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Isomorphisms Between Big Mapping Class Groups

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We show that any isomorphism between mapping class groups of orientable infinite-type surfaces is induced by a homeomorphism between the surfaces. Our argument additionally applies to automorphisms between finite-index subgroups of these "big" mapping class groups and shows that each finite-index subgroup has finite outer automorphism group. As a key ingredient, we prove that all simplicial automorphisms between curve complexes of infinite-type orientable surfaces are induced by homeomorphisms.

1 Introduction

All *surfaces* in this paper will be connected, orientable, and without boundary. A surface S is said to be of *finite-type* if its fundamental group is finitely generated; otherwise S has *infinite-type*. The *(extended) mapping class group* of S is the group $\mathrm{Map}(S)$ of isotopy classes of possibly orientation-reversing homeomorphisms of S. An *end* of S is a nested choice of connected components of $S \setminus K_i$ for some compact exhaustion $K_1 \subset K_2 \subset \cdots$ of S. More formally, the set of ends is the inverse limit $\mathrm{End}(S) = \varprojlim \pi_0(S \setminus K)$ over the directed (via inclusion) system of compact subsets K of S. The *pure mapping class group* is the subgroup $\mathrm{PMap}(S) \leq \mathrm{Map}(S)$ that fixes $\mathrm{End}(S)$ pointwise. We also have the index 2 subgroups $\mathrm{PMap}^+(S)$ and $\mathrm{Map}^+(S)$ consisting of orientation-preserving elements.

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In the case of finite-type surfaces, an old result of Ivanov [14] shows that the automorphism group of Map(S) is isomorphic to Map(S) itself; the closed case being independently obtained by McCarthy [19]. It is a related folk-theorem (implicit in [14] and following in most cases from [2] and [10]) that, aside from low-complexity exceptions, non-homeomorphic finite-type surfaces cannot have isomorphic mapping class groups; for a full discussion and proof see [22, Appendix A]. Thus, the group Map(S) determines the surface S when S has finite-type.

Here we focus on the so-called "big" mapping class groups, that is, groups Map(S) and PMap(S) for S of infinite-type. Unlike mapping class groups of finite-type surfaces, these big mapping class groups have uncountably many elements and inherit a non-discrete topology from the compact open topology on Homeo(S). Despite a recent growing interest in big mapping class groups (e.g., [4, 1, 5, 21, 13]), the above properties have remained open in this setting. Our main results establish them for all infinite-type surfaces.

Theorem 1.1. Let S_1 and S_2 be infinite-type surfaces. For $i = 1, 2$, let G_i be a finite-index subgroup of either Map(S_i) or PMap(S_i) and let $\Phi\colon G_1 \to G_2$ be any algebraic isomorphism. Then there is a homeomorphism $h\colon S_1 \to S_2$ so that $\Phi(f) = h \circ f \circ h^{-1}$. In particular, Φ is automatically continuous.

Thus, mapping class groups—and even their finite-index subgroups—distinguish infinite-type surfaces. This answers Question 1.1 and generalizes Theorem 1 in the recent paper of Patel and Vlamis [21], who treat the special case of PMap for infinite-type surfaces of finite genus at least 4.

The *abstract commensurator* of G is the group Comm(G) of all equivalence classes of isomorphisms $H_1 \to H_2$ between finite-index subgroups of G, where two such isomorphisms are equivalent if they agree on a finite-index subgroup. There are natural maps $G \to \mathrm{Aut}(G) \to \mathrm{Comm}(G)$ arising from the fact that every conjugation or automorphism of G is itself a commensuration. However, Comm(G) is in general much larger than Aut(G); for example, $\mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ whereas $\mathrm{Comm}(\mathbb{Z}) \cong \mathbb{Q}^*$ is not even finitely generated. We view Comm(G) as capturing the "hidden" symmetries of G; an assertion that Comm(G) is small thus conveys a strong *algebraic rigidity* that is reminiscent of superrigidity for lattices Γ in a semisimple Lie group $G \neq \mathrm{PSL}(2,\mathbb{R})$. Indeed, the work of Margulis, Mostow and Prasad (see [18, 25]) implies that $[\mathrm{Comm}(\Gamma) : \Gamma] < \infty$ when Γ is nonarithmetic and that Comm(Γ) virtually embeds into G when Γ is arithmetic. Theorem 1.1 implies this strong algebraic rigidity for Map(S), generalizing

Ivanov's result computing Comm(Map(*S*)) for finite-type surfaces [15], as well as the following consequences that, in particular, establish Conjecture 1.2 of [21].

Corollary 1.2. Let S be an infinite-type surface. Then

(i) The natural maps $\mathrm{Map}(S) \to \mathrm{Aut}(\mathrm{Map}(S)) \to \mathrm{Comm}(\mathrm{Map}(S))$ are isomorphisms.
(ii) $\mathrm{PMap}(S)$, $\mathrm{Map}^+(S)$, and $\mathrm{PMap}^+(S)$ are characteristic in $\mathrm{Map}(S)$.
(iii) $\mathrm{Out}(G)$ is finite for every finite-index subgroup of $\mathrm{Map}(S)$.
(iv) Finite-index subgroups of $\mathrm{Map}(S)$ or $\mathrm{PMap}(S)$ are isomorphic iff they are conjugate.

Our proof of Theorem 1.1 follows Ivanov's approach [15] and has two main ingredients. The first is an algebraic characterization of Dehn twists in terms of centralizers of elements (see Section 4). This is related to the characterization of "algebraic twist subgroups" used by Ivanov [14] and others and further relies on a new characterization (Proposition 4.2) of finitely-supported elements by the cardinality of their conjugacy classes.

The second ingredient comes from curve complexes. By a *curve* in S, we mean the equivalence class of an embedding $\mathbb{S}^1 \hookrightarrow S$ of the circle that is neither nullhomotopic nor homotopic into an end of S, where embeddings are equivalent if they are homotopic or differ by precomposition with an orientation-reversing homeomorphism of $\mathbb{S}^1$.

A *multicurve* is a finite set of distinct curves that admits representative embeddings with disjoint images. The *curve complex* of S is the simplicial complex $\mathcal{C}(S)$ whose simplices correspond to multicurves of S and face maps to inclusions of multicurves.

The curve complex of a surface was first introduced by Harvey [11] as a Teichmüller-theoretic analog of the Tits building for symmetric spaces. A powerful theorem of Ivanov [15], Korkmaz [16], and Luo [17] in the finite-type setting, analogous to a fundamental theorem of Tits [24], states that every simplicial automorphism of $\mathcal{C}(S)$ is induced by an element of Map(*S*). Ivanov originally used this to give a new proof of Royden's famous theorem that Map(*S*) is the isometry group of the Teichmüller space of S [23], and it is now known that many (indeed most) other complexes built from S have automorphism group equal to Map(*S*) (e.g., see [20] or [3] and the references therein). Our final theorem extends this result to infinite-type surfaces.

Theorem 1.3. Let S and S' be surfaces and suppose S has infinite type. Then any simplicial isomorphism $\mathcal{C}(S) \to \mathcal{C}(S')$ is induced by a homeomorphism $S \to S'$.

Theorem 1.3 was independently proven in a very recent paper [12] by Hernández, Morales, and Valdez. We give a proof based on finite-type exhaustions and a simple observation, already present in [15, Lemma 1], that a multicurve's link in $\mathcal{C}(S)$ is able to detect the components of its complement in S (Lemma 3.1).

2 Preliminaries

Let us briefly establish some terminology for dealing with an infinite-type surface S. A *domain* Y in S is a connected component of $S \setminus \alpha$ for some multicurve α; we then define ∂Y to be the smallest sub-multicurve β of α so that Y is a component of $S \setminus \beta$. Note that domains are only defined up to isotopy and that each domain Y is itself a surface. A curve in S is *essential* in Y if its equivalence class contains an embedding that defines a curve in Y. A curve and a domain are *disjoint* if they have disjoint representatives; thus the curves of ∂Y are disjoint from Y.

Definition 2.1. A domain Y of S is said to be *principal* if Y has finite type with $\chi(Y) \leq -3$ and if every component X of $S \setminus \partial Y$ with $X \neq Y$ has infinite type.

Notice that Map(S) respectively acts on the sets of curves, multicurves, and domains of S. We make frequent implicit use of the following result of Hernández, Morales, and Valdez extending the well-known Alexander method (see [7, Section 2.3]) to the infinite-type setting:

Theorem 2.2. (Hernández–Morales–Valdez [13]) Let S be an infinite-type surface. If $f \in$ Map(S) fixes each curve of S, then f is trivial in Map(S).

Accordingly, we say that $f \in$ Map(S) has *finite support* if there is a finite-type domain Y of S such that f fixes every curve disjoint from Y.

Lemma 2.3. If $f \in$ Map(S) has finite support, then f is orientation-preserving.

Proof. By definition, there is an infinite-type domain Y such that $f(Y) = Y$ and f fixes each curve in Y. By Theorem 2.2, $f|_Y$ is isotopic to the identity. Thus, f evidently preserves the orientation on Y and, consequently, all of S. ■

Following Handel and Thurston [9, Section 2], for X any surface and $f \in$ Map(X) we write $\mathcal{O}(f)$ for the set of curves α of X such that $\{f^k(\alpha) \mid k \in \mathbb{Z}\}$ is finite and write ∂f

for the set of curves in $\mathcal{O}(f)$ that are disjoint from all other elements of $\mathcal{O}(f)$. It is clear that ∂f is a canonical set of disjoint curves in X for which $f(\partial f) = \partial f$.

Definition 2.4. Say that $f \in \mathrm{Map}(S)$ is *multi-annular* if

- f has finite support,
- f fixes each component of ∂f, and
- f fixes every curve disjoint from ∂f.

If ∂f is a single curve, we further say that f is *annular*.

Each curve α of S determines an associated pair $D_\alpha, D_\alpha^{-1} \in \mathrm{PMap}(S)$ of *Dehn twists* about α defined as follows: cut S on α to obtain a 2-manifold with two boundary components, rotate one component a full revolution to the left (for D_α) or right (for D_α^{-1}), and reglue; for details see [7, Chapter 3]. The Dehn twists D_α and D_α^{-1} are distinguished from each other by the choice of an orientation on S; thus, in writing D_α we have implicitly specified an orientation. As the distinction is not pertinent for us, we often (e.g., in Corollary 4.8) consider the pair $\{D_\alpha, D_\alpha^{-1}\}$, which is well-defined irrespective of orientation. We call α a *pants curve* if one component of $S \setminus \alpha$ is a thrice-punctured sphere. In this case there are also *half-twists* $H_\alpha, H_\alpha^{-1} \in \mathrm{Map}(S)$ satisfying $H_\alpha^{\pm 2} = D_\alpha^{\pm}$ and defined by fixing α and swapping the other two punctures in the thrice-punctured sphere component of $S \setminus \alpha$; see [7, §9.1.3]. Note that $H_\alpha^{\pm} \notin \mathrm{PMap}(S)$. To streamline notation, for each curve α of S we define the associated *twists* about α to be

$$T_\alpha^{\pm} = \begin{cases} H_\alpha^{\pm}, & \text{if } \alpha \text{ is a pants curve} \\ D_\alpha^{\pm}, & \text{otherwise.} \end{cases}$$

For a multicurve β with components $\beta_1, \ldots, \beta_k$, we similarly define the associated *twists* $T_\beta^{\pm} = \prod_{i=1}^{k} T_{\beta_i}^{\pm}$ about β. We note the following trivialities:

Lemma 2.5. Let α, β be multicurves on a surface S. Then

(1) T_α is multi-annular with $\partial(T_\alpha) = \alpha$.
(2) T_α and T_β commute iff α and β are disjoint.
(3) If $n, m \in \mathbb{Z} \setminus \{0\}$ are such that $T_\alpha^n = T_\beta^m$, then $\alpha = \beta$.
(4) If $\sigma(f) \in \{1, -1\}$ records whether $f \in \mathrm{Map}(S)$ preserves orientation, then

$$f \circ T_\alpha \circ f^{-1} = T_{f(\alpha)}^{\sigma(f)}.$$

The following fact will play a crucial role in our proof of Lemma 4.5.

Lemma 2.6. If $f \in \mathrm{Map}(S)$ is nontrivial and has finite support, then ∂f is a nonempty multicurve in S.

As ∂f is clearly empty when f has finite-order, it will help to first establish

Lemma 2.7. If $f \in \mathrm{Map}(S)$ is nontrivial and has finite support, then f has infinite-order. Furthermore, if Y is a principal domain in S so that f fixes every curve disjoint from Y and no power of f is a nontrivial product of Dehn twists about curves of ∂Y, then the restriction $g = f|_Y$ is an infinite-order element of $\mathrm{Map}(Y)$.

Remark 2.8. A domain Y as in Lemma 2.7 may always be obtained by enlarging a finite-type domain for which f fixes every curve disjoint from it. Further, the restriction $f|_Y$ is well defined in $\mathrm{Map}(Y)$: indeed, f induces an automorphism of $\mathcal{C}(Y)$ that, according to [17] (and using $\chi(Y) \le -3$), is equivalent to an element of $\mathrm{Map}(Y)$.

Proof of Lemma 2.7. Fix a particular subset $Y \subset S$ representing the domain in the statement, and let $\overline{Y}$ be its closure in S. We similarly let the subset $\partial Y = \overline{Y} \setminus Y$ represent the multicurve $\alpha = \partial Y$. Let $\Gamma = \mathrm{Homeo}(\overline{Y}, \partial Y)$ denote the group of homeomorphisms of $\overline{Y}$ that fix ∂Y pointwise, and write Γ_0 for its identity component. Also let Y' be the compactification of $\overline{Y}$ obtained by "plugging" each end of $\overline{Y}$ with a point. That is, Y' is a compact 2-manifold with boundary such that $\overline{Y} = Y' \setminus P$ for some finite (and possibly empty) set $P \subset \mathrm{int}(Y')$. We then similarly have $\Gamma' = \mathrm{Homeo}(Y', \partial Y)$ with identity component Γ'_0. We now have (see [8, Section 2.4]) an exact sequence

$$1 \longrightarrow B_k(Y') \longrightarrow \Gamma/\Gamma_0 \longrightarrow \Gamma'/\Gamma'_0 \longrightarrow 1,$$

where $B_k(Y')$ is the braid group on $k = |P|$ strands in Y'. (Note that $B_k(Y')$ is trivial when P is empty.) The group $B_k(Y')$ is torsion-free by [8, Corollary 9] (see also [6, Theorem 8]), and the quotient Γ'/Γ'_0 is torsion-free by [7, Corollary 7.3]. Therefore, the middle group Γ/Γ_0 is torsion-free as well.

Since f fixes every curve disjoint from Y, we may use Theorem 2.2 to choose a representative $\varphi \in \mathrm{Homeo}(S)$ that restricts to the identity on $S \setminus Y$. In particular, φ fixes ∂Y pointwise. Restricting to $\overline{Y}$ now yields an element $\mu = \varphi|_{\overline{Y}} \in \Gamma$ such that the further restriction of μ to $Y = \mathrm{int}(\overline{Y})$ represents $g = f|_Y \in \mathrm{Map}(Y)$.

We caution that the coset of μ in Γ/Γ_0 is not canonically defined, as it depends on the chosen representative φ. Nevertheless, μ is nontrivial in Γ/Γ_0, as otherwise a path from μ to $\mathrm{Id}_{\overline{Y}}$ in Γ would extend to an isotopy between φ and Id_S, contradicting the nontriviality of f. Thus, $\mu\Gamma_0 \in \Gamma/\Gamma_0$ has infinite-order.

We now prove that $g = f|_Y$ has infinite-order in Map(Y); as $f^n|_Y = g^n$, this will imply that f has infinite-order as well. If instead $g^k \simeq \mu^k|_Y$ is trivial for $k \geq 1$, then we may adjust μ^k by an isotopy in $Y = \mathrm{int}(\overline{Y})$ to obtain some $\psi \in \Gamma$ that is supported in a neighborhood of ∂Y and is in fact a nontrivial (since $\mu^k\Gamma_0 \neq \Gamma_0$) product of Dehn twists about the curves of ∂Y; see [7, Proposition 3.19]. Extending this isotopy $\mu^k \simeq \psi$ via the identity gives an isotopy from $f^k \simeq \varphi^k$ to a nontrivial element of the form

$$D_{\gamma_1}^{k_1} \ldots D_{\gamma_n}^{k_n} \in \mathrm{Map}(S),$$

where $\gamma_1, \ldots, \gamma_n$ are the component curves of ∂Y. This contradicts our assumption on f. ■

Proof of Lemma 2.6. Fix an exhaustion $Y_1 \subset Y_2 \subset \ldots$ of S by domains Y_i satisfying the hypothesis of Lemma 2.7 and such that ∂Y_i is essential in Y_{i+1} for each i. Let $g_i \in \mathrm{Map}(Y_i)$ be the restriction $f|_{Y_i}$ to Y_i (see Remark 2.8) and note that g_i has infinite order by Lemma 2.7.

Consider the sets $\mathcal{O}(g_i)$ and ∂g_i. Since g_i has infinite order and $\mathcal{O}(g_{i+1})$ is nonempty (as it contains ∂Y_i), we may apply [9, Lemma 2.2] to conclude that ∂g_i is nonempty for each $i > 1$. Note also that ∂g_i is finite, as Y_i has finite-type. It is clear from the definitions that $\mathcal{O}(g_i) \subset \mathcal{O}(g_{i+1})$ for each i and that

$$\mathcal{O}(f) = \bigcup_i \mathcal{O}(g_i) \qquad \text{and} \qquad \partial f = \bigcup_i \bigcap_{j \geq i} \partial g_j.$$

Since $\mathcal{O}(g_{i+1})$ contains all curves of Y_{i+1} that are disjoint from Y_i by construction, we see that each element of ∂g_{i+1} must in fact be an essential curve of Y_i. Therefore, we have $\partial g_{i+1} \subset \partial g_i$ and may consequently conclude that $\partial f = \cap_i \partial g_i$ is a nonempty finite set of disjoint curves of S. ■

3 Automorphisms of Curve Complexes

In this section we prove Theorem 1.3. If α is a multicurve in a surface S, the *link of* α is the full subcomplex $\mathrm{link}(\alpha) \subset \mathcal{C}(S)$ spanned by the set of vertices of $\mathcal{C}(S) \setminus \alpha$ that are adjacent to α (i.e., the curves β that are distinct and disjoint from each curve of α). Define a relation $\sim$ on the vertices of $\mathrm{link}(\alpha)$ by declaring $\beta \sim \delta$ if there exists a vertex in $\mathrm{link}(\alpha)$

that is nonadjacent to both β and δ. For β a vertex of link(α), we denote by $[\beta]$ the set of curves related to β, and write $\mathrm{link}(\alpha)|_{[\beta]}$ for the full subcomplex of link(α) spanned by $[\beta]$. The following shows that $\sim$ is an equivalence relation and gives a bijection between the equivalence classes of link(α) and the components of $S \setminus \alpha$ that are not thrice-punctured spheres (as such components have no essential curves).

Lemma 3.1. Let α be a multicurve of an infinite-type surface S. Let β be a vertex of link(α), and let Y be the component of $S \setminus \alpha$ containing β. Then $[\beta]$ is equal to the set of curves that are essential in Y and $\mathrm{link}(\alpha)|_{[\beta]} = \mathcal{C}(Y)$.

Proof. Each vertex of link(α) corresponds to a curve disjoint from α and so lies in some connected component of $S \setminus \alpha$. If δ and γ are nonadjacent vertices in link(α), then their corresponding curves intersect and so necessarily lie in the same component. In particular, if γ is nonadjacent to both δ and β, then δ (and γ) and β lie in the same component of $S \setminus \alpha$. This proves that the curves of $[\beta]$ lie in Y. Conversely, for any curve δ contained in Y we may choose a third curve γ in Y that intersects both δ and β. Thus, $\delta \sim \beta$ and we have proven that $[\beta]$ is the set of curves in Y. The fact that $\mathrm{link}(\alpha)|_{[\beta]} = \mathcal{C}(Y)$ is now immediate from the definitions. ∎

We now prove that isomorphisms of curve complexes are geometric.

Proof of Theorem 1.3. Let $\Psi\colon \mathcal{C}(S) \to \mathcal{C}(S')$ be an isomorphism. Fix an exhaustion $Y_1 \subset Y_2 \subset \dots$ of S by principal domains Y_i (Definition 2.1) and set $\alpha_i = \partial Y_i$. By enlarging the domains Y_i if necessary, we may assume the curves of α_i are essential in Y_{i+1}. Since Y_i is principal, Lemma 3.1 implies that the equivalence class E_i corresponding to Y_i is the unique equivalence class of link(α_i) with finite clique number.

For each i, we set $\alpha_i' = \Psi(\alpha_i)$ and observe that Ψ restricts to an isomorphism $\mathrm{link}(\alpha_i) \to \mathrm{link}(\alpha_i')$ that maps equivalence classes to equivalence classes. The image E_i' of E_i is therefore the unique equivalence class of link(α_i') with finite clique number. Writing Y_i' for the component of Y_i' of $S' \setminus \alpha_i'$ corresponding to E_i' (Lemma 3.1), it follows that Y_i' has finite-type and that Ψ restricts to an isomorphism

$$\mathcal{C}(Y_i) \cong \mathrm{link}(\alpha_i)\big|_{E_i} \xrightarrow{\Psi} \mathrm{link}\left(\alpha_i'\right)\big|_{E_i'} \cong \mathcal{C}\left(Y_i'\right).$$

By the original result for finite-type curve complexes (e.g., [17]), each of these isomorphisms is induced by a homeomorphism

$$\phi_i\colon Y_i \to Y_i' \subset S'.$$

We note that ϕ_{i+1} is compatible with ϕ_i by construction. That is, $\phi_{i+1}(Y_i) = Y_i'$ with the restriction of ϕ_{i+1} to Y_i agreeing with ϕ_i. Since S is the union of the Y_i, the direct limit of (ϕ_i) now gives a homeomorphism $\phi\colon S \to S'$ inducing Ψ. ■

4 Algebraic Characterization of Twists

For the entirety of this section, fix an infinite-type surface S and let Γ denote either Map(S) or PMap(S). Fix also a finite-index subgroup G of Γ. Our goal in this section is to give an algebraic characterization of certain "generating twists" of G (Definition 4.7). The first step is to characterize finitely-supported elements.

Definition 4.1. Set $\mathcal{F}_G = \{g \in G \mid \text{the conjugacy class of g in G is countable}\} \leq G$.

Proposition 4.2. An element $f \in G$ has finite support if and only if $f \in \mathcal{F}_G$.

Proof. Assume f does not have finite support. Then there exists a curve a_1 such that $f(a_1) \neq a_1$. Now suppose we have chosen distinct disjoint curves $a_1, \ldots, a_n$ such that, for every $1 \leq i \leq n$, $b_i = f(a_i)$ is distinct from all a_j and so that the curves a_i and b_j are disjoint except possibly when $i = j$. Then take a finite-type domain that contains the curves a_i, $b_i = f(a_i)$, and $f^{-1}(a_i)$ for $i = 1, \ldots, n$. Since f has infinite support, we can find a new curve a_{n+1} outside of Y that is not fixed. By induction, we thus get an infinite list of curves a_i not fixed by f, with the property that all a_i and $b_j = f(a_j)$ are distinct and disjoint except maybe when $i = j$.

Since G has finite-index in Γ, for each i we may choose $k_i \geq 1$ so that $T_{a_i}^{k_i} \in G$. For each sequence $\epsilon = (\epsilon_1, \epsilon_2 \ldots)$ with $\epsilon_i \in \{1, -1\}$, we consider the infinite product

$$\phi_\epsilon = \prod_i T_{a_i}^{\epsilon_i k_i} \in G.$$

The associated conjugates $f_\epsilon = \phi_\epsilon^{-1} f \phi_\epsilon$ are then all distinct. Indeed, if $\epsilon' = (\epsilon_1', \epsilon_2', \ldots)$, then our choice of a_i and $b_i = f(a_i)$ allows us to easily observe that

$$\phi_{\epsilon'} \left(f_\epsilon f_{\epsilon'}^{-1}\right) \phi_{\epsilon'}^{-1} = \phi_{\epsilon'} \phi_\epsilon^{-1} (f \phi_\epsilon f^{-1}) \left(f \phi_{\epsilon'}^{-1} f^{-1}\right) = \prod_{\{i \mid \epsilon_i \neq \epsilon_i'\}} T_{a_i}^{(\epsilon_i' - \epsilon_i) k_i} T_{b_i}^{\sigma(f)(\epsilon_i - \epsilon_i') k_i}$$

is nontrivial when $\epsilon \neq \epsilon'$. Therefore, the conjugacy class of f in G is uncountable.

Conversely, every finitely-supported mapping class may be written as a finite product of Dehn twists and half-twists (see, e.g., [7, Corollary 4.15]). As there are only countably many curves, it follows that Map(S) has only countably many finitely-supported elements. Therefore, when f has finite support, its conjugacy class in G is countable. ■

Given an element $f \in G$, we write

$$C_G(f) = \{g \in G \mid gf = fg\} \leq G$$

for the centralizer of f and write $Z(\mathcal{F}_G \cap C_G(f))$ for the center of the subgroup $\mathcal{F}_G \cap C_G(f)$. The following notation will help us algebraically identify twists.

Definition 4.3. Write $\mathcal{M}_G \subset G$ for the set of of elements $f \in G$ satisfying

(1) $f \in \mathcal{F}_G$,
(2) $Z(\mathcal{F}_G \cap C_G(f))$ is infinite cyclic, and
(3) $C_G(f) = C_G(f^k)$ for all $k \geq 1$.

For each $f \in \mathcal{M}_G$, set $(\mathcal{M}_G)_f = \{h \in \mathcal{M}_G \mid fh = hf\}$.

Lemma 4.4. Let $f \in G$ be annular and consider the twist T_α (i.e., Dehn twist or half-twist) about the curve $\alpha = \partial f$. Then

$$Z(\mathcal{F}_G \cap C_G(f)) = \langle T_\alpha \rangle \cap G \cong \mathbb{Z} \qquad \text{and} \qquad C_G(f) = C_G(f^k)$$

for each $k \geq 1$. In particular, $f \in \langle T_\alpha \rangle$ and furthermore $f \in \mathcal{M}_G$.

Proof. Choose $j \geq 1$ so that T_α^j generates $\langle T_\alpha \rangle \cap G$. First observe that because f is annular, it is a power of T_α. Indeed, if α is not a pants curve, then according to Alexander's method in its finite and infinite versions (see Theorem 2.2), f is homotopic to the identity on each component of $S \setminus \alpha$, thus f is a nonzero power of D_α; if α is a pants curve, then f is homotopic to the identity on one component of $S \setminus \alpha$, and the other component is a three-punctured sphere, on which f is either homotopic to the identity or f is a nonzero power of a half-twist. In both cases, $f = T_\alpha^{jm}$ for some $m \in \mathbb{Z} \setminus \{0\}$.

By Lemma 2.5(4) we have for each $k \in \mathbb{Z} \setminus \{0\}$ that

$$C_G\left(T_\alpha^j\right) = \{g \in G \mid g(\alpha) = \alpha \text{ and } g \text{ preserves orientation}\} = C_G\left(T_\alpha^{jk}\right).$$

Therefore,

$$C_G(f) = C_G\left(T_\alpha^{jm}\right) = C_G(f^k)$$

for each $k \geq 1$. Since $T_\alpha^j \in \mathcal{F}_G \cap Z(C_G(f))$, we clearly have

$$T_\alpha^j \in Z\big(\mathcal{F}_G \cap C_G(f)\big).$$

Conversely, let $g \in Z\big(\mathcal{F}_G \cap C_G(f)\big)$ be nontrivial. Then g has finite support by Proposition 4.2. If g is not annular with $\partial g = \alpha$, then by definition there is a curve β in $S \setminus \alpha$ with $g(\beta) \neq \beta$. But then $D_\beta^i \in \mathcal{F}_G \cap C_G(f)$ for some i by the above and $gD_\beta^i g^{-1} \neq D_\beta^i$ by Lemma 2.5; contradicting our choice of g. Therefore, g must be annular with $\partial g = \alpha$; by the above this implies $g \in \langle T_\alpha^j \rangle$ and so proves $Z(\mathcal{F}_G \cap C_G(f)) = \langle T_\alpha^j \rangle \cong \mathbb{Z}$. ■

Lemma 4.5. Let f be an element of G. If $f \in \mathcal{M}_G$, then f is multi-annular.

Proof. Since $f \in \mathcal{F}_G$, we know that f has finite support and, by Lemma 2.6, that $\alpha = \partial f$ is a nonempty multicurve. Consider the twist T_α about α. Let $g \in \mathcal{F}_G \cap C_G(f)$ be arbitrary. Then g preserves orientation (Proposition 4.2 and Lemma 2.3) and we have:

$$g(\partial f) = \partial(gfg^{-1}) = \partial f.$$

Thus, g commutes with T_α by Lemma 2.5(4), showing that T_α is in $Z\big(\mathcal{F}_G \cap C_G(f)\big)$. Since $f \in Z\big(\mathcal{F}_G \cap C_G(f)\big)$ as well and this group is infinite cyclic by assumption, there necessarily exist $m, n \in \mathbb{Z} \setminus \{0\}$ so that $f^m = T_\alpha^n$.

We claim that f is multi-annular. First, to see that f fixes each curve comprising α, let γ be one such curve and choose $k \geq 1$ so that $f^k(\gamma) = \gamma$; this is possible since f permutes the finitely many curves of α. Then f^k commutes with T_γ by Lemma 2.5. Choosing $j \geq 1$ so that $T_\gamma^j \in G$, it follows that

$$T_\gamma^j \in C_G(f^k) = C_G(f).$$

But this is only possible if $f(\gamma) = \gamma$, as required.

It remains to show that f fixes each curve disjoint from α. Let β be one such curve and choose $i \geq 1$ so that $T_\beta^i \in G$. Since β and α are disjoint, we then have

$$T_\beta^i \in C_G(T_\alpha^n) = C_G(f^m) = C_G(f).$$

Hence, again by Lemma 2.5, we have $f(\beta) = \beta$. ■

Proposition 4.6. An element $f \in G$ is annular if and only if $f \in \mathcal{M}_G$ and $\left(\mathcal{M}_G\right)_f$ is a maximal (with respect to inclusion) member of the collection $\{\left(\mathcal{M}_G\right)_h\}_{h \in \mathcal{M}_G}$.

Proof. First suppose f is annular and let $\alpha = \partial f$. We have seen (Lemma 4.4) that $f \in \mathcal{M}_G$. Let $h \in \mathcal{M}_G$ be such that $\left(\mathcal{M}_G\right)_f \subset (\mathcal{M}_G)_h$. Let β be any curve disjoint from α and choose $k \geq 1$ so that $T_\beta^k \in G$. Then $T_\beta^k \in \mathcal{M}_G$ and evidently $T_\beta^k \in (\mathcal{M}_G)_f$. By assumption, this gives $hT_\beta^k = T_\beta^k h$, thus $h(\beta) = \beta$ by Lemma 2.5. Therefore, h fixes every curve disjoint from α, proving that h is annular with $\partial h = \alpha$. It now follows from Lemma 4.4 that $f^m = h^n$ for some $m, n \in \mathbb{Z}$. Thus, we may conclude the desired maximality of $(\mathcal{M}_G)_f$ by noting

$$(\mathcal{M}_G)_f = C_G(f) \cap \mathcal{M}_G = C_G(f^m = h^n) \cap \mathcal{M}_G = C_G(h) \cap \mathcal{M}_G = (\mathcal{M}_G)_h.$$

Next suppose $f \in \mathcal{M}_G$ and that f is not annular. Then ∂f contains two distinct curves δ and γ. Pick a curve β that intersects δ but is disjoint from γ. Choose $k \geq 1$ so that $T_\gamma^k, T_\beta^k \in G$ and consequently $T_\gamma^k, T_\beta^k \in \mathcal{M}_G$. Let $h \in (\mathcal{M}_G)_f$ be arbitrary. Then $h(\partial f) = \partial f$ so we may choose a power h^i that fixes each component of ∂f. In particular, we have $h^i(\gamma) = \gamma$ so that $hT_\gamma = T_\gamma h$. Thus, $h \in (\mathcal{M}_G)_{T_\gamma^k}$ and we have proven

$$(\mathcal{M}_G)_f \subset (\mathcal{M}_G)_{T_\gamma^k}.$$

However, T_β^k lies in $(\mathcal{M}_G)_{T_\gamma^k}$ (since γ and β are disjoint) but not in $(\mathcal{M}_G)_f$ (since, e.g., the orbit of $\delta \subset \partial f$ under T_β^k is infinite). Thus, $(\mathcal{M}_G)_f$ is not maximal. ■

Definition 4.7. (Generating twist) Say that $f \in G$ is a *generating twist* of G if

(1) $f \in \mathcal{F}_G$,
(2) $Z(\mathcal{F}_G \cap C_G(f))$ is infinite cyclic and generated by f,
(3) $C_G(f) = C_G(f^k)$ for all $k \geq 1$, and
(4) $(\mathcal{M}_G)_f$ is maximal in the collection $\{(\mathcal{M}_G)_h\}_{h \in \mathcal{M}_G}$.

Note that these are algebraic conditions in terms of the group structure of G.

The following is a now consequence of Lemmas 2.5 and 4.4 and Proposition 4.6.

Corollary 4.8. For each curve α of S there is a unique $j_\alpha \geq 1$ so that $T_\alpha^{j_\alpha}$ and $T_\alpha^{-j_\alpha}$ are generating twists of G. This assignment $\alpha \mapsto \{T_\alpha^{\pm j_\alpha}\}$ gives a bijection between curves and inverse pairs of generating twists under which two curves are disjoint if and only if their associated generating twists commute.

5 Isomorphisms Between Big Mapping Class Groups

We may now easily prove our main results.

Proof of Theorem 1.1. For $i = 1, 2$ let S_i be an infinite-type surface and G_i a finite-index subgroup of $\mathrm{PMap}(S_i)$ or $\mathrm{Map}(S_i)$. For each curve α of S_1, let $T_\alpha^{j_\alpha}$ be the associated generating twist from Corollary 4.8. Since generating twists are defined algebraically, they are preserved by the given isomorphism $\Phi\colon G_1 \to G_2$. Therefore, for each curve α of S we have

$$\Phi\left(T_\alpha^{j_\alpha}\right) = T_{h(\alpha)}^{i_\alpha} \qquad \ddagger$$

for some unique curve $h(\alpha)$ of S_2 and power $i_\alpha \in \mathbb{Z} \setminus \{0\}$. Since the isomorphism Φ preserves commutativity, Corollary 4.8 ensures that α and β are disjoint if and only if $h(\alpha)$ and $h(\beta)$ are disjoint. The assignment $\alpha \mapsto h(\alpha)$ thus extends to a simplicial automorphism $\mathcal{C}(S_1) \to \mathcal{C}(S_2)$ and is consequently, by Theorem 1.3, induced by some homeomorphism $h\colon S_1 \to S_2$.

We show, for each $f \in G_1$, that

$$\Phi(f) = h \circ f \circ h^{-1}\colon S_2 \to S_2$$

Following [15, Section 3], for each $f \in G_1$ and curve α of S_1, (‡) and Lemma 2.5(4) give

$$\Phi\left(fT_\alpha^{j_\alpha}f^{-1}\right) = \Phi(f)\Phi\left(T_\alpha^{j_\alpha}\right)\Phi(f^{-1}) = \Phi(f)T_{h(\alpha)}^{i_\alpha}\Phi(f)^{-1} = T_{\Phi(f)(h(\alpha))}^{\sigma(\Phi(f))i_\alpha}$$

and similarly

$$\Phi\left(fT_\alpha^{j_\alpha}f^{-1}\right) = \Phi\left(T_{f(\alpha)}^{j_\alpha}\right) = T_{h(f(\alpha))}^{i_{f(\alpha)}}.$$

Since twists have a common power if and only if their supporting curves agree (Lemma 2.5(3)), this proves $\Phi(f)(h(\alpha)) = h(f(\alpha))$ for all curves α and all $f \in G_1$. Applying this with $\alpha = h^{-1}(\beta)$, we conclude that

$$\Phi(f)(\beta) = h \circ f \circ h^{-1}(\beta)$$

for every curve β of S_2. Therefore, $\Phi(f) = h \circ f \circ h^{-1}$ by Theorem 2.2, as claimed. ■

Proof of Corollary 1.2. For (i), let $\hat{\iota}\colon \operatorname{Aut}(\operatorname{Map}(S)) \to \operatorname{Comm}(\operatorname{Map}(S))$ be the natural map sending an automorphism to its equivalence class of commensurations, and let

$$\iota\colon \operatorname{Map}(S) \to \operatorname{Aut}(\operatorname{Map}(S))$$

be the homomorphism sending f to $g \mapsto fgf^{-1}$. If $f \in \ker(\hat{\iota}\circ\iota)$, then there is a finite-index subgroup $G \leq \operatorname{Map}(S)$ such that $\iota(f)|_G$ is the identity. Then for every curve α we may choose $n \geq 1$ so that $T_\alpha^n \in G$ and consequently

$$T_\alpha^n = \iota(f)\left(T_\alpha^n\right) = fT_\alpha^n f^{-1} = T_{f(\alpha)}^{n\sigma(f)}.$$

Thus, f is trivial by Lemma 2.5(3) and Theorem 2.2, showing that $\hat{\iota}\circ\iota$ is injective. On the other hand, for each isomorphism $\Phi\colon G \to G'$ of finite-index subgroups, Theorem 1.1 provides $h \in \operatorname{Map}(S)$ so that $\Phi = \iota(h)|_G$, showing that ι and $\hat{\iota}\circ\iota$ are surjective as well. For (ii), since every automorphism of $\operatorname{Map}(S)$ is inner, the normality of these subgroups implies they are characteristic. For (iii), Theorem 1.1 gives a surjection

$$N(G)/G \to \operatorname{Aut}(G)/\operatorname{Inn}(G) = \operatorname{Out}(G),$$

where $N(G)$ is the normalizer of G in $\operatorname{Map}(S)$. Thus, when G has finite-index, $[N(G):G]$ and $\operatorname{Out}(G)$ are finite. Finally, (iv) is a special case of Theorem 1.1. ■

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