of index 0. Denote this geodesic by γ_2 . The length of γ_2 does not exceed $4d$. The open disc bounded by γ_2 inside the last triangle cannot contain a simple periodic geodesic of index 0 and length $\leq 6d$. Therefore, another application of Lemma 4.1 implies that we can contract γ_2 to a point inside this disc through simple pairwise non-intersecting curves of length $\leq 10d + \varepsilon$, and we are done.

6. CONCLUDING REMARKS.

Observe that, in fact, we proved the following theorem:

Theorem 6.1. *Let M be a Riemannian 2-sphere of diameter d . There exist three distinct simple periodic geodesics on M that satisfy exactly one of the following four sets of (mutually exclusive) conditions:*

- (i) *All three geodesics have index 0. Their lengths do not exceed $2d$, $4d$ and $6d$, correspondingly.*
- (ii) *Two of these geodesics have index 0. Their lengths do not exceed $2d$ and $4d$. The third geodesic has a positive index and length $\leq 10d$.*
- (iii) *One of these geodesics has index 0 and length $\leq 2d$. Two other geodesics have positive indices and lengths $\leq 8d$ and $\leq 20d$, correspondingly.*
- (iv) *All three geodesics have positive indices. Their lengths do not exceed $5d$, $10d$ and $20d$, correspondingly.*

Our last remark pertains to the situation, when one does not have a meridional slicing of M into nonself-intersecting curves of length $\leq Cd$ for some large C . (Recall, that the existence of such a slicing would imply an upper bound $\text{const } d$ for lengths of three simple

periodic geodesics of positive index that appear in the original Lyusternik-Shnirelman proof.) Note, that iterating the same idea as in the last section, we will be able to conclude the existence of k simple pairwise non-intersecting geodesics with length $\leq 2d, \leq 4d, \dots, \leq 2kd$, where $k \sim \frac{C}{4}$.

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