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The Length of a Shortest Closed Geodesic on a Two-Dimensional Sphere and Coverings by Metric Balls

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Abstract. In this paper we will present upper bounds for the length of a shortest closed geodesic on a manifold M diffeomorphic to the standard two-dimensional sphere. The first result is that the length of a shortest closed geodesic $l(M)$ is bounded from above by $4r$, where r is the radius of M. (In particular that means that $l(M)$ is bounded from above by $2d$, when M can be covered by a ball of radius $d/2$, where d is the diameter of M.) The second result is that $l(M)$ is bounded from above by $2(\max\{r_1, r_2\} + r_1 + r_2)$, when M can be covered by two closed metric balls of radii r_1, r_2 respectively. For example, if $r_1 = r_2 = d/2$, then $l(M) \leqslant 3d$. The third result is that $l(M) \leqslant 2(\max\{r_1, r_2, r_3\} + r_1 + r_2 + r_3)$, when M can be covered by three closed metric balls of radii r_1, r_2, r_3. Finally, we present an estimate for $l(M)$ in terms of radii of k metric balls covering M, where $k \geqslant 3$, when these balls have a special configuration.

Key words. closed geodesic, Riemannian manifolds, curvature-free bounds.

1. Introduction

The main purpose of this paper is to improve the known upper bounds for the length of a shortest closed geodesic on a manifold diffeomorphic to the standard two-dimensional sphere. The existence of a closed geodesic on any closed Riemannian manifold M was proven by Lusternik and Fet, by looking at the space ΛM of all continuous maps $f\colon S^1 \to M$ taking the smallest i, such that $\pi_i(\Lambda M) \neq \{0\}$ and $\pi_i(M) = \{0\}$, considering a noncontractible sphere $f\colon S^i \to \Lambda M$ and then trying to deform it into M along the gradient flow of the energy functional. Any sphere of dimension at most i is contractible to a point in M. So, we cannot succeed in deforming $f\colon S^i \to \Lambda M$ into M because it will contradict the initial assumption about this sphere being noncontractible. Therefore, the energy functional should have a critical point, which is a closed geodesic (see [1], for a detailed proof). As a corollary $l(M) \leqslant \sup_{t \in S^i}$ length $f(t)$. The first curvature free upper bounds for the length of a shortest closed geodesic on a manifold M diffeomorphic to the 2-sphere are due to

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Croke who proved that (1) $l(M) \leqslant 9d$ and (2) $l(M) \leqslant 31\sqrt{A}$, where A is the area of M (see [2]). His approach can be described as the effectivization of the existence theorem of Lusternik and Fet. In [2] Croke constructed a noncontractible map $g : S^1 \to \Lambda M$ and estimated the length of curves through which it passes, thus obtaining an upper bound on $l(M)$. The map g is obtained by 'slicing' M into curves of small length. Later Maeda observed that the method of Croke can be improved to yield $l(M) \leqslant 5d$ (see [3]).

Both (1) and (2) were later improved by Nabutovsky and Rotman, and, independently, by Sabourau who proved that $l(M) \leqslant 4d$, and $l(M) \leqslant 8\sqrt{A}$ (see [4, 5]). The later bounds were obtained by considering a new space Γ (initially defined in [6]) instead of ΛM, which, informally speaking can be viewed as a space, where each element is a one-dimensional cycle that is either a closed curve or consists of two closed curves. In the new approach one constructs a noncontractible $g : S^1 \to \Gamma$, estimates the length of curves through which this loop passes, thus, obtaining an upper bound for $l(M)$. In particular, in [4] the loop is obtained by taking a noncontractible map $\tilde{g} : S^2 \to M$ and 'slicing' it into 1-cycles that consist of one or two closed curves, where $\tilde{g} : S^2 \to M$ appears as an obstruction to the extension process: one starts with any diffeomorphism $F: S^2 \to M$, and then tries to extend it to a disc D^3, triangulated as a cone over a very fine triangulation of S^2.

Nevertheless, even in this simplest case of M diffeomorphic to S^2 the optimal bounds remain unknown. In particular, it is unknown whether $l(M) \leqslant 2d$, as it can be expected. Observe that for the canonical metric on S^2 $l(M) = 2d$.

In this paper, we find a method that in most cases will allow one to obtain estimates better than the previously known estimate of $4d$, in many cases provides the bound $2d$, and in some cases (when M is long and narrow) allows to obtain estimates that are significantly better than $2d$. Observe that our upper bounds are quite easy to evaluate for specific metrics on S^2, and that they are not too sensitive to small perturbation of the Riemannian metric in C^0 topology.

The bounds are found by improving the extension process in [4]. In [4] we triangulated D^3 in the most obvious way: as a cone over its boundary and did not care where we mapped the new vertex. In the present paper we consider cell subdivisions of D^3 that take into account the shape of M, and map new zero-dimensional cells in the optimal way.

In this paper we will prove the following theorems:

THEOREM 1. *Let M be a manifold diffeomorphic to the two-dimensional sphere. Then $l(M) \leqslant 4r$, where r is the radius of M.*

Note that $r = \min_{x\in M} \max_{y\in M} \operatorname{dist}(x, y) \leqslant d = \max_{x\in M} \max_{y\in M} \operatorname{dist}(x, y)$. Therefore, the above estimate is stronger than the previous best known estimate $l(M) \leqslant 4d$. Observe that for a wide class of metrics on S^2 $r = d/2$, so we obtain $l(M) \leqslant 2d$ for these metrics.

THEOREM 2. *Let M be a manifold diffeomorphic to the two-dimensional sphere, and suppose M can be covered by k $(k \in \{2,3\})$ closed metric balls of radii $\{r_i\}_{i=1}^{k}$. Then $l(M) \leqslant 2(\max_{i=1}^{k} r_i + \sum_{i=1}^{k} r_i)$.*

We observe that we do not know any examples of Riemannian metrics on S^2 such that Theorems 1 and 2 do not lead to a better estimate for $l(M)$ than $l(M) \leqslant 4d$ proved in [4, 5]. For some Riemannian metrics on S^2 Theorems 1 and 2 yield upper bounds for $l(M)$ that are better than $2d$ (which is the best theoretically possible estimate on $l(M)$ in terms of the diameter).

One can prove similar results in the case when M is covered by $k > 3$ balls. Yet we do not see any practical applications for such results unless a configuration of these balls is especially simple. In such special cases one can find quite good upper bounds for $l(M)$, for example:

THEOREM 3. *Let M be diffeomorphic to S^2. Assume that M is covered by $k \geqslant 3$ metric balls $\{B_i(p_i)\}_{i=1}^{k}$ of radii r_i, such that $B_i(p_i) \cap B_j(p_j) \neq \emptyset$ if and only if $|i-j| \leqslant 1$ (see Figure 7). Then $l(M) \leqslant 2\max_{i\in\{2,3,\cdots,k-1\}}(r_i + \max\{r_{i-1}, r_{i+1}\} + \max\{r_{i-1}, r_i, r_{i+1}\})$.*

EXAMPLES

(1) Suppose M can be covered by two closed metric balls of radii $d/2$. Then $l(M) \leqslant 3d$.
(2) Assume that in the situation of Theorem 3 all $r_i = d/2$. Then $l(M) \leqslant 3d$ independently of k.
(3) If M is long and thin, so that $r_i \leqslant \epsilon d$ for some small ϵ (see Figure 7) then $l(M) \leqslant 6\epsilon d$.

As in papers [4, 5] one of the key ideas is to use the methods coming from the geometric measure theory developed by Almgren and Pitts (see [7, 8]) and adapted by Calabi and Cao to studies of closed geodesics on surfaces. Following Calabi and Cao we will consider the space that consists of parametrized closed curves and pairs of

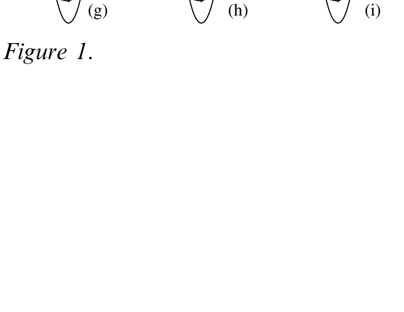

Figure 1.

closed curves, formally defined as follows: let $\Gamma = \{\Phi | \Phi = (\phi_1, \phi_2), \phi_i : [0,1] \to M$ is a piecewise smooth path and $\{\phi_1(1), \phi_2(1)\} = \{\phi_1(0), \phi_2(0)\}\}$. One can define the length functional $L(\Phi)$ simply as $L(\Phi) = L(\phi_1) + L(\phi_2)$ (see [6]). Assume that M is isometrically embedded in a Euclidean space. The distance between (ϕ_1, ϕ_2) and (ψ_1, ψ_2) is defined as

$$\sup_{t \in [0,1], i \in \{1,2\}} |\phi_i(t) - \psi_i(t)| + \Sigma_{i=1}^2 \sqrt{\int_0^1 |\phi_i'(t) - \psi_i'(t)|^2 \mathrm{d}t}$$

(this definition is somewhat different from the definition in [6].) This distance then induces the topology on Γ that makes L into a continuous function. Now, let $\chi(M)$ denote the set of all smooth vector fields on M. Any $X \in \chi(M)$ generates a one-parameter group of diffeomorphisms h_t. The derivation of L at $\phi \in \Gamma$ in direction of X is defined by $\delta L_\Phi(X) = \mathrm{d}/\mathrm{d}t(L(h_t \circ \Phi))|_{t=0}$. If $\delta L_\Phi(X) = 0$ for all $X \in \chi(M)$, then Φ is called a critical point of L on Γ and $L(\Phi)$. As Pitts, Calabi and Cao have shown, almost all critical points of L correspond to closed geodesics in the case when M is two-dimensional (see [6] as well as [9] for more details). In particular, let f: $S^1 \to \Gamma$ be a closed curve in Γ. Consider $B(f) = \max_{t \in S^1} L(f(t))$. Calabi and Cao observed that the infimum of $B(f)$ over all noncontractible closed curves f in Γ is achieved at some noncontractible loop f_0 such that $B(f_0)$ is achived at $t_0 \in S^1$ so that $f_0(t_0)$ is either a nontrivial closed geodesic or consists of two geodesics at least one of which is nontrivial. One can find a detailed proof of this fact in the Proposition 5 in [9]. Therefore for any noncontractible closed curve f in Γ, $B(f)$ is an upper bound for the length of a shortest closed geodesic.

So our goal will be to construct a noncontractible f: $S^1 \to \Gamma$ and to estimate $B(f)$ which will be automatically an upper bound for $l(M)$.

2. Proofs of the Theorems

We will now proceed with the proofs of the theorems. Note that the proof of Theorem 1 is parallel to the proof of Theorem 1 in [4] with one important improvement explained below.

Proof of Theorem 1. Let us assume that $l(M) > 4r$. Then all curves of length $\leqslant 4r$ can be contracted to a point without length increase using the Birkhoff curve shortening process. We will begin by considering an arbitrary diffeomorphism $h : S^2 \to M$ between the standard sphere S^2 and our manifold M. Let D^3 be the standard three-dimensional disc that has S^2 as its boundary. It should be clearly impossible to extend the map h to that disc. So, we will try to construct a new map H: $D^3 \to M$, such that $H|_{S^2} = h$ and as an obstruction to this extension obtain a map H: $S^2 \to M$, so that there will be a control over the size of curves that compose that sphere.

We will proceed as follows: triangulate S^2 so that the diameter of any simplex of the induced triangulation on M is at most as large as some small ϵ. Triangulate D^3 as the cone over S^2. The procedure that we perform will be inductive to the skeleta of D^3. To extend to the 0-skeleton, let $p \in M$ be a point such that $d(p,q) \leqslant r$ for any $q \in M$. Let $\tilde{p}$ be the center of the disc D^3. Then we will let $H(\tilde{p}) = p$. To extend to the 1-skeleton we will map all edges in D^3/S^2, i.e. the ones that connect $\tilde{p}$ with the vertices of S^2 to shortest geodesic segments that connect the point p with the corresponding vertices on M. To extend to the 2-skeleton, we have to extend to all of the 2-simplices of the form $[\tilde{p}, \tilde{v}_1, \tilde{v}_2]$ where $\tilde{v}_1, \tilde{v}_2$ are vertices of the triangulation of S^2. By our assumption, the closed curve that is formed by two geodesics $[p, v_1]$, $[p, v_2]$ that connect the point p with vertices $v_1 = g(\tilde{v}_1)$ and $v_2 = g(\tilde{v}_2)$, respectively, and by the edge $[v_1, v_2]$ is contractible to a point along the curves of length $\leqslant 2r + \epsilon$. We let the image of $[\tilde{p}, \tilde{v}_1, \tilde{v}_2]$ be the surface generated by the above homotopy.

Now, we observe that we cannot extend our map to the 3-skeleton. That means that there exists at least one 3-simplex $\sigma^3 = [\tilde{p}, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3]$, such that the map $H\colon \partial[\tilde{p}, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3] \to ([v_1, v_2, v_3] \cup [p, v_1, v_2] \cup [p, v_2, v_3] \cup [p, v_1, v_3])$, is a non-contractible sphere. There $[v_1, v_2, v_3]$ denotes a 2-simplex in the triangulation of the sphere, and $[p, v_i, v_j]$ is a surface generated by the homotopy that connects the closed curve formed by the 1-simplex $[v_i, v_j]$ and two geodesic segments $[p, v_i], [p, v_j]$ to a point.

Let γ_i be a minimal geodesic connecting p and v_i, $i = 1, 2, 3$. Let e_{12} be the edge connecting v_1 and v_2; e_{13} be the edge connecting v_1 and v_3 and e_{23} be the edge connecting v_2 and v_3. Let $\alpha_1 = \gamma_2 \cup -e_{12} \cup -\gamma_1$; $\alpha_2 = \gamma_1 \cup e_{13} \cup -\gamma_3$; $\alpha_3 = \gamma_3 \cup -e_{23} \cup -\gamma_2$; $\alpha_4 = e_{12} \cup e_{23} \cup -e_{13}$. Note that the length of $\alpha_i \leqslant 2r + \epsilon$, $i = 1, 2, 3$ and the length of $\alpha_4 \leqslant 3\epsilon$. Assume that ϵ is less than the convexity radius of the sphere, so that α_4 is contractible to v_1 without length increase inside $[v_1, v_2, v_3]$.

Observe that we have homotopies contracting α_i to p_i for $i = 1, 2, 3$ without length increase inside $[p, v_1, v_2], [p, v_2, v_3], [p, v_1, v_3]$, respectively.

Next, we will use these homotopies to construct a loop $f\colon S^1 \to \Gamma$ and show that it is noncontractible. We will write $(\phi_1, \phi_2) \sim (\psi_1, \psi_2)$, when the pairs of curves are homotopic. We will start with a pair of constant maps $\Phi = (\phi_1, \phi_2)$, where $\phi_1(t) = p_1$ and $\phi_2(t) = p_2$. Now, we see that $(p_1, p_2) \sim (\alpha_1, \alpha_2) \sim (\gamma_2 \cup -e_{12}, e_{13} \cup -\gamma_3) \sim (\alpha_3, \alpha_4) \sim (p_4, p_3) \sim (p_1, p_2)$ (see Figure 1). (Each of these homotopies is a path in Γ. The union of these paths will be the desired loop in Γ.)

Figure 1(b)–(d) shows that $(p_1, p_2) \sim (\alpha_1, \alpha_2)$ and the total length of curves in the homotopy is bounded from above by $4r + 2\epsilon$. Figure 1(d)–(f) shows that $(\alpha_1, \alpha_2) \sim (\gamma_2 \cup -e_{12}, e_{13} \cup -\gamma_3)$ and the total length of curves in the homotopy is bounded from above by $4r + 2\epsilon$. Figure 1(f) and (g) shows that $(\gamma_2 \cup -e_{12}, e_{13} \cup -\gamma_3) \sim (\alpha_3, \alpha_4)$ and the length of the curves in the homotopy is bounded from above by $2r + 4\epsilon$, and finally, Figure 1(g)–(i) shows that $(\alpha_3, \alpha_4) \sim (p_3, p_4)$ and the length of curves in the homotopy is bounded above by $2r + 4\epsilon$. One can also see that the length of curves in the homotopy is bounded from above by $4r + 4\epsilon$.

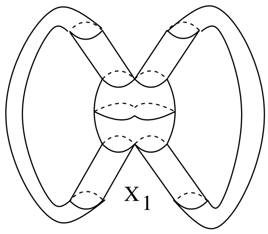

Figure 2.

It remains to check that the obtained closed curve in Γ is noncontractible in order to conclude that $l(M) \leqslant 4r + 4\epsilon$. This can be done as follows: We are going to assign a singular 2-cycle in M to the loop $f\colon S^1 \to \Gamma$. We will show that this cycle does not bound and that it implies a noncontractibility of $f\colon S^1 \to \Gamma$. (The detailed proof of a very general version of the last assertion can be found in Section 4 of [9].) The singular 2-cycle is constructed as follows: Observe that Γ is a subspace of the space of maps of the disjoint union of two copies of [0,1] into M (with the appropriate topology). So, we can assign a map of a union of two cylinders $S^1 \times [0,1]$ into M. This map can be factored through the quotient X_1 of $\bigcup_{i=1}^2 S_i^1 \times [0,1]_i$, where we identify points on their boundaries in accordance with the identification of the end points of two curves forming $f(t)$ for each $t \in S^1$. In our case X_1 can be constructed in the following way: identify the boundaries of the two disjoined cylinders $S_i^1 \times [0,1]_i$, $i = 1, 2$ in the following way: $[0, \frac{1}{3}]_i \times \{0\}_i \sim [0, \frac{1}{3}]_i \times \{1\}_i$ for $i = 1, 2$; $[\frac{1}{3}, \frac{2}{3}]_1 \times \{0\}_1 \sim [\frac{1}{3}, \frac{2}{3}]_2 \times \{1\}_2$ and $[\frac{1}{3}, \frac{2}{3}]_1 \times \{1\}_1 \sim [\frac{1}{3}, \frac{2}{3}]_2 \times \{0\}_2$; finally $[\frac{2}{3}, 1]_i \times \{0\}_i \sim [\frac{2}{3}, 1]_i \times \{1\}_i$ for $i = 1, 2$. Let F be the induced map from X_1 to M (see Figure 2). Note that F factors into the composition of $\phi : X_1 \to \partial\sigma^3$ and the original map $H\colon \partial\sigma^3 \to M$. X_1 is clearly a polytope, so for any triangulation of X_1, F generates a singular 2-cycle on M. This cycle does not bound, because the original sphere $H : \partial\sigma^3 \to M$ is noncontractible. Now suppose that $f\colon S^1 \to \Gamma$ is contractible. Then it can be contracted over a disc $\tilde{f}\colon D^2 \to \Gamma$. In that case we can construct a singular 3-chain on M that has the 2-cycle as its boundary. This 3-chain can be constructed as follows: assign a map $\tilde{F}$ from the quotient space X_2 obtained from $D^2 \times ([0,1] \bigcup [0,1])$ with four boundaries identified in accordance with the identification of endpoints of two segments forming $\tilde{f}(x)$, where $x \in D^2$. We claim that $\tilde{f}$ can be chosen so that X_2 is a polyhedron (see Section 4 of [9] for the details). Triangulating X_2 we obtain a singular 3-chain that corresponds to a map of D^2 into Γ. Observe that if we consider the singular 3-chain corresponding to a map of D^2 into Γ under this correspondence, then its boundary will be a singular 2-cycle corresponding to the restriction of this map to $S^1 = \partial D^2$. Therefore, a map of S^1 into Γ is contractible only if the corresponding 2-cycle represents 0 in $H_2(M)$. Thus, $f\colon S^1 \to \Gamma$ is noncontractible. Finally, let ϵ go to 0 to obtain $l(M) \leqslant 4r$. □

Proof of Theorem 2. Let $\{B_i(p_i)\}_{i=1}^k$ be the set of closed metric balls of radii $\{r_i\}_{i=1}^k$ respectively. The two cases: $k = 2$, and $k = 3$ will have to be considered separately,

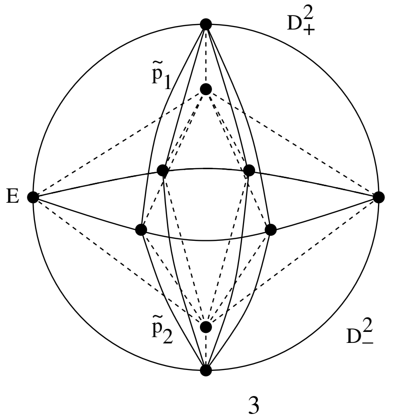

Figure 3.

however, in both of them we will begin with a homeomorphism $h\colon S^2 \to M$, where S^2 is triangulated so that diameter of any simplex in the induced triangulation of M is smaller than ϵ. Attempt to extend h to the disc D^3 and as an obstruction to the extension obtain a noncontractible sphere $H\colon S^2 \to M$, which we will then interprete as a noncontractible loop in the space Γ.

Part I: *Subdividing* D^3.
Case 1: $k = 2$. Let $B_1(p_1), B_2(p_2)$ be two closed metric balls of radii, r_1, r_2 centered at p_1, p_2 respectively, such that $M = B_1(p_1) \cup B_2(p_2)$. Without loss of generality we can

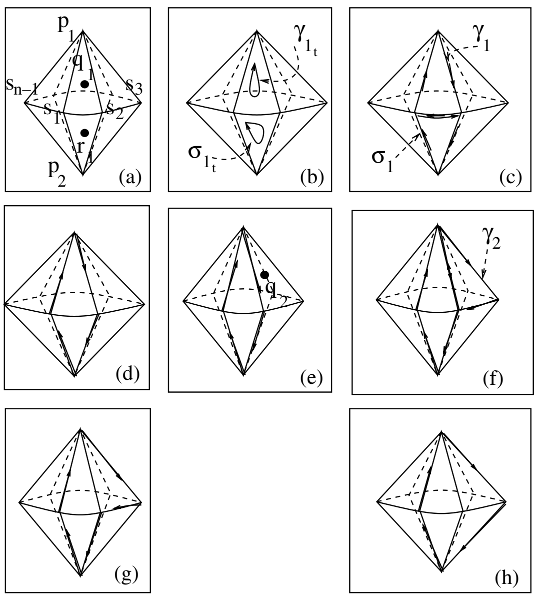

Figure 4.

assume that open balls $\operatorname{int} B_1(p_1) \cup \operatorname{int} B_2(p_2)$ also cover M. (Otherwise we can just consider concentric open balls centered at p_1, p_2 of radii $r_1 + \delta, r_2 + \delta$, where δ can be made arbitrarily small.)

Then by Mayer-Vietoris sequence there exists a set S in $B_1(p_1) \cap B_2(p_2)$ that is homeomorphic to S^1 and separates M into two sets $M_1 \subset B_1(p_1)$ and $M_2 \subset B_2(p_2)$. Both M_1 and M_2 are diffeomorphic to the 2-disc (see Lemma 4 in the Appendix). Consider the standard sphere S^2. Let D^2_+, D^2_- be closed sets that denote its northern and southern hemispheres respectively, and let E be its equator. Let $h\colon S^2 \to M$ be a homeomorphism, such that $h|_{D^2_+}, h|_{D^2_-}, h|_E$ are homeomorphisms onto M_1, M_2 and S, respectively. We are going to try to extend h to the disc D^3 that has S^2 as its boundary. We triangulate S^2, so that D^2_+, D^2_- and E become subcomplexes of S^2. Next we assign to D^3 the structure of a CW-complex: We will let its 0-skeleton consist of the 0-skeleton of S^2 and two additional vertices: $\tilde{p}_1$ that lies in the northern part of D^3 and $\tilde{p}_2$ that lies in its southern part. The 1-skeleton will consist of the 1-skeleton of S^2, the 1-skeleton of $\tilde{p}_1 * D^2_+$ and the 1-skeleton of $\tilde{p}_2 * D^2_-$.

Similarly, its 2-skeleton will consist of the 2-skeleton of S^2, the 2-skeleton of $\tilde{p}_1 * D^2_+$ and the 2-skeleton of $\tilde{p}_2 * D^2_-$. Finally, the disc's 3-skeleton will consist of the 3-skeleton of $\tilde{p}_1 * D^2_+$ and $\tilde{p}_2 * D^2_-$ and one additional cell that is the part of the disc which is bounded by the suspension of $\tilde{p}_1, \tilde{p}_2$ over E (see Figure 3). We will denote this cell Cl.

Case 2: $k = 3$. Let $i_1, i_2, i_3 \in \{1, 2, 3\}$. There exist two balls $B_{i_2}(p_{i_2}), B_{i_3}(p_{i_3})$ that have a nonempty intersection. Let $C = B_{i_2}(p_{i_2}) \cup B_{i_3}(p_{i_3})$. Then by Mayer-Vietoris sequence there exists a set $S_1 \subset B_{i_1}(p_{i_1}) \cap C$ that separates M into two discs: $D^2_1 \subset B_{i_1}(p_{i_1})$ and $D^{2'}_2 \subset C$ (see the proof of Lemma 4 in the Appendix). Next by the second application of the Mayer-Vietoris sequence there exists a finite collection of sets $\{T_i\}$ that separates the $D^{2'}_2$ into the discs and annuli, where each disc and each annuli fully lies either in $B_{i_2}(p_{i_2})$ or in $B_{i_3}(p_{i_3})$, and where each T_i is homeomorphic either to S^1 or to the unit interval (see Lemma 5 in the Appendix). Let us first consider the two simplest cases, when $\{T_i\}$ consists of one element only. The proof in the general case (denoted Case 2C) is analogous to the proofs of those simple cases. In the first case T_1 is homeomorphic to [0,1]. Let us denote it by S_2. In that case it separates $D^{2'}_2$ into the two discs: $D^2_2 \subset B_{i_2}(p_{i_2})$ and $D^2_3 \subset B_{i_3}(p_{i_3})$. We will call this Case 2A (see Figure 5(a)). In the second case — Case 2B (see Figure 5 (b)) T_1 is homeomorphic to S^1. We will denote it by $S_2{}' \subset B_{i_2}(p_{i_2}) \cap B_{i_3}(p_{i_3}) \cap D^{2'}_2$ and it separates $D^{2'}_2$ into a disc D^2_2 and an annulus A. Without loss of generality we can assume that in the first case $i_1 = 1, i_2 = 2, i_3 = 3$ and in the second case that $D^2_2 \subset B_{i_3}(p_{i_3}) \cap D^{2'}_2$ and that $A \subset B_{i_2}(p_{i_2}) \cap D^{2'}_2$.

Case 2A: Let E be the equator of the standard 2-sphere, and let R be the lower half of a meridean. Let D^2_N be the northern hemisphere of S^2, and D^2_{SW}, D^2_{SE} be its south-western and south-eastern quaters respectively. Let $h\colon S^2 \to M$ be a homeomorphism that maps E onto S_1, R onto S_2 and $D^2_N, D^2_{SW}, D^2_{SE}$ onto D^2_1, D^2_2, D^2_3 respectively.

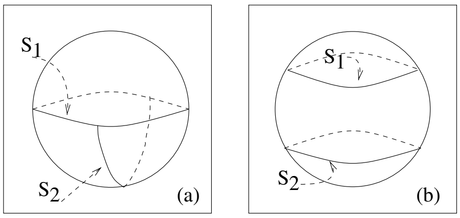

Figure 5.

Let us now give D^3 the structure of a CW-complex. Assume that $D^2_N, D^2_{SW}, D^2_{SE}, E, R$ are subcomplexes of S^2. We construct, the 0-skeleton of D^3 by combining the 0-skeleton of S^2 with the points $\tilde{p}_1, \tilde{p}_2, \tilde{p}_3$, where, $\tilde{p}_1$ is a point that lies in the northern half of D^3, and the points $\tilde{p}_2$, $\tilde{p}_3$ lie in the south-western and the south-eastern quarters, respectively. The construction of the 1- and the 2-skeleta of D^3 is analogous to that in the Case 1. The 3-skeleton will consist of the 3-simplices of the form $[\tilde{p}_1, \tilde{v}_{i_1}, \tilde{v}_{i_2}, \tilde{v}_{i_3}]$, or $[\tilde{p}_2, \tilde{w}_{j_1}, \tilde{w}_{j_2}, \tilde{w}_{j_3}]$, or $[\tilde{p}_3, \tilde{q}_{k_1}, \tilde{q}_{k_2}, \tilde{q}_{k_3}]$ and of one additional cell Cl described below. Let E_L be the left half of the equator and E_R be the right half of the equator. Then Cl will have $\tilde{p}_1 * E \cup \tilde{p}_2 * (E_L \cup R) \cup \tilde{p}_3 * (E_R \cup R)$ as its boundary.

Case 2B: In the second case, let $S_1, S_2{}'$ be as described before. Let P_1, P_2 be two parallels on S^2. Let $P_1 \in D^2_+$, the upper hemisphere of S^2 and $P_2 \in D^2_-$, its lower hemisphere. Two planes passing through P_1 and through P_2 subdivide D^3 into three parts: U, M, L and subdivide the sphere into three parts: $\tilde{U}, \tilde{M}, \tilde{L}$. Let $\tilde{p}_1 \in U, \tilde{p}_2 \in M, \tilde{p}_3 \in L$. Let $h\colon S^2 \to M$ be a homeomorphism that maps P_1, P_2 onto $S_1, S_2{}'$ respectively and that maps $\tilde{U}$ onto D^2_1; $\tilde{M}$ onto A and $\tilde{L}$ onto $D^{2'}_2$. Assume that $\tilde{U}, \tilde{M}, \tilde{L}, P_1, P_2$ are subcomplexes of S^2 that the diameter of any simplex in the induced triangulation on M is smaller than ϵ.

We will let 0-skeleton of D^3 consist of three points $\tilde{p}_1, \tilde{p}_2, \tilde{p}_3$ in addition to the vertices of S^2. Its 1-skeleton will be the 1-skeleton of $\tilde{p}_1 * \tilde{U} \cup \tilde{p}_2 * \tilde{M} \cup \tilde{p}_3 * \tilde{L}$ together with the 1-skeleton of S^2. Likewise its 2-skeleton will consist of the 2-skeleton of $\tilde{p}_1 * \tilde{U} \cup \tilde{p}_2 * \tilde{M} \cup \tilde{p}_3 \cup \tilde{L}$ and the 2-skeleton of S^2. Finally, its 3-skeleton will consist of the 3-skeleton of $\tilde{p}_1 * \tilde{U} \cup \tilde{p}_2 * \tilde{M} \cup \tilde{p}_3 * \tilde{L}$ and two cells bounded by the suspensions of $\tilde{p}_1, \tilde{p}_2$ over P_1 and of $\tilde{p}_2, \tilde{p}_3$ over P_2. Let us denote them Cl_1, Cl_2 respectively.

Part II: Extending to D^3 and constructing $f\colon S^1 \to \Gamma$.

Now we are going to try to extend our map to the disc. The procedure will be inductive to the disc's skeleta. To extend to the 0-skeleton, we will let the points $\tilde{p}_i$ be mapped to the corresponding centers of the balls p_i.

To extend to the 1-skeleton, we let each 1-simplex be mapped to a minimal geodesic segment between previously constructed images of its endpoints.

To extend to the 2-skeleton, consider any 2-simplex $\tilde{\sigma}^2$. Its boundary has been already mapped to a closed piecewise geodesic curve of length at most $2r_i + \epsilon$. Assuming, there is no closed geodesics of length $2\max_i r_i + \epsilon$ when ϵ is small, we can use the Birkhoff curve shortening process to contract this curve to a point without the length increase. We will map the simplex to the surface generated by the above homotopy. Likewise, we can extend to any 2-simplex.

We cannot extend this map further to the 3-skeleton, which means that there exists a 3-cell $\tilde{\sigma}^3$, such that $H\colon \partial\tilde{\sigma}^3 \to M$ is a noncontractible sphere.

In the Case 1: this cell $\tilde{\sigma}^3$ can have one of two shapes. It can either be a three-dimensional simplex (Case 1(a)) or it can be a cell Cl described above (Case 1(b)). We are going to interprete $H\colon \partial\tilde{\sigma}^3 \to M$ as a noncontractible loop in Γ as follows:

Case 1(a): Without loss of generality, assume $\tilde{\sigma}^3 = [\tilde{v}_0, \tilde{v}_1, \tilde{v}_2, \tilde{p}_1]$. Then $\partial\tilde{\sigma}^3 = [\tilde{v}_1, \tilde{v}_2, \tilde{p}_1] - [\tilde{v}_0, \tilde{v}_2, \tilde{p}_1] + [\tilde{v}_0, \tilde{v}_1, \tilde{p}_1] - [\tilde{v}_0, \tilde{v}_1, \tilde{v}_2]$. Let $\gamma_1 = [v_1, v_2] + [v_2, p_1] + [p_1, v_1]$, $\gamma_2 = [p_1, v_2] + [v_2, v_0] + [v_0, p_1]$, $\gamma_3 = [p_1, v_0] + [v_0, v_1] + [v_1, p_1]$, and finally $\gamma_4 = [v_0, v_2] + [v_2, v_1] + [v_1, v_0]$. We have fixed curve shortening homotopies that connect curves γ_i with points q_i for $i = 1, \cdots, 4$. We will denote curves in these homotopies γ_{i_t}. We will now present a noncontractible loop in the space Γ: $(q_1, q_2) \sim (\gamma_{1_t}, \gamma_{2_t}) \sim (\gamma_1, \gamma_2) \sim ([v_1, v_2] + [v_2, v_0], [v_0, p_1] + [p_1, v_1])$ and the length of curves in these homotopies is bounded from above by $4r_1 + 2\epsilon$. $([v_1, v_2] + [v_2, v_0], [v_0, p_1] + [p_1, v_1]) \sim (-\gamma_3, -\gamma_4) \sim (-\gamma_{3_t}, -\gamma_{4_t}) \sim (p_3, p_4) \sim (q_1, q_2)$, and the length of curves in the homotopy is bounded from above by $2r_1 + 4\epsilon$. So, we conclude that the length of curves in the loop $f\colon S^1 \to \Gamma$ is bounded from above by $4r_1 + 4\epsilon$.

Case 1(b): Let σ^3 be the cell Cl bounded by the suspension of $\tilde{p}_1$ and $\tilde{p}_2$ over E. We will represent the map from its boundary to M as a nontrivial loop in Γ.

Let $\{\tilde{s}_i\}_{i=1}^n$ be a sequence of vertices of E, where $\tilde{s}_n = \tilde{s}_1$ and let $\{s_i\}_{i=1}^{n-1}$ be a corresponding sequence of S. Let $[s_i, s_{i+1}]$ be 1-simplices of S, and finally, let $\gamma_i = [s_{i+1}, s_i] + [s_i, p_1] + [p_1, s_{i+1}]$, $\sigma_i = [s_i, s_{i+1}] + [s_{i+1}, p_2] + [p_2, s_i]$, where $[s_i, p_1]$, $[p_1, s_{i+1}], [s_{i+1}, p_2], [p_2, s_i]$ are minimal geodesic segments connecting corresponding points and directed from first to the second point. Assume γ_i is contractible to q_i and σ_i is contractible to r_i without length increase. The nontrivial loop in the space Γ follows (see Figure 4). $(q_1, r_1) \sim (\gamma_{1_t}, \sigma_{1_t})$ (see Figure 4(a) and (b)), and the length of curves in the homotopy is bounded from above by $2r_1 + 2r_2 + 2\epsilon$. $(\gamma_{1_t}, \sigma_{1_t}) \sim (\gamma_1, \sigma_1)$, (see Figure 4(c)). $(\gamma_1, \sigma_1) \sim ([s_1, p_1] + [p_1, s_2], [s_2, p_2] + [p_2, s_1])$, the curve of length $2r_1 + 2r_2$, (see Figure 4(d)). The above curve is homotopic to the curve $([s_1, p_1] + [p_1, s_2] + [s_2, p_2] + [p_2, s_1], q_2) \sim ([s_1, p_1] + [p_1, s_2] + [s_2, p_2] + [p_2, s_1], \gamma_{2_t}) \sim ([s_1, p_1] + [p_1, s_2] + [s_2, p_2] + [p_2, s_1], \gamma_2)$ (see Figure 4(e) and (f)). The length of curves in these homotopies is bounded from above by $4r_2 + 2r_1 + \epsilon$. The above curve is homotopic to $([s_1, p_1] + [p_1, s_3] + [s_3, s_2], [s_2, p_2] + [p_2, s_1])$ of length at most $2r_1 + 2r_2 + \epsilon$ (see Figure 4(g)). Now this curve is homotopic to the curve $([s_1, p_1] + [p_1, s_3] + [s_3, s_2] + [s_2, p_2] + [p_2, s_1], r_2) \sim ([s_1, p_1] + [p_1, s_3] + [s_3, s_2] + [s_2, p_2] + [p_2, s_1], \sigma_2) \sim ([s_1, p_1] + [p_1, s_3], [s_3, p_2] + [p_2, s_1])$ (see Figure 4(h)) and the length of curves in these homotopies is bounded from above by $2r_1 + 4r_2 + 2\epsilon$. Similarly, we obtain that $([s_1, p_1] + [p_1, s_3], [s_3, p_2] + [p_2, s_1]) \sim \cdots \sim ([s_1, p_1] + [p_1, s_n], [s_n, p_2] + [p_2, s_1])$, where $s_n = s_1$. The last pair of curves is homotopic to $(s_1, s_1) \sim (q_1, r_1)$.

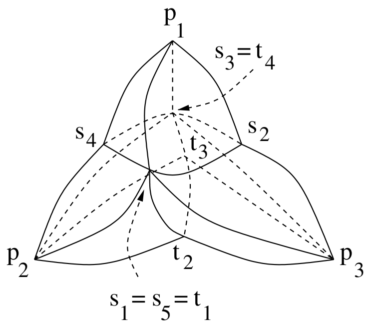

Figure 6.

Case 2A: There is at least one 3-cell, such that the map H restricted to its boundary is a nontrivial sphere. It can either be a 'small' cell, like $[\tilde{p}_1, \tilde{v}_{i_1}, \tilde{v}_{i_2}, \tilde{v}_{i_3}]$, $[\tilde{p}_2, \tilde{w}_{j_1}, \tilde{w}_{j_2}, \tilde{w}_{j_3}]$ or $[\tilde{p}_3, \tilde{q}_{k_1}, \tilde{q}_{k_2}, \tilde{q}_{k_3}]$ or a 'large' cell Cl. If it is a 'small' cell one can interpret its boundary as a loop $f: S^1 \to \Gamma$ as it was done in Case 1(a). The loop will pass through the curves of length at most $4r_i + 4\epsilon$, $i = 1, 2, 3$. Now suppose it is a 'large' cell. Then $H: \partial Cl \to M$ is noncontractible. We will next interprete this sphere as $f: S^1 \to M$.

Let $\{\tilde{s}_i\}_{i=1}^n$ be a sequence of vertices of E, such that $\tilde{s}_1 = \tilde{s}_n$. Let $\{\tilde{t}_j\}_{j=1}^m$ be a sequence of vertices of R, such that $\tilde{t}_1 = \tilde{s}_1$ and $\tilde{t}_m = \tilde{s}_k$ for some k. Let $\{s_i\}_{i=1}^n$ and $\{t_j\}_{j=1}^m$ be corresponding sequences in M. Let $[p_1, s_i], [p_2, s_i], [p_2, t_j], [p_3, s_i], [p_3, t_j]$ denote minimal geodesic segments in M (see Figure 6). Let $\gamma_i = [p_1, s_{i+1}] + [s_{i+1}, s_i] + [s_i, p_1]$ for $i = 1, \dots, n-1$ and assume that it can be contracted to a_i without length increase. Let $\sigma_i = [p_2, s_i] + [s_i, s_{i+1}] + [s_{i+1}, p_2]$ for $i = k, \dots, n-1$ and suppose it contracts to b_i. Let $\alpha_j = [p_2, t_{j+1}] + [t_{j+1}, t_j] + [t_j, p_2]$ for $j = 1, \dots, m-1$ and suppose it contracts to x_j. Let $\omega_j = [p_3, t_j] + [t_j, t_{j+1}] + [t_{j+1}, p_3]$ for $j = 1, \dots, m-1$ and suppose it contracts to y_j, and finally, suppose that $\beta_i = [p_3, s_i] + [s_i, s_{i+1}] + [s_{i+1}, p_3]$ for $i = 1, \cdots, k-1$ and contractible to z_i. A loop $f: S^1 \to \Gamma$ will follow. We will begin with a pair of constant maps: $(a_1, z_1) \sim (\gamma_1, \beta_1) \sim ([s_2, p_1] + [p_1, s_1], [s_1, p_3] + [p_3, s_2]) \sim ([s_2, p_1] + [p_1, s_1] + [s_1, p_3] + [p_3, s_2], a_2) \sim ([s_2, p_1] + [p_1, s_1] + [s_1, p_3] + [p_3, s_2], \gamma_2) \sim ([p_1, s_1] + [s_1, p_3] + [p_3, s_2], [s_2, s_3] + [s_3, p_1]) \sim ([p_1, s_1] + [s_1, p_3] + [p_3, s_2] + [s_2, s_3] + [s_3, p_1], z_2) \sim ([p_1, s_1] + [s_1, p_3], [p_3, s_3] + [s_3, p_1])$. Notice that the length of curves in this homotopy is bounded from above by $2(\max\{r_1, r_3\} + r_1 + r_3 + \epsilon)$. Similarly, we see that $([p_1, s_1] + [s_1, p_3], [p_3, s_3] + [s_3, p_1]) \sim ([p_1, s_1] + [s_1, p_3], [p_3, s_k] + [s_k, p_1])$ and the length of curves in the homotopy is still bounded by $2(\max\{r_1, r_3\} + r_1 + r_3 + \epsilon)$. Now note that $([p_1, s_1] + [s_1, p_3], [p_3, s_k] + [s_k, p_1]) \sim ([p_1, s_1] + [s_1, p_3] + [p_3, s_k] + [s_k, p_1], a_k) \sim ([p_1, s_1] + [s_1, p_3] + [p_3, s_k] + [s_k, p_1], \gamma_k) \sim ([p_1, s_1] + [s_1, p_3] + [p_3, s_k], [s_k, s_{k+1}] + [s_{k+1}, p_1]) \sim ([p_1, s_1] + [s_1, p_3] + [p_3, s_k] + [s_k, s_{k+1}] + [s_{k+1}, p_1], b_k) \sim ([p_1, s_1] + [s_1, p_3] + [p_3, s_k] + [s_k, s_{k+1}] + [s_{k+1}, p_1], \gamma_k) \sim ([p_1, s_1] + [s_1, p_3] + [p_3, s_k], [s_k, p_2] + [p_2, s_{k+1}] + [s_{k+1}, p_1]) \sim \cdots \sim ([s_1, p_3] + [p_3, s_k], [s_k, p_2] + [p_2, s_1])$, which

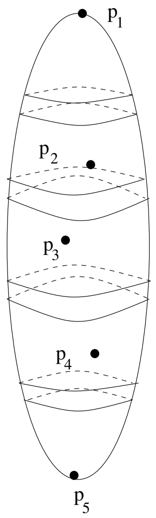

Figure 7.

is the same as $([t_1, p_3] + [p_3, t_m], [t_m, p_2] + [p_2, t_1])$. Note that the length of curves in this homotopy is bounded from above by $2(\max\{r_1, r_2\} + r_1 + r_2 + r_3 + \epsilon)$. Similarly, we can show that $([t_1, p_3] + [p_3, t_m], [t_m, p_2] + [p_2, t_1]) \sim ([t_1, p_3] + [p_3, t_m] + [t_m, p_2] + [p_2, t_1], x_1) \sim ([t_1, p_3] + [p_3, t_m] + [t_m, p_2] + [p_2, t_1], \alpha_1) \sim ([t_1, p_3] + [p_3, t_m] + [t_m, p_2], [p_2, t_2] + [t_2, t_1]) \sim ([t_1, p_3] + [p_3, t_m] + [t_m, p_2] + [p_2, t_2] + [t_2, t_1], y_1) \sim ([t_1, p_3] + [p_3, t_m] + [t_m, p_2] + [p_2, t_2] + [t_2, t_1], \omega_1) \sim ([t_2, p_3] + [p_3, t_m], [t_m, p_2] + [p_2, t_2]) \sim \cdots \sim ([t_m, p_3] + [p_3, t_m], [t_m, p_2] + [p_2, t_m]) \sim (p_3, p_2) \sim (a_1, z_1)$ and the length of curves in the homotopy is bounded from above by $2(\max\{r_2, r_3\} + r_2 + r_3)$. We have thus, constructed $f : S^1 \to \Gamma$ such that the length of curves through which it passes is bounded from above by $2(\max\{r_1, r_2, r_3\} + r_1 + r_2 + r_3 + \epsilon)$.

Case 2B: Since, we cannot extend to the 3-skeleton, there should be a 3-cell such that $H : \partial\sigma^3 \to M$ is a noncontractible sphere. This cell can either be a 3-simplex, or Cl_1 or Cl_2. In the first case, this sphere correspond to $f : S^1 \to \Gamma$ that passes through curves of length at most either $4r_1 + 4\epsilon$ or $4r_2 + 4\epsilon$ or $4r_3 + 4\epsilon$ depending on the type of the simplex. In the second case the corresponding loop passes through the curves that are at most $2(\max\{r_{i_1}, r_{i_2}\} + r_{i_1} + r_{i_2} + \epsilon)$ long and in the third case the length of curves in the $f : S^1 \to M$ is bounded from above by $2(\max\{r_{i_2}, r_{i_3}\} + r_{i_2} + r_{i_3} + \epsilon)$. The loop f is constructed exactly as in Case 1(b). Observe that in the situation of Case 2B, when M is covered by 3 metric balls, two of which do not intersect, we obtained a somewhat better estimate than in the general case. It coincides with the estimate provided by Theorem 3, when $k = 3$. For example, if M is covered by balls of radii $d/2, d/2$ and $d/3$ Theorem 4 implies that $l(M) \leqslant 11d/3$. However, if two of these balls do not intersect, we can conclude that $l(M) \leqslant 3d$.

It remains to show that in all of the cases the constructed loops are noncontractible. The proof, however is completely analogous to the proof of the same fact in the Theorem 1. This follows from the fact that the corresponding 2-cycles on M are not null-homotopic (see also Section 4 of [9] for a proof of a very general version of the last assertion). Now let ϵ approach 0 and obtain the bound.

Finally, let us look at the general case.

Case 2C: In the general case, when $\{T_i\}$ consists of several elements, S_1 and $\{T_i\}$ divide M into the domains D_{i_1}; $\{M_{i_2,k}\}_{k=1}^{m}$, where $M_{i_2,k} \subset B_{i_2}$ for all $k = 1, \ldots, m$ and $\{M_{i_3,j}\}_{j=1}^{l}$, where $M_{i_3,j} \subset B_{i_3}$, (see Lemma 5 in the Appendix). Each domain is either a disc on an annulus. Similarly, decompose S^2, and consider the homeomorphism $f : S^2 \to M$, so that the equator E is mapped homeomorphically onto S_1, the southern hemisphere $\tilde{D}_1^2$ is mapped homeomorphically to D_{i_1} each interval or circle $\tilde{T}_i$ is mapped homeomorphically onto the corresponding T_i and finally each domain: $\tilde{M}_{2,k}$ is mapped homeomorphically onto the corresponding $M_{i_2,k}$ and each domain $\tilde{M}_{3,j}$ is mapped onto the corresponding $M_{i_3,j}$. We will try to extend $h : S^2 \to M$ to the disc D^3 and as an obstruction obtain a noncontractible $H : S^2 \to M$, that can be interpreted as a nontrivial loop in Γ. In order to perform the extension process D^3 is given the structure of a CW-complex. 0-Skeleton is constructed by placing inside D^3, $l + m + 1$ additional vertices that will be denoted $\tilde{p}_1; \tilde{p}_{2,k}$ and $\tilde{p}_{3,j}$. Vertex $\tilde{p}_1$ will be placed below the equator; each vertex of the form $\tilde{p}_{2,k}$ will be placed 'close' to the corresponding domain $\tilde{M}_{2,k}$ and each vertex of the form $\tilde{p}_{3,j}$ will be placed 'close' to the domain $\tilde{M}_{3,j}$. We assume that sphere S^2 has a fine triangulation and that all relevant curves and domains are the subcomplexes. The new vertices $\tilde{p}_1, \tilde{p}_{2,k}, \tilde{p}_{3,j}$ should be constructed so that when they are joined with the vertices of the corresponding domain, the new edges do not intersect, except at the endpoints. The construction of the 1, 2, and 3-skeleta is similar to that in cases A and B. We next perform the extension process. To extend to the 0-skeleton, we will map $\tilde{p}_1$ to p_{i_1}, $\tilde{p}_{2,k}$ to p_{i_2} for all $k = 1, \ldots, m$ and $\tilde{p}_{3,j}$ to p_{i_3} for all $j = 1, \ldots, l$. The rest of the proof is quite similar to that in cases A and B, but is more awkward. Therefore we will skip the details. □

Proof of Theorem 3. The proof of Theorem 3 is completely analogous to Case 1B in the proof of Theorem 2. The idea is to finely triangulate S^2 and then extend this triangulation to a cell subdivision of D^3 with k new vertices $\tilde{p}_1, \ldots, \tilde{p}_k$ inside the ball and $(k-1)$ cells Cl_i bounded by suspensions of $\tilde{p}_i$ and $\tilde{p}_{i+1}$ over parallels on S^2 (see Figure 8). Then proceed as in Case 1(b). □

Appendix

LEMMA 4. *Let M be a Riemannian manifold diffeomorphic to the standard 2-sphere. Let B_1 and B_2 be two open metric balls that jointly cover M, such that neither B_1 nor B_2 covers M by itself. Then there exists a set $S \in M$ that is homeomorphic to S^1 that separates M into two discs $D_1^2 \in B_1$ and $D_2^2 \in B_2$.*

Proof. First, we will show that $B_1 \cap B_2$ is connected. Consider the following segment of the reduced Mayer-Vietoris sequence: $\tilde{H}_1(B_1 \cup B_2) \to \tilde{H}_0(B_1 \cap B_2) \to \tilde{H}_0(B_1) \oplus \tilde{H}_0(B_2)$. Since M is simply connected and both B_1, B_2 are connected, $\tilde{H}_0(B_1 \cap B_2) = \{0\}$. Therefore, $B_1 \cap B_2$ is connected. Next, we will consider another segment of the Mayer-Vietoris sequence: $H_2(B_1) \oplus H_2(B_2) \to H_2(B_1 \cup B_2) \to H_1(B_1 \cap B_2)$. Since, $H_2(B_1)$ and $H_2(B_2)$ are both trivial, and since $H_2(B_1 \cup B_2)$ is Z then Z is a subgroup of $H_1(B_1 \cap B_2)$. Moreover, if we examine the corresponding zig-zag lemma, we will see that the generator of this subgroup can be represented by a cycle c_1 that is a common boundary of two 2-chains of B_1 and B_2 that jointly represent the fundamental class of M. c_1 can be represented by a finite union of simple closed curves $\{\alpha_i\}$, which, with a little modification can be arranged to be pairwise disjoint. If c_1 is represented by one curve only then we are done, since it is then homeomorphic to S^1, and thus, by the Jordan curve theorem separates M into the two discs. Suppose the number of curves representing c_1 is two: α_1, α_2. Let $a_1 \in \alpha_1$ and $a_2 \in \alpha_2$. Since, $B_1 \cap B_2$ is connected, there exists a path P in the intersection that connects the points a_1 and a_2. It is possible to find such a_1, a_2, P that P intersects α_i only at a_i for $i = 1, 2$. Now, the chain $\tilde{c}_1 = \alpha_1 \cup P \cup \alpha_2 \cup -P$ is homologous to c_1, where $-P$ is P with the opposite orientation. Since $B_1 \cap B_2$ is open we can perturb the above curve, so that it will become simple, and so that the corresponding chain is homologous to the original chain. The rest of the proof follows by induction on the number of curves. □

LEMMA 5. *Let M be a Riemannian manifold diffeomorphic to the standard 2-dimensional sphere. Let B_1, B_2, B_3 be three metric balls that jointly cover M, but no two of them cover M by themselves. Then there exists a set S_1 homeomorphic to S^1 that separates M into two discs D_1^2 and $D_2^{2'}$, where $D_1^2 \subset B_{i_1}, D_2^{2'} \subset B_{i_2} \cup B_{i_3}$, $i_j \in \{1, 2, 3\}$, $j = 1, 2, 3$, and a finite collection of sets $\{T_i\}$ (where each T_i is either homeomorphic to S^1 or the interval) that decomposes $D_2^{2'}$ into the collection of discs and annuli, where*

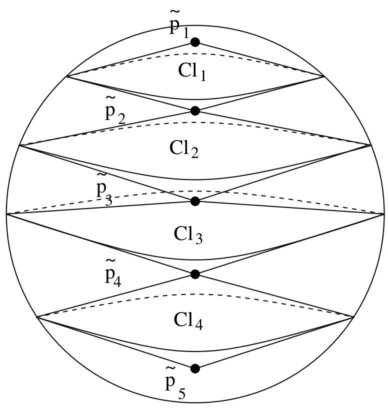

Figure 8.

each disc and annulus is a subset of B_{i_2} or of B_{i_3}.

Proof. There exists at least one pair of balls with a non-empty intersection. We will denote them as B_{i_2} and B_{i_3}. Let $B_{i_2} \cup B_{i_3} = C$. Then by Mayer-Vietoris sequence there exists a set $S_1 \subset B_{i_1}(p_{i_1}) \cap C$ homeomorphic to S^1 that separates M into two discs: $D_1^2 \subset B_{i_1}(p_{i_1})$ and $D_2^{2'} \subset C$. Next, let us denote $D_2^{2'} \cap B_{i_2}(p_{i_2})$ as K_2 and $D_2^{2'} \cap B_{i_3}(p_{i_3})$ as K_3 and consider the relative Mayer-Vietoris sequences: $H_2(K_2, K_2 \cap S_1) \oplus H_2(K_3, K_3 \cap S_1) \to H_2(D_2^{2'}, \quad S_1) \to H_1(K_2 \cap K_3, K_2 \cap K_3 \cap S_1)$. Since the first term in this sequence is trivial and the second is isomorphic to Z, there exists a nontivial 1-cycle in the group of the relative chains of the intersection $K_2 \cap K_3$, which is a common boundary of the relative 2-chains of K_2 and K_3, which sum represents the relative fundamental class of the disc. This 1-cycle can be represented by a finite union of the intervals and circles, and with a small modification those intervals and circles can be made pairwise disjoint. □

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