Short geodesic loops on complete Riemannian manifolds with finite volume.

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Abstract

In this paper we will show that on any complete noncompact Riemannian manifold with a finite volume there exist geodesic loops of arbitrarily small length.

Introduction.

In the paper we will prove the following theorem:

Theorem 0.1 *Let M^n be a complete noncompact Riemannian manifold of a finite volume. Then, given $\varepsilon > 0$, there exists a geodesic loop on M^n of length $\leq \varepsilon$.*

This theorem provides an answer to one of many questions about relationship between the volume of a complete non compact Riemannian manifold and lengths of various stationary objects.

Previously, questions of a similar nature were investigated by C. B. Croke, who has established a volume upper bound for the length of a shortest periodic geodesic on a surface of finite volume, (see [C]) and by S. Sabourau, who has indicated how to bound the length of a shortest geodesic loop on a complete Riemannian manifold by its volume, (see [S2]). Note that it is not known whether on any complete Riemannian manifold of finite volume of dimension greater than two there exists a periodic geodesic, (though it was shown by V. Bangert and G. Thorbergsson that there exist infinitely many geodesics on a complete surface of a finite volume, (see [B], [T])).

On the other hand, when one assumes that a Riemannian manifold M^n is closed, there are numerous results that connect the size, (i. e. the length or the area) of various stationary objects: geodesic loops, minimal geodesic cycles and nets, minimal surfaces or submanifolds to the size of a manifold as measured either by its volume or the diameter, (see, for instance, [Bl], [NR2], [NR3], [NR4], [R2], [R3], [R4], [S2]). Presently there are no general curvature-free upper bounds of this nature for the length of a shortest periodic geodesic on a simply connected manifold, except in dimension two, (see [C], [M], [NR1], [R1], [S1]), though many results for manifolds with non-trivial fundamental group are known, (see [BrZ], [CK] for surveys of these results). The most notable is the result of M. Gromov, which gives a volume estimate for the length of a shortest periodic geodesic on closed Riemannian manifolds that are essential, (see [G]).

Our proof will make use of the following definition and result by M. Gromov, (see [G] as well as recent papers by S. Wenger ([W]) and by L. Guth, ([Gt1]) for simplifications of the original proof. The paper of Wenger contains some improvements of the original result.) We will also use some ideas from Gromov's paper [G] and our approach of constructing "fillings" of cycles in the absence of short geodesic loops used in [R3].

Definition 0.2 *Filling Radius. Let M^n be an n-dimensional Riemannian manifold in an arbitrary metric space X. Then the filling radius $FillRad(M^n \subset X)$ is the infimum of $\varepsilon > 0$, such that M^n bounds in the ε-neighborhood $N_\varepsilon(M^n)$, that is $i_*(H_n(Z_n)) = \{0\}$, where i_* is induced by the inclusion $i : M^n \longrightarrow N_\varepsilon(M^n)$ and, where $H_n(M^n)$ is taken with coefficients in $\mathbf{Z}$, when M^n is orientable, and with coefficients in $\mathbf{Z}_2$, when M^n is nonorientable. The filling radius of an abstract Riemannian manifold is then defined to be $FillRad(M^n \subset L^\infty(M^n))$, where the embedding of M^n into $L^\infty(M^n)$ is a map that to each point p of M^n assigns a distance function $p \longrightarrow f_p = d(p,q)$, (see [G]). Equivalently, $FillRadM^n$ can be defined as the infimum of $FillRad(M^n \subset X)$ over all metric spaces X and isometric embeddings of M^n into X.*

Theorem 0.3 *(***[G]***) Let M^n be an n-dimensional manifold. Then $FillRadM^n \leq k(n)vol(M^n)^{\frac{1}{n}}$, where $k(n)$ is an explicit function of the dimension of a manifold.*

Note that L. Guth has recently improved the above result by showing that a complete Riemannian manifold with the filling radius R contains a ball of radius R of volume bounded from below by $c(n)R^n$, (see [Gt2]).

1 Three simple lemmas.

We will begin the proof of the main result with the following three lemmas:

Lemma 1.1 *Let M^n be a complete noncompact Riemannian manifold of a finite volume. Let $\sigma(t)$ be a geodesic ray, starting at a point p. Then given $\varepsilon, T > 0$ there exists a connected $(n-1)$-dimensional submanifold Z_ε, such that $vol_{n-1}(Z_\varepsilon) < \varepsilon$, Z_ε does not bound in $M^n \backslash \{p\}$ and the distance between the point p and Z_ε is greater than T.*

Proof. Let $\varrho_\delta : M^n \longrightarrow \mathbf{R}$ be a function that is smooth on $M^n \backslash \{p\}$ and that approximates a distance function ϱ_p from the point p in the following way: (1) $\varrho_\delta = \varrho$ on a geodesic ball centered at p of radius smaller than the injectivity radius of M^n at p; (2) $|\varrho_p - \varrho_\delta| \leq \delta$ and (3) $|\text{grad}\varrho_\delta| \leq 1 + \delta$. The details of constructing such a function can be found in M. P. Gaffney's work [Ga].

Let us now consider the sublevel sets of ϱ_δ. For some small values of $t \in \mathbf{R}$, they will be geodesic spheres, because ϱ_δ agrees with the distance function in some neighborhood of p. Let $S_r(p)$ be a geodesic sphere centered at p with radius r smaller than the injectivity radius at the point p. Then $S_r(p)$ is homeomorphic to S^{n-1}.

By the virtue of Mayer-Vietoris exact sequence it follows that $S_r(p)$ does not bound in $M^n \backslash \{p\}$. Otherwise, $H_n(M^n) \neq \{0\}$, which would contradict the assumption that M^n is not compact.

Let us denote a 1-parameter family of sublevel sets of ϱ_δ as $S_t(p), t \in (0, \infty)$. They are homologous for all t. Thus, for all t, $S_t(p)$ does not bound in $M^n \backslash \{p\}$, which means that for all t there is a connected component of $S_t(p)$ that does not bound in $M^n \backslash \{p\}$. Furthermore, $dist(p, S_t(p)) \geq t - \delta$. Now the coarea formula implies that $\int_0^\infty vol_{n-1} S_t(p) dt \leq (1+\delta)V$, where V is the volume of M^n. From above equality it follows that for every $\varepsilon > 0$, and $T > 0$, there exists $t > T + \delta$ such that $vol_{n-1}(S_t(p)) < \varepsilon$. Moreover, such $t's$ form a set of infinite measure. Since, by Sard's theorem, $S_t(p)$ is an $(n-1)$-dimensional submanifold of M^n for almost all t, we can choose such a value of t so that $S_t(p)$ is a smooth manifold, which proves the lemma. □

The following two lemmas were used in [R3]. We will present them here for the sake of completeness.

The first is a Morse-theoretic type lemma asserting that the space of loops based at a fixed point q of length smaller than the length of a minimal geodesic loop at q is contractible.

Lemma 1.2 *Let M^n be a complete Riemannian manifold. Let $q \in M^n$. Suppose that the length of a shortest geodesic loop $l_q(M^n)$ based at q is greater than L. Then given any piecewise differentiable loop $\gamma : [0,1] \longrightarrow M^n$ of length $\leq L$ such that $\gamma(0) = \gamma(1) = q$ there exists a length decreasing path homotopy connecting this curve with q that depends continuously on initial loop γ.*

Proof. There is a standard explicit length shortening deformation of the space of loops based at q of length $\leq L$ to the constant loop via the Birkhoff Curve Shortening Process, (see [C] for the detailed description of the Birkhoff Curve Shortening Process (BCSP) for closed curves. The only difference between the BCSP for closed curves and the BCSP for loops is that one fixes a base point during the homotopies in the latter case.)

□

The third lemma can be viewed as an effective version of an elementary assertion that two curves γ_1, γ_2 connecting points q_1, q_2 are path homotopic if and only if the loop $\gamma_2 * -\gamma_1$ is path homotopic to a point. Lemma 1.3 is analogous to a similar statement in [C], namely, Lemma 3.1.

Lemma 1.3 *Let γ_1, γ_2 be two curves with $\gamma_1(0) = \gamma_2(0) = q_1$ and $\gamma_1(1) = \gamma_2(1) = q_2$ on a complete Riemannian manifold M^n of length l_1, l_2 respectively.*

Let $\gamma_2 * -\gamma_1$ be the product of γ_2 and $-\gamma_1$ based at q_1. If this curve is contractible to q_1 as a loop along the curves of length $\leq l_1 + l_2$ then there is a path homotopy, (i.e. a homotopy that fixes the end points), $h_\tau(t), \tau \in [0,1]$, such that $h_0(t) = \gamma_1(t), h_1(t) = \gamma_2(t)$ and the length of curves during this homotopy is bounded above by $2l_1 + l_2$. Alternatively there exists a path homotopy with the same properties, such that the length of curves in it is bounded by $l_1 + 2l_2$. Moreover, when M^n has no geodesic loops of length $\leq l_1 + l_2$, this path homotopy can be made to continuously depend on a digon formed by γ_1 and γ_2, see (1.2).

Proof. Let $\tilde{h}_\tau(t)$ be a homotopy that connects $\gamma_2 * -\gamma_1$ with a point p, (see fig. 1 (a) and (b)). Then let us consider the following homotopy $\gamma_1 \sim \tilde{h}_{1-\tau} * \gamma_1 \sim \gamma_2 * -\gamma_1 * \gamma_1 \sim \gamma_2$, (see fig. 1 (a)-(g)). The length of curves during this homotopy is $\leq 2l_1 + l_2$.

Note that, assuming there are no geodesic loops of length $\leq l_1 + l_2$, one can contract $\gamma_2 * -\gamma_1$ via the BCSP, which continuously depends on the initial curve, (see Lemma 1.2). Thus, the path homotopy between $\gamma_1(t)$ and $\gamma_2(t)$ will also continuously depend on the initial digon.

Also, one can reverse the role of γ_1 and γ_2 and construct a path homotopy between γ_2 and γ_1 passing through curves of length $l_1 + 2l_2$. Then we reverse the direction of this path homotopy obtaining a path homotopy from l_1 to l_2 with the required properties.

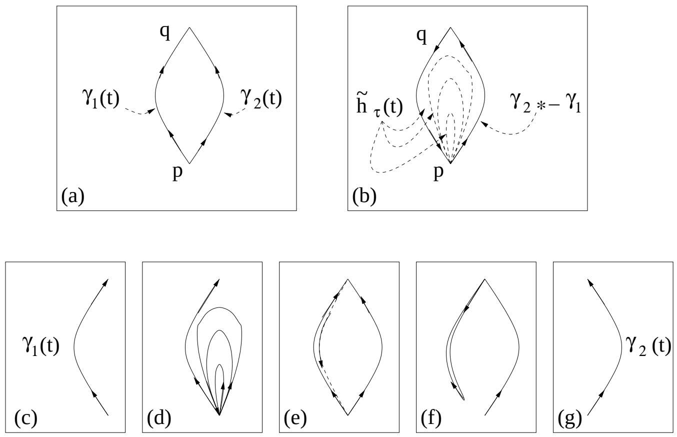

Figure 1: Illustration of the proof of Lemma 1.3.

□

2 Proof of Theorem 0.1.

In the following definition, (Definition 3.1 in [R3]), we let σ^{m+1} denote the standard $(m+1)$-dimensional simplex, and $C(X, Y)$ denote the space of continuous maps from X into Y.

Definition 2.1 *Given $l > 0$ and a positive integer m, let $K_{m,l}$ be a space of piecewise smooth maps of the 1-skeleton of the complete graph with $(m+2)$ vertices into M^n, such that each edge is mapped into a curve of length $\leq l$. We define an N-filling of $K_{*,l}$ as a a collection of continuous maps $\phi_m : K_{m,l} \longrightarrow C(\sigma^{m+1}, M^n)$,$m = 1, 2, \ldots, N$, satisfying the following properties:*
(1) For every $k \in K_{m,l}$ the restriction of $\phi_m(k)$ to the 1-skeleton of σ^{m+1} coincides with k, that is, each $\phi_m(k)$ extends k.
(2) For every $k \in K_{m,l}$, $(m \leq N)$, the restriction of $\phi_m(k)$ to an m-dimensional face of σ^{m+1} coincides with ϕ_{m-1} evaluated on the element

of $K_{m-1,l}$ obtained from k by restricting k to the set of all 1-dimensional simplices in the 1-skeleton of this face of σ^m.

Here is an informal description of the above definition: we are "filling" graphs with "short" edges, (i.e. of length $\leq l$) that correspond to the immersed 1-skeleton of a simplex of dimension $m+1$ by discs of dimension $m+1$, so that the map of the disc extends the map from 1-skeleton. Moreover, this extension is done in a coherent way, that is, if we consider the restriction of this map to a face of the simplex, it will be a "filling" of the 1-skeleton of the face, in particular, that means that each N-filling agrees with its subfillings and depends continuously on its 1-skeleton.

Lemma 2.2 *Suppose that the length of a shortest geodesic loop on M^n is greater than $3 \cdot 4^{n-1}l$. Then there exists an n-filling of $K_{*,l}$. Moreover, if $k \in K_{m,l}$, ($m \leq n$), the disc that fills k will lie in $6 \cdot 4^{n-2}l$-neighborhood of vertices of k, that is, the maximal distance between points of the disc and the set of vertices of k is at most $6 \cdot 4^{n-2}l$.*

Proof. We will prove the existence of the i-fillings of $K_{*,l}$ for every $i \leq n$. The proof will be by induction with respect to i. The base step corresponds to $i = 1$. Let $k_1 \in K_{1,l}$. By Definition 2.1 it is an immersion of a full graph that consists of three vertices and three edges. Let v_0, v_1, v_2 be the vertices of this immersed graph. The three edges form a loop based at v_0. Since we have assumed that there are no "short" geodesic loops, (and, in particular, no geodesic loops of length $\leq 3l$), this loop is contractible to v_0 via shorter loops based at v_0. This homotopy generates a disc that "fills" k^1.

At each subsequent step, to construct ϕ_j we consider its restriction to $\partial\sigma^{j+1}$. This restriction is uniquely determined by the definition of N-fillings and, if $i > 1$, by the previous steps of the induction. That is, the previous step of the induction results in a filling of elements of $K_{j-1,l}$ obtained from elements of $K_{j,l}$ by deleting a vertex. Consider $k \in K_{j,l}$. Then the fillings of

$j+2$ elements of $K_{j-1,l}$ that are obtained from k by deleting a vertex are discs of dimension j provided by the previous step of the construction. They together form a j-dimensional sphere, which, according to our definition, is a restriction of ϕ_j to $\partial\sigma^{j+1}$. The required disc is then generated by a homotopy that contracts this sphere to a point. To construct this homotopy we begin by constructing a 1-parameter family k_t of immersed graphs connecting $k = k_0$ with a complete graph k_1 with $(j+2)$ vertices immersed in M^n such that all of its edges are mapped to some paths in a tree. This path k_t

should continuously depend on the initial graph k. Next, we construct a 1-parameter family of spheres S^j_t by filling all k_ts. This result in a homotopy between the sphere $\phi_j(\partial\sigma^{j+1})$ and the degenerate sphere S^j_1 that lives in a tree and is, therefore, contractible, (to contract this degenerate sphere we contract k_1 over itself and fill it by the n-sphere at each moment of the homotopy).

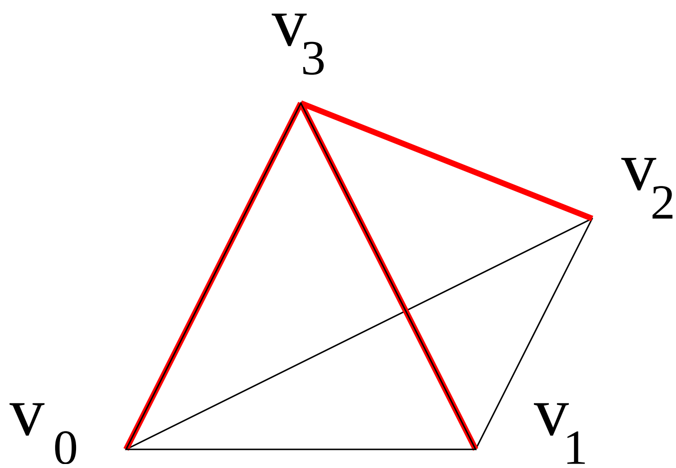

Figure 2: Collapsing triangles

k_t is constructed by several applications of an operation of a *collapsing of a triangle*: Let k_a, k_b, k_c be any of the three edges of k. As there are no geodesic loops of length $\leq$ length k_a+ length k_b+ length k_c in M^n, we can apply Lemma 1.3 to construct a path homotopy between k_a and $k_b * k_c$. This homotopy passes through paths of length $\leq 2length(k_a) + length(k_b) + length(k_c) \leq 4l$. This homotopy induces a homotopy of triangles $(k_a)_t, k_b, k_c, t \in [0,1]$ that we call a collapsing of the triangle k_a, k_b, k_c. At the end of this homotopy k_a is being replaced by another edge that goes along $k_b * k_c$, and the considered triangle becomes thin.

After collapsing finitlely many triangles, we can obtain an element of $K_{j,4l}$, where all edges run along the tree-shaped union k_1 of edges of k adjacent to one vertex of k, let's say the vertex with the highest number, (see fig. 2 , which illustrates that the edge $[v_0, v_1]$ is being collapsed to the path $[v_0, v_3, v_1]$, the edge $[v_1, v_2]$ is being collapsed to $[v_1, v_3, v_2]$, and the edge $[v_0, v_2]$ is being collapsed to $[v_0, v_3, v_2]$).

Now we can continue the homotopy of complete graphs by contracting all edges of k_1 to a point, (to v_3 on fig. 2) along the tree by a length non-increasing homotopy.

The resulting graphs are filled by j-spheres using the induction assumption on $K_{m,4l}$, since the length of edges that result in the process of collapsing of triangles is bounded above by $4l$.

Let $k \in K_{m,l}$. Then k is a (map of) the complete graph with $m+2$ vertices $v_0, v_1, ..., v_{m+1}$. Let k_{t_1} denote a one parameter family of graphs obtained from k by collapsing triangles. We define $k_{t_1}^{i_1}$ to be a subgraph of k_{t_1} obtained from it by removing a vertex v_{i_1}. In general, let $k_{t_1,...,t_j}^{i_1,...,i_{j-1}}$ be a family of complete graphs with $m+3-j$ vertices obtained from $k_{t_1,...,t_{j-1}}^{i_1,...,i_{j-1}}$ by collapsing triangles and let $k_{t_1,...,j_k}^{i_1,...,i_j}$ be complete graph with $m+2-k$ vertices obtained from $k_{t_1,...,t_j}^{i_1,...,j-1}$ by removing a vertex v_j. Let $a(j)$ be the maximal possible length of an edge of $k_{t_1,...,t_j}^{i_1,...,i_j}$. Note that $a(1) \leq 4l$ and that $a(j+1) \leq 4a(j)$. Thus, $a(m-1) \leq 4^{m-1}l$. So, the length of loops that one contracts in the recursive process described above is at most $3 \times 4^{n-1}l$.

Note also, that as all the homotopies are constructed by contracting loops to one of the vertices of k, the maximal distance from the points of the resulting disc to one of the vertices is at most half the maximal length of such loops. □

Here is the informal description of the above proof when $m = 2$. We would like to show that in the case when the length of a shortest geodesic loop is $> 12l$ we can fill $K_{2,l}$. Let us recall that $K_{2,l}$ is the space of immersed 1-skeleta of simplices of dimension 3, such that the length of each edge does not exceed l. We would like to extend each of the immersions to a 3-simplex, so that these extensions are continuous with respect to the original graph, and so that they are coherent. The last requirement means that if we consider a restriction of the immersion to a subcomplex, which is a 1-skeleton of a 2-face, it will agree with the earlier extension. Thus, the procedure is inductive and we will begin by filling $K_{1,4l}$. In this case, if $k \in K_{1,4l}$ then its total length is at most $12l$. However, since the length of a shortest geodesic loop is greater than $12l$ each such curve is contractible via the BCSP as a loop to any of the vertices of k. Let us, however, choose to contract to vertices with the biggest index. Here we use Lemma 1.3 to construct the required path homotopy between one side of k and its two other sides. Next let us consider $k^2_{v_0,v_1,v_2,v_3} \in K_{2,l}$. Note that, as we know how to extend each $k^1_{v_0,,,,\hat{v}_i,...,v_3}$ we, as the result of these extensions and a natural identifications of the four 2-discs have a map of the 2-sphere into M^n naturally assigned to $k^2_{v_0,v_1,v_2,v_3}$. Let us denote this (map of the) 2-sphere by S_0^2. We would like to construct a map of a 3-disc that fills 2-sphere. It will be constructed as a 1-parameter family of 2-spheres S_τ^2 that begins with the original sphere obtained in the previous step S_0^2 and ends with a point. Here is how we will construct S_τ^2. Let us begin by constructing a 1-parameter family of graphs $k_\tau^2, \tau \in [0,2]$. We will let $k_0^2 = k^2_{v_0,v_1,v_2,v_3}$. Next, by Lemma 1.3 there is a homotopy that moves edges $[v_i, v_{(i+1) \bmod 3}], 0 \leq i \leq 2$ to $[v_i, v_3] + [v_3, v_{(i+1) \bmod 3}]$. This path homotopy passes through curves of length $\leq 4l$. Let us denote the curves in these homotopies by e_τ^i. So, we will continuously replace edges $e^i = [v_i, v_{(i+1) \bmod 3}]$ by the edges e_τ^i respectively, thus forming k_τ^2. Let us now consider all the subcomplexes of k_τ^2 that correspond to elements of $K_{1,4l}$. By the previous step they can all be "filled" by 2-discs. Gluing these discs together results in a 2-sphere S_τ^2. When $\tau = 1$ this sphere will degenerate to (a map of the 2-sphere into) a tree with root at v_3 and three edges connecting v_3 with v_0, v_1, v_2. This sphere fills a degenerate element of $K_{2,2l}$ where all edges are mapped into this tree. This element can be contracted over itself to the constant map of the complete graph into v_3. Filling the resulting homotopy by spheres we obtain a family of 2-spheres $S_\tau^2, \tau \in [1,2]$ that connects S_1^2 and $S_2^2 = \{v_3\}$. Thus, we obtain a 3-disc that "fills" any $k^2_{v_0,\ldots,v_3} \in K_{2,l}$.

Proof of Theorem 0.1. Let $\varepsilon > 0$ be given. By Lemma 1.1 there exists a connected $(n-1)$-dimensional manifold $Z_\varepsilon \subset M^n$, such that $vol_{n-1}(Z_\varepsilon) \leq \varepsilon$. Moreover Z_ε does not bound in $M^n - p$, where p is some point of M^n that is located at the distance greater than $T = 4^n k(n-1)\varepsilon^{\frac{1}{n-1}}$ from Z_ε. Here $k(n-1)$ is a constant of Theorem 0.3. Let $X = L^\infty(Z_\varepsilon)$. By Definition 0.2, Z_ε isometrically embedds into X and for every $\delta > 0$ there exists a singular chain c in the $(FillRad(Z_\varepsilon) + \delta)$-neighborhood of Z_ε in X, such that Z_ε bounds c. Without loss of generality we can take c to be an n-dimensional polyhedron, (see Statement 1.2.C on page 10 in [G].) Also, recall that the Kuratowski embedding of Z_ε in X is an isometry, (see Def. 0.2).

Assume that lengths of all nontrivial geodesic loops in M^n are greater than

$$\tilde{\varepsilon} = 6 \cdot 4^{n-1} k(n-1)\varepsilon^{\frac{1}{n-1}} > 6 \cdot 4^{n-1} k(n-1) vol_{n-1}(Z_\varepsilon)^{\frac{1}{n-1}} \ (*).$$

Furthermore, Gromov's Theorem 0.3 implies that $\tilde{\varepsilon} > 6 \cdot 4^{n-1}\ FillRad(Z_\varepsilon)$

We will extend the inclusion map of $Z_\varepsilon = \partial c$ into $M^n - p$ to a map $\tau : c \longrightarrow M^n - p$, which would imply Z_ε bounds in $M^n - p$, thus reaching a contradiction.

Let us begin with a triangulation of c into simplices of diameter at most $\delta > 0$, that will eventually approach zero. The extension will be done by induction on the skeleta of c. Each of the 0-simplices of c, (excluding those in Z_ε), will be mapped to one of the closest points in Z_ε (in the metric of the ambient space X). Each of the 1-simplices of $c \setminus Z_\varepsilon$ will be mapped into a minimizing geodesic in M^n between the images v_{i_1}, v_{i_2} of the vertices $\tilde{v}_{i_1}, \tilde{v}_{i_2}$ respectively of this simplex. Then $dist(v_{i_1}, v_{i_2}) \leq dist(v_{i_1}, \tilde{v}_{i_1}) + dist(\tilde{v}_{i_1}, v_{i_2}) \leq dist(v_{i_1}, \tilde{v}_{i_1}) + dist(\tilde{v}_{i_1}, \tilde{v}_{i_2}) + dist(\tilde{v}_{i_2}, v_{i_2}) \leq 2FillRad + \tilde{\tilde{\delta}}$, where $\tilde{\tilde{\delta}} = 3\delta$. Thus, the length of the image of each 1-simplex of $c \setminus Z_\varepsilon$ is at most $2FillRad(Z_\varepsilon) + \tilde{\delta}$, where $\tilde{\delta}$ can be made arbitrarily small by selecting a sufficiently small δ and by refining the chosen triangulation of c.

The desired extension of τ to any closed n-dimensional simplex of $c \setminus Z_\varepsilon$ is accomplished by filling the image of its 1-skeleton described in Lemma 2.2. One, however, has to take care to fill every k-simplex identically, when it is considered as a k-face of different n-simpleces.

Let us begin by enumerating all the vertices of the chosen triangulation of c by increasing successive integers. We will apply Lemma 2.2 to previously constructed images of 1-skeleta of all n-dimensional simplices of c. In order to do that we need to number vertices of every n-dimensional simplex of c by numbers $0, 1, ..., n$. To do this we take the numbering of all of the vertices of c and then renumbering the vertices of every n-simplex by $\{0, 1, ..., n\}$ in the increasing order. Next apply Lemma 2.2 using $(*)$ and taking $\tilde{\tilde{\delta}}$ to be sufficiently small. As the result, we obtain an extension to the n-skeleton of c. Note that the resulting image does not pass through the point p, because the distance between p and the image is sufficiently large. Thus, we have reached a contradiction refuting the assumption in $(*)$. Finally, note that as ε becomes arbitrarily small so does $\tilde{\varepsilon}$, and so there must be a geodesic loop of arbitrarily small length. $\square$

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