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Geodesic loops and periodic geodesics on a Riemannian manifold diffeomorphic to S^3

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Abstract Let M^n be a closed 2-connected Riemannian manifold, such that $\pi_3(M^n) \neq \{0\}$. In this paper we prove that either there exists a periodic geodesic on M^n of length $\leq 6d$, where d is the diameter of M^n, or at each point $p \in M^n$ there exists a geodesic loop of length $\leq 2d$.

Introduction and main results

In 1951, L. Lusternik and A. Fet proved that on any closed Riemannian manifold there exists at least one periodic geodesic. Their proof uses Morse theory on the space ΛM^n of all continuos maps $f : S^1 \longrightarrow M^n$. In a similar way one can show that at every point p of a closed Riemannian manifold there exists a non-trivial geodesic loop based at that point. The later statement also follows from a well-known result by Serre [19] that states that for any two points of a closed Riemannian manifold there exist infinitely many geodesics connecting them.

It is, therefore, reasonable to ask whether there is a connection between the length of a shortest periodic geodesic/geodesic loop at a point and other geometric parameters of a manifold. For example, in 1983 M. Gromov asked whether one can bound above the length of a shortest periodic geodesic $l(M^n)$ on M^n by $c(n)vol(M^n)^{\frac{1}{n}}$, where $vol(M^n)$ is the volume of M^n and $c(n)$ is a constant that depends on the dimension of M^n only, (see [7, p. 135]). A similar question can be asked about the relationship between $l(M^n)$ and the diameter of a manifold d. In particular, one can state the following

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Conjecture A Let M^n be a closed Riemannian manifold of dimension n. There exists a periodic geodesic of length $\leq \tilde{c}(n)d$, where d is the diameter of the manifold.

Moreover, it is possible that the length of a shortest periodic geodesic is always bounded by $2d$.

Similarly, one can conjecture

Conjecture B Let M^n be a closed Riemannian manifold of dimension n. Then there exists a geodesic loop of length $\leq 2d$, where d is the diameter of M^n at every point $p \in M^n$.

Theorem 0.1 below asserts that for every closed Riemannian manifold with $\pi_1(M^n) = \pi_2(M^n) = \{0\}$ and $\pi_3(M^n) \neq \{0\}$ either the conclusion of Conjecture A or the Conjecture B is true.

In Rotman [14] we have established that for every point $p \in M^n$ the length of a shortest geodesic loop at p, where M^n is a closed Riemannian manifold of dimension n is bounded by $2nd$ and even $\leq 2qd$, where $q = \min\{i : \pi_i(M^n) \neq \{0\}\}$. However, the question about whether the length of a shortest geodesic loop is bounded by $2d$ at each point of M^n still remains unanswered for simply connected manifolds, even when M^n is diffeomorphic to S^2 and even in the case of convex metrics on a 2-dimensional sphere.

Note that prior to Rotman [14] there existed only the result of Sabourau [18] asserting that the shortest length of a non-trivial geodesic loop on the whole manifold is $\leq \tilde{k}(n)d$, where $\tilde{k}(n) = \frac{(8 \cdot 3^n - 2)d}{3}$. In this case the base point p is not prescribed.

We would also like to mention that the length of a shortest geodesic loop cannot be uniformly bounded in terms of the volume of a manifold at each point $p \in M^n$. That is, there does not exist a constant $k(n)$, such that $l_p(M^n)$ is bounded above by $k(n)vol(M^n)^{\frac{1}{n}}$ for every $p \in M^n$. As an example consider an ellipsoid E that is also a surface of revolution, that is a surface generated by an ellipse that is rotated around its major axis. Let R denote its polar radius and $p \in E$ be its north pole. All the geodesic loops based at p are ellipses, (in this case the geodesic loops also happen to be periodic geodesics) (see Fig. 1). Therefore, when the smaller semiaxis is fixed and R goes to infinity, the ratio $\frac{l_p(E)}{\sqrt{A(E)}}$ approaches infinity as well.

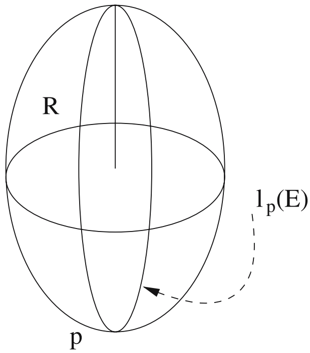
However, one can estimate the shortest length of a geodesic loop on the whole manifold by volume, as it was first indicated by Sabourau [18].

It is easy to see to prove the following

Observation C *Let M^n be a non-simply connected closed Riemannian manifold. Then both the length of a shortest periodic geodesic and the length of a shortest geodesic loop at each point $p \in M^n$ are bounded by twice the diameter of a manifold.*

Proof Let us begin by taking an arbitrary non-contractible map $f : S^1 \longrightarrow M^n$, where S^1 is subdivided into small segments, in such a way that the diameter of each edge in the triangulation of $f(S^1)$ induced by f is smaller than some small positive δ. Let D^2 be the 2-disc that is triangulated as a cone over the triangulation of S^1. We will attempt to extend $f : S^1 \longrightarrow M^n$ to D^2, which is impossible. Thus, as an obstruction to this extension we will obtain a non-contractible loop based at a prescribed point $p \in M^n$ of length $\leq 2d + \delta$. Then we can obtain a non-contractible periodic geodesic by minimizing the length in the free homotopy class of the loop. We can also obtain

Fig. 1 There is no uniform volume bound for the length of a shortest geodesic loop at a point



a non-contractible geodesic loop based at p by shortening this loop while keeping p fixed. The extension process will be inductive on skeleta of D^2. Let us begin by extending to the 0-skeleton of D^2. The center of the disc $\tilde{p} \in D^2$ is the only additional vertex of D^2, (the rest of the vertices are coming from S^1). We will map $\tilde{p}$ to the point $p \in M^n$. Next we will extend to the 1-skeleton. Consider an edge $[\tilde{p}, \tilde{v}_i]$, where $\tilde{v}_i$ is the vertex in the triangulation of S^1. We will map this edge to a minimal geodesic segment $[p, v_i]$ connecting the point p with $v_i = f(\tilde{v}_i)$. Finally, we will extend to the 2-skeleton of D^2. Take a 2-simplex $[\tilde{p}, \tilde{v}_i, \tilde{v}_{i+1}]$. The boundary of this simplex is mapped to a closed curve of length $\leq 2d + \delta$, made of two minimizing geodesic segments and a "small" edge $[v_i, v_j]$ of length $\leq \delta$.

The map of the boundary of at least one of such simplices must be non-contractible, since, otherwise, we could have extended $f : S^1 \longrightarrow M^n$ to the whole disc D^2. Thus, we obtain a non-contractible loop based at p of length $\leq 2d + \delta$.

Letting δ approach 0 we obtain a non-contractible loop based at p of length $\leq 2d$. □

We would like to note in passing that there are numerous volume estimates for the length of a shortest periodic geodesic on non-simply connected manifolds, especially in the case of surfaces. The first such results are due to C. Loewner, P. Pu, followed by R. Accola, C. Blatter, C. Bavard, Ju. Burago and V. Zalgaller, J. Hebda, T. Sakai and others (see [3,6]) M. Gromov's generalized the above results for the class of 1-essential manifolds, which include all aspherical manifolds as well as manifolds homotopy equivalent to real projective spaces.

In the case of simply connected manifolds, the only curvature-free estimates exist for manifolds diffeomorphic to the 2-dimensional sphere ([2,4,8,9,12,13,17]). In particular, these results imply Conjecture B for surfaces.

At present there are no known similar curvature-free upper bounds for the length of a shortest closed geodesic $l(M^n)$ in the general case of a closed Riemannian manifold M^n, though such bounds do exist for stationary 1-cycles and geodesic nets ([10,15,16]), and minimal surfaces ([11]). Moreover, one does not know if the length of a shortest periodic geodesic can be estimated in terms of diameter or volume for any diffeomorphism type of simply-connected manifold M^n for $n \geq 3$.

Even for a manifold diffeomorphic to S^3 the only known estimate is that of Croke ([5]), who found that on a manifold M^n diffeomorphic to S^3 the volume is bounded below by $\min\{l(M^n), D(M^n)\}$, where $D(M^n)$ is the minimal distance between the antipodal points.

In this paper we will prove the following

Theorem 0.1 *Let M^n be a 2-connected closed Riemannian manifold with a non-trivial 3rd homotopy group. (For example, M^n can be diffeomorphic to S^3). Then either the length of a shortest periodic geodesic on M^n is bounded above by 6d, or at each point $p \in M^n$ there exists a geodesic loop of length $\leq 2d$.*

In particular, we obtain curvature-free upper bounds for the $\min\{\beta(M^n), l(M^n)\}$, where $\beta(M^n)$ denotes the supremum of the length of a shortest geodesic loop at p taken over all of p and $l(M^n)$ denotes the length of a shortest periodic geodesic on M^n.

Balacheff [1] proved an analog of our result in the case of simply connected manifold M^n with a non-trivial second homotopy group. He showed that in this case, either there exists a periodic geodesic of length $\leq 4d$ or for every $p \in M^n$ there exists a geodesic loop of length $\leq 2d$ based at p. In Rotman [14] we strengthened his result by showing that on closed Riemannian manifolds with a non-trivial second homology group with no periodic geodesics of length $\leq 4d$ for every $p \in M^n$ there exists at least three geodesic loops of length $\leq 2d$ based at p.

The proof of our theorem uses a modified version of Gromov [7] extension technique. It also uses our technique of contraction of k-spheres in M^n (in our case, for $k \leq 3$) using continuous homotopies of 1-dimensional objects. Note that the homotopies do not use sweep-outs of spheres by these 1-dimensional objects. As in Rotman [14] the proof will make a repeated use of Lemma 1.1 and of the following well-known

Observation D *Let M^n be a complete Riemannian manifold. Let $p \in M^n$. Suppose that the length of a shortest periodic geodesic $l(M^n)$ is greater than L. Then given any piecewise differentiable closed curve $\gamma : S^1 \longrightarrow M^n$, of length $\leq L$ there exists a length decreasing homotopy H_γ contracting γ to a point that depends continuously on a curve γ. (In other words the space of closed curves of length $\leq L$ on M^n can be contracted to M^n, regarded as the space of closed curves of length 0 on M^n by a homotopy that decreases lengths.) Moreover, the homotopy H_γ can be chosen so that closed curves obtained during this homotopy are geometrically the same, but have opposite orientation when compared to the curves obtained during the homotopy H_γ contracting the curve $-\gamma(t) = \gamma(1-t)$.*

A standard proof of this assertion involves the Birkhoff curve-shortening process, (cf. [4] for the details).

The starting point for the proof of Theorem 0.1 will be a non-contractible map $f : S^3 \longrightarrow M^n$ from the standard sphere with a fine triangulation, that we will then try to extend to D^4 triangulated as a cone over S^3. That will be done by induction on the skeleta of D^4. Technically, the main difference with the approach of Rotman [14] will be in the way we extend to the 4-skeleton of D^4. We would like to note that a similar technique will not lead to a similar estimate for the length of a shortest periodic geodesic in the case of a closed Riemannian manifold without other topological restrictions, as it will require a stronger assumptions about the length of a geodesic loops (other than that their length is $> 2d$ at some point $p \in M^n$).

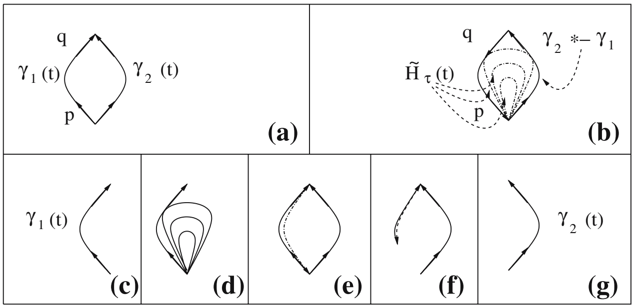


Fig. 2 Curves γ_1 and γ_2 are path homotopic

1 The proof of Theorem 0.1

Lemma 1.1 *Let M^n be a Riemannian manifold.*

Let $\gamma_1(t), \gamma_2(t)$ be two curves connecting the points $p, q \in M^n$ of lengths l_1, l_2 respectively. Consider a (not geodesic) loop based at p $\gamma_2 * -\gamma_1$, that is a product of γ_2 and $-\gamma_1$. If this loop is contractible to p by a path homotopy (that is, as a loop based at p) without the length increase then there is a path homotopy (that is a homotopy that fixes the end points) $H_\tau(t), \tau \in [0, 1]$, such that $H_0(t) = \gamma_1(t), H_1(t) = \gamma_2(t)$ and the length of curves during this homotopy is bounded above by $2l_1 + l_2$.

Proof Let $\tilde{H}_\tau(t)$ denote a homotopy that connects $\gamma_2 * -\gamma_1$ with a p (see Fig. 2a, b). Then below is a path homotopy between $\gamma_1(t)$ and $\gamma_2(t)$ satisfying the required properties. $\gamma_1 \sim \tilde{H}_{1-\tau} * \gamma_1 \sim \gamma_2 * -\gamma_1 * \gamma_1 \sim \gamma_2$ (see Fig. 2a–g). The length of curves during this homotopy is $\leq 2l_1 + l_2$. □

(A similar argument is used by Croke [4] to prove Lemma 3.1.)

Proof of Theorem 0.1 Let $f : S^3 \longrightarrow M^n$ be a non-contractible map from the 3-dimensional sphere S^3 that is triangulated so that the diameter of simplices in the triangulation of $f(S^3)$ induced by f is smaller than some positive small δ. Let D^4 denote the 4-disc triangulated as a cone over the triangulation of S^3. The proof will be by contradiction. Assume that $l(M^n) > 6d$ and that $l_p(M^n) > 2d$ at some point p of M^n. We will extend f to D^4, thus reaching a contradiction. Firstly, we will extend f to the 0-skeleton of D^3. Let $\tilde{p} \in D^3$ be the center of this disc. We will let the image of $\tilde{p}$ be the given point $p \in M^n$. Secondly, we will extend to the 1-skeleton by assigning to an edge $[\tilde{p}, \tilde{v}_i]$ that connects the center of the disc with the vertex $\tilde{v}_i$ a minimal geodesic segment $[p, v_i]$ that connects the point p with the vertex $v_i = f(\tilde{v}_i)$. Next we will extend to the 2-skeleton. Take an arbitrary 2-simplex $\tilde{\sigma}_i = [\tilde{p}, \tilde{v}_{i_1}, \tilde{v}_{i_2}]$. Its boundary $\partial\tilde{\sigma}_i^2 = [\tilde{p}, \tilde{v}_{i_1}] - [\tilde{p}, \tilde{v}_{i_2}] + [\tilde{v}_{i_1}, \tilde{v}_{i_2}]$ is mapped to a closed curve of length $\leq 2d + \delta$ by the previous step of the extension procedure. Using our assumption that the length of a shortest periodic geodesic is greater than $2d + \delta$, this closed curve can be contracted to a point by a length-decreasing homotopy.

The 2-simplex is then mapped using this homotopy Its image will be denoted as σ_i^2. At the next step we will extend to the 3-skeleton. Take an arbitrary 3-simplex $\tilde{\sigma}_i^3 = [\tilde{p}, \tilde{v}_{i_1}, \tilde{v}_{i_2}, \tilde{v}_{i_3}]$. By the previous step of the extension, its boundary $\partial\tilde{\sigma}_i^3 = \sum_{j=0}^{3}(-1)^j[\tilde{v}_{i_0}, \ldots \hat{\tilde{v}}_{i_j}, \ldots, \tilde{v}_{i_3}]$, where $\tilde{v}_{i_0} = \tilde{p}$. We will denote $[\tilde{p}, \tilde{v}_{i_j}] = \tilde{e}_j$ and $[p, v_{i_j}]$ as e_j. Since $[v_{i_1}, v_{i_2}, v_{i_3}]$ can be made arbitrarily small we will pretend here that it is a

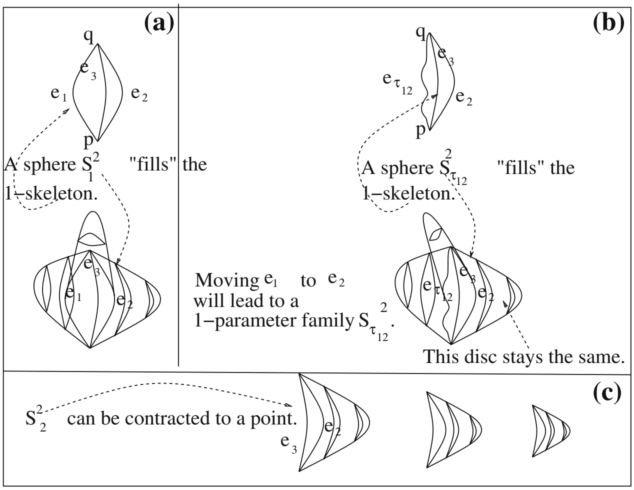


Fig. 3 Construction of $S^2_{\tau_{12}}$

point q for the sake of simplicity of the exposition. The details of why it is possible to treat small simplices as points will be explained in the remark.

Consider the image of its 1-skeleton. It consists of the three edges e_1, e_2, e_3. Here is the main idea behind the extension to the 3-skeleton:

1. In order to extend to the 3-skeleton, it is necessary to contract $f|_{\partial\tilde{\sigma}_i^3}$ to a point.
2. This homotopy can be viewed as a 1-parameter family of 2-spheres, that begins with $f(\partial\tilde{\sigma}_i^3)$ and ends with a point.
3. Here is how we will obtain this 1-parameter family of spheres. We will begin with the 1-skeleton formed by e_1, e_2, e_3 and will apply Lemma 1.1. This lemma allows us to "move" e_1 to e_2, thus resulting in a 1-parameter family of "skeleta". Using this family, we construct a 2-sphere that "fills" any given skeleton, by gluing the three discs, that are obtained by contracting three pairs of curves without the length increase.

More specifically, by assuming that there are no "short" periodic geodesics and no "short" geodesic loops based at p, we can contract $f : \partial\tilde{\sigma}_i^3 \longrightarrow M^n$ to a point as follows:

By our assumption, there is no geodesic loops based at p of length $\leq 2d$. Therefore, we can apply Lemma 1.1 to conclude that there is a path homotopy between e_1 and e_2 that passes through curves $e_{\tau_{12}}, 1 \leq \tau_{12} \leq 2$ of length $\leq 3d$. Now, let us define $S^2_{\tau_{12}}$. Take the two points p and q joined by two geodesic segments e_2, e_3 and the curve $e_{\tau_{12}}$, that now replaces e_1 in the 1-skeleton of the future sphere $S^2_{\tau_{12}}$, (see Fig. 3a). Furthermore, we have assumed that there is no periodic geodesics of length $\leq 4d (< 6d)$, therefore, both curves $e_2 * -e_{\tau_{12}}$ and $e_{\tau_{12}} * -e_3$ are contractible to a point by the length decreasing homotopies mentioned in Observation D, (e.g. by the Birkhoff curve-shortening process). Let us call the discs obtained during these homotopies $(D^2_2)_{\tau_{12}}$, (see Fig. 3b) and $(D^2_3)_{\tau_{12}}$, respectively. These discs change continuously with τ_{12}. This is due to the fact that in the absence of "short" periodic geodesics, the length-decreasing homotopy is continuous with respect to the initial curve.

$S^2_{\tau_{12}}$ is obtained by gluing the three discs: $\sigma^2_{i_0,i_2,i_3}$, $(D^2_2)_{\tau_{12}}$ and $(D^2_3)_{\tau_{12}}$ along their boundaries, just as we glue simplices in the boundary of a 3-dimensional simplex, where one of the simplices is a point.

Note that at the time $\tau_{12} = 1$, $S^2_{\tau_{12}}$ is the original sphere and at the time $\tau_{12} = 2$ it is a sphere that is essentially obtained from two discs that are geometrically the same, but have opposite orientations, which are then glued along their common boundary. That is, we begin with two points p and q, join them with three segments two of which coincide: e_2, e_2, e_3. Next obtain three discs, one of which is degenerate: it is obtained by contracting a curve $e_2 * -e_2$ along itself. As it is mentioned above, the other two discs coincide, but have opposite orientation: one is obtained by contracting $e_2 * -e_3$ and the second one, by contracting $e_3 * -e_2$ (see Fig. 3c). So, obviously, the sphere that we obtain is contractible along itself. Thus, we obtain a homotopy between $f|_{\partial(\tilde{\sigma}^3_i)}$ ans a constant map. We will map simplex $\tilde{\sigma}^3_i$ using the above homotopy.

Let us denote the image of a 3-simplex $\tilde{\sigma}^3_{i_0,\dots,i_3} = [\tilde{v}_{i_0}, \tilde{v}_{i_1}, \dots, \tilde{v}_{i_3}]$, where $\tilde{v}_{i_0} = \tilde{p}$ by $\sigma^3_{i_0,\dots,i_3}$.

Finally, we will extend our map to the 4-skeleton. Let us consider an arbitrary simplex $\tilde{\sigma}_{i_0,i_1,i_2,i_3,i_4} = [\tilde{p}, \tilde{v}_{i_1}, \tilde{v}_{i_2}, \tilde{v}_{i_3}, \tilde{v}_{i_4}]$. Its boundary is mapped to the following 3-sphere $\sum_{j=0}^4 (-1)^j \sigma_{i_0,\dots,\hat{i}_j,\dots,i_4}$.

Here are the main ideas behind the extension to the 4-skeleton:

1. In order to extend to the 4-skeleton, it is necessary to construct the homotopy that contracts this sphere to a point. Again, without loss of generality, assume that simplex $[v_{i_1}, \dots, v_{i_4}]$ is so small that it can, for our purposes, be treated as a point q. Each of the four edges $[p, v_{i_j}]$ will be denoted by e_j.
2. This homotopy can be viewed as a 1-parameter family of 3-spheres, that starts with the sphere $f(\partial\tilde{\sigma}^3_i)$ and ends with a point.
3. This 1-parameter family is constructed as follows: We consider the 1-skeleton that consists of the edges e_1, e_2, e_3, e_4 and apply Lemma 1.1 to "move" e_4 to e_3. This corresponds to a 1-parameter family of 1-skeleta. During the previous step we have already learned how to construct 3-discs from 1-skeleton. The desired 3-spheres are constructed by generating four 3-discs, that are then glued as in the boundary of a 4-simplex, (taking into account that the fifth disc is just a point).

Here are the details of the proof.

By applying Lemma 1.1 we can see that e_4 is homotopic to e_3 by a path homotopy along the curves $e_{\tau_{43}}$, $1 \le \tau_{43} \le 2$ of length $\le 3d$. This results in a 1-parameter family of curves. Let us consider the new 1-skeleton of the future 3-sphere $S^3_{\tau_{43}}$ in which $e_{\tau_{43}}$ has replaced e_4. For each τ_{43} we will then construct a 3-sphere that "fills" the new 1-skeleton. Than we will construct a 1-parameter family of 3-spheres $S^3_{\tau_{43}}$ that continuously depends on τ_{43}. This family of spheres will generate the required 4-disc.

The sphere $S^3_{\tau_{43}}$ will be made of four discs $(D^3_i)_{\tau_{43}}, i = 1, 2, 3, 4$ that will be glued as four simplices in the boundary of the 4-simplex, in which the fifth simplex is taken to be a point.

As τ_{43} will change from 1 to 2 the disc $(D^3_1)_{\tau_{43}}$ will be constantly equal to $\sigma^3_{i_1,i_2,i_3}$.

Next, let us construct $(D^3_3)_{\tau_{43}}$. This will be done by taking two points p, q, connected by three segments: $e_1, e_2, e_{\tau_{43}}$, (see Fig. 4a), by constructing a 2-sphere that "fills" this 1-skeleton and, finally, by constructing a family of spheres that begins with this given sphere and ends with a point. A sphere $S^2_{\tau_{43}}$ is constructed, of course, by considering the three closed curves that result from $e_1, e_2, e_{\tau_{43}}$ and by contracting each to a point

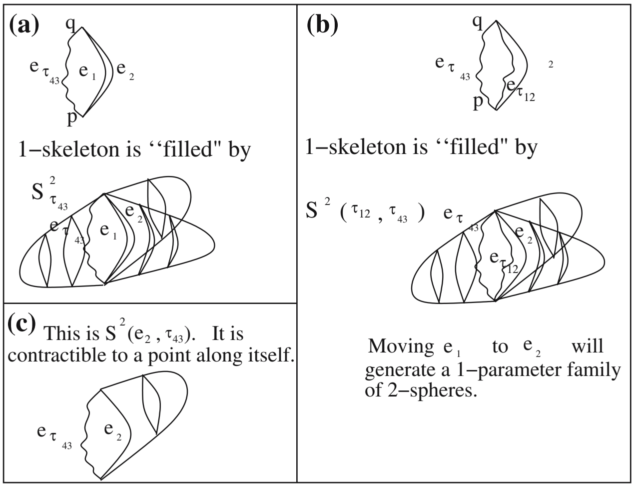


Fig. 4 Constructing $(D^3_3)_{\tau_{43}}$

without the length increase, using the assumption that all the periodic geodesics are "long".

Just as we have done when we extended the map f to the 3-skeleton, we can continuously deform this sphere to a point as follows:

1. We have constructed $S^2_{\tau_{43}}$ by taking three closed curves $e_2 * -e_1, e_{\tau_{43}} * -e_2, e_1 * -e_{\tau_{43}}$ and contracting them to a point, (see Fig. 4b).
2. At the next step we will apply Lemma 1.1 again. It implies that there exists a path homotopy that connects e_1 with e_2 along the curves $e_{\tau_{12}}, 1 \leq \tau_{12} \leq 2$ of length $\leq 3d$. This is due to the fact, that the loop $e_2 * -e_1$ is contractible to p without the length increase, (see Fig. 4c), since all the loops based at p have length greater than $2d$.
3. As e_1 moves to e_2, we construct a family of 2-dimensional spheres $S^2(\tau_{12}, \tau_{43})$ that continuously depends on τ_{12} and that coincides with $S^2_{\tau_{43}}$, when $\tau_{12} = 1$. That is, we consider a 1-skeleton consisting of $e_{\tau_{12}}, e_2, e_{\tau_{43}}$ and we obtain a 2-sphere by gluing the three discs that result by length-decreasing homotopies contracting the three closed curves obtained from this 1-skeleton to a point. This is analogous to the first step, but applied to $e_{\tau_{12}}, e_2, e_{\tau_{43}}$ instead of $e_1, e_2, e_{\tau_{43}}$. Note that when $\tau_{12} = 2$, we obtain a sphere, consisting essentially of a 2-disc taken twice with the opposite orientation, that can be contracted to a point. This family of spheres corresponds to a 3-disc $(D^3_3)_{\tau_{43}}$. Note that at $\tau_{43} = 1$ it is $\sigma^3_{i_1,i_2,i_4}$ and at $\tau_{43} = 2$ it is $-\sigma^3_{i_1,i_2,i_3}$.
4. The other two discs $(D^3_2)_{\tau_{43}}$ and $(D^3_1)_{\tau_{43}}$ are similarly constructed. $(D^3_2)_{\tau_{43}}$ is constructed by "filling" the 1-skeleton that consists of $e_1, e_3, e_{\tau_{43}}$ and $(D^3_1)_{\tau_{43}}$ is constructed by "filling" the 1-skeleton that consists of $e_2, e_3, e_{\tau_{43}}$. Note that at the time $\tau_{43} = 2$, both $(D^3_2)_2$ and $(D^3_1)_2$ degenerate into two discs of dimension 2 that are geometrically the same, but have opposite orientation that are contracted to a point along themselves.
5. We finally, glue the four 3-discs to obtain a sphere $S^3_{\tau_{43}}$. Note that S^3_1 is the original sphere and S^3_2 is a sphere that is obtained by gluing $\sigma^3_{i_1,i_2,i_3}$ and $-\sigma^3_{i_1,i_2,i_3}$, and so it is contractible to a point. We will map $\tilde{\sigma}^4_{i_0,\ldots,i_4}$ to the disc $\sigma^4_{i_0,\ldots,i_4}$ generated by this family of 3-spheres. We have, thus, extended the map to the 4-skeleton, reaching a contradiction. □

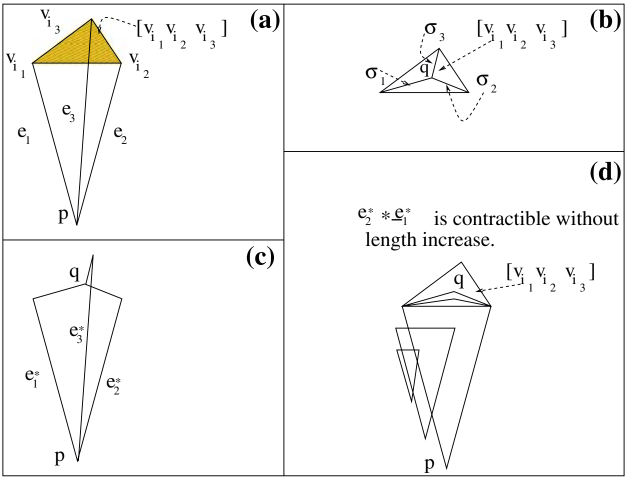
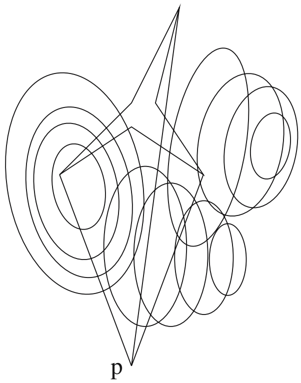


Fig. 5 Small 2-simplex can be ignored

Fig. 6 Spheres S_t



Remark. Here we explain why we can treat small simplices coming from the fine triangulation of S^k as points, (in our case $k = 2$ or 3).

The explanation will be provided only in the two-dimensional case. It is, however, similarly true for $k = 3$. Let us look at a 2-sphere that is constructed from a small singular 2-simplex $[v_{i_1}, v_{i_2}, v_{i_3}]$ in M^n and a point $p \in M^n$ as follows: Connect the point

p with $v_{i_j}, j = 1,2,3$ by some minimal geodesic segments $e_j, j = 1,2,3$. Next contract each of the three closed curves $e_j + [v_{i_j}, v_{i_{j \bmod 3+1}}] - e_{j \bmod 3+1}$, where $j = 1,2,3$ without the length increase, (see Fig. 5c).

In order to construct a new sphere S_1, take a point $q \in [v_{i_1}, v_{i_2}, v_{i_3}]$ and connect it with each vertex v_{i_j} by a small segment σ_j in $[v_{i_1}, v_{i_2}, v_{i_3}], j = 1,2,3$. Define $e_j^* = e_j * \sigma_j, j = 1,2,3$.

The closed curves of the form $e^*_{j \bmod 3+1} * -e_j^*$ are contractible to a point without length increase, thus generating the three discs. Gluing them as in the boundary of a 3-simplex results in S_1, (see Fig. 5d).

Also Lemma 1.1 can be applied to the digons $e^*_{j \bmod 3+1} * -e_j^*, j = 1,2,3$ that are contractible as loops to p without length increase and thus, e_j^* is path homotopic to $e^*_{j \bmod 3+1}$ and the length of curves in this path homotopy is bounded by $3d + 3\delta$.

One can easily show that S_0 and S_1 are homotopic, when δ is small enough. The intermediate spheres S_t are depicted on Fig. 6. Therefore, if S_1 is not contractible S_0 is not contractible as well.

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