

Quiz 1

MAT475H1F Fall 2026

Name: _____

Student number: _____

Grade: _____ / 5

Question.

[5 marks]

You start with a pile of n coins. At each step, choose a pile containing at least two coins and split it into two nonempty piles. If the new piles contain a and b coins, you earn ab points. Continue until every pile contains exactly one coin.

For example, starting with four coins, you could split them into piles of two and two, earning four points. Splitting each of those piles earns one more point, for a total of six.

Prove by strong induction that, regardless of how you split the piles, your total score is always

$$\frac{n(n-1)}{2}.$$

Hint: After the first split, think of the two resulting piles as two smaller games.

Solution. We use strong induction on n . When $n = 1$, there are no splits, so the score is $0 = 1(1-1)/2$.

Now let $n \geq 2$ and assume that, for every $k < n$, any complete sequence of splits starting from k coins earns $k(k-1)/2$ points.

Consider any sequence of splits starting from n coins. Suppose its first split produces piles of sizes a and b , where $a, b \geq 1$ and $a + b = n$. This split earns ab points. Since $a, b < n$, the induction hypothesis applies to both smaller piles. Their subsequent splits earn $a(a-1)/2$ and $b(b-1)/2$ points, regardless of their order or how they are performed. Thus the total score is

$$\begin{aligned} ab + \frac{a(a-1)}{2} + \frac{b(b-1)}{2} &= \frac{2ab + a^2 - a + b^2 - b}{2} \\ &= \frac{a^2 + 2ab + b^2 - (a+b)}{2} \\ &= \frac{(a+b)^2 - (a+b)}{2} \\ &= \frac{n^2 - n}{2} \\ &= \frac{n(n-1)}{2}. \end{aligned}$$

This proves the claim for every $n \geq 1$ by strong induction. □