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Limit sets of Weil–Petersson geodesics with nonminimal ending laminations

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In this paper, we construct examples of Weil–Petersson geodesics with nonminimal ending laminations which have 1-dimensional limit sets in the Thurston compactification of Teichmüller space.

Keywords: Weil–Petersson metric; Thurston compactification; non-positive curvature.

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1. Introduction

A number of authors have studied the limiting behavior of Teichmüller geodesics in relation to the Thurston compactification of Teichmüller space, [2, 9, 14, 16–19, 23]. This work has highlighted the delicate relationship between the vertical foliation of the quadratic differential defining the geodesic and the limit set in the Thurston boundary.

The *ending lamination* of a Weil–Petersson (WP) geodesic ray was introduced by Brock, Masur and Minsky in [5] and in some sense serves as a rough analogue of the vertical foliation of the quadratic differential defining a Teichmüller geodesic ray. Ending laminations have been used to study the behavior of WP geodesics [3, 5, 6, 27, 28] and dynamics of the WP geodesic flow on moduli spaces [6, 7, 13]. In this paper, complementing our work in [3], we provide examples of WP geodesic rays with nonminimal, and hence nonuniquely ergodic, ending laminations whose limit sets in the Thurston compactification of Teichmüller space is larger than a single point.

Theorem 3.1. *There exist Weil–Petersson geodesic rays with nonminimal, nonuniquely ergodic ending laminations whose limit set in the Thurston compactification of Teichmüller space is* 1*-dimensional.*

See also Theorem 3.17 for a more precise statement. Our construction closely follows that of Lenzhen [17] who gave the first examples of Teichmüller geodesics having 1-dimensional limit sets in the Thurston compactification.

2. Preliminaries

Notation 2.1. Let $K \geq 1$, $C \geq 0$, and let X be any set. For two functions $f, g : X \to [0, \infty)$, we write $f \asymp_{K,C} g$ if $\frac{1}{K}g(x) - C \leq f(x) \leq Kg(x) + C$ for all $x \in X$. Similarly, we write $f \overset{*}{\asymp}_K g$ if $\frac{1}{K}g(x) \leq f(x) \leq Kg(x)$ for all $x \in X$, and $f \overset{+}{\asymp}_C g$ if $g(x) - C \leq f(x) \leq g(x) + C$ for all $x \in X$. Moreover, $f \overset{*}{\prec}_K g$ means that $f(x) \leq Kg(x)$ for all $x \in X$ and $f \overset{+}{\prec}_C g$ means that $f(x) \leq g(x) + C$ for all $x \in X$. We drop K, C from the notation when the constants are understood from the context.

Teichmüller space. Given a finite type surface S, we denote its Teichmüller space by $\mathrm{Teich}(S)$. The points in $\mathrm{Teich}(S)$ are isotopy classes of (finite type) Riemann surface structures on S. When the Euler characteristic $\chi(S) < 0$, we also view $X \in \mathrm{Teich}(S)$ as an isotopy class of complete, finite area, hyperbolic metric on S. In this case, given a homotopy class of closed curve α and $X \in \mathrm{Teich}(S)$, we write $\ell_\alpha(X)$ for the length of the X-geodesic representative of α. If α is simple, we let $w_\alpha(X)$ denote the *width* of α in X, defined by

$$w_\alpha(X) = 2\sinh^{-1}(1/\sinh(\ell_\alpha(X)/2)). \tag{2.1}$$

The term “width” is justified by the following, see e.g. [8, Sec. 4].

Lemma 2.2. (Collar Lemma) *Given any $X \in \mathrm{Teich}(S)$ and distinct homotopy classes of disjoint simple closed curves α_1, α_2, let $w_i = w_{\alpha_i}(X)$, for $i = 1, 2$. Then $N_{w_1/2}(\alpha_1)$ and $N_{w_2/2}(\alpha_2)$, the $w_i/2$-neighborhoods of the X-geodesic representative of the α_i, are pairwise disjoint, embedded annuli.*

For $w = w_\alpha(X)$, we call $N_{w/2}(\alpha)$, the *standard collar*, and note that the distance inside $N_{w/2}(\alpha)$ between the boundary components is w. An important consequence is that for any other homotopy class of curve β, we have $\ell_\beta(X) \geq i(\alpha,\beta)w_\alpha(X)$, where $i(\alpha,\beta)$ is the geometric intersection number of α and β (c.f. Theorem 2.3).

Weil–Petersson metric. When $\chi(S) < 0$, the Weil–Petersson (WP) metric is a negatively curved, incomplete, geodesically convex, Riemannian metric on $\mathrm{Teich}(S)$. Its completion, $\overline{\mathrm{Teich}(S)}$, is a stratified CAT(0) space, with a stratum $\mathcal{S}(\sigma)$ for each (possibly empty isotopy class of) multicurve σ, consisting of appropriately marked Riemann surfaces pinched precisely along σ. The stratum $\mathcal{S}(\sigma)$ is totally geodesic and isometric to the product of the Teichmüller spaces of the connected components of $S\backslash\sigma$ with their WP metric. The completion of $\mathcal{S}(\sigma)$ is the union of all strata $\mathcal{S}(\sigma')$ for which $\sigma \subset \sigma'$; see [22]. The stratification has the so-called *non-refraction property*: the interior of a geodesic segment with end points in two strata $\mathcal{S}(\sigma_1)$ and $\mathcal{S}(\sigma_2)$ lies in the stratum $\mathcal{S}(\sigma_1 \cap \sigma_2)$; see [11, 31].

Curve complexes, markings, and projections. We refer the reader to [24, 25] for definitions of the objects described in this subsection — our objective here is to fix notation and terminology. In this paper we denote the *curve complex* of a subsurface Y by $\mathcal{C}(Y)$. The set of vertices of $\mathcal{C}(Y)$, denoted by $\mathcal{C}_0(Y)$, is the set of curves on Y (more precisely, the set of isotopy classes of essential simple closed curves on Y). A *partial marking* μ on S consists of a pants decomposition, $\mathrm{base}(\mu)$, and a transversal for some curves in $\mathrm{base}(\mu)$. A *marking* is a partial marking such that every curve in $\mathrm{base}(\mu)$ has a transversal. For a curve or partial marking μ, we denote the *subsurface projection of* μ to the subsurface Y by $\pi_Y(\mu)$ (see [25, Sec. 2]), and for two μ, μ' define

$$d_Y(\mu,\mu') := \mathrm{diam}_{\mathcal{C}(Y)}(\pi_Y(\mu) \cup \pi_Y(\mu')). \tag{2.2}$$

An important property of d_Y is that it satisfies the triangle inequality when the associated projections are nonempty. If Y is an annulus with core curve α, we also write $\mathcal{C}(\alpha)$ for $\mathcal{C}(Y)$, π_α for π_Y, and $d_\alpha(\mu,\mu')$ for $d_Y(\mu,\mu')$; see again [25, Sec. 2].

There exists a constant $L_S > 0$, called the *Bers constant*, depending on S, such that for any $X \in \mathrm{Teich}(S)$ there is a pants decomposition such that every curve in the pants decomposition has hyperbolic length at most L_S with respect to X; see e.g. [8]. Such a pants decomposition is called a *Bers pants decomposition* for X. A *Bers curve* for X is a curve α for which $\ell_\alpha(X) \leq L_S$. A *Bers marking* for X is a marking μ such that $\mathrm{base}(\mu)$ is a Bers pants decomposition for X and transversal curves have minimal lengths.

Given a point $X \in \mathrm{Teich}(S)$ and a curve α, the *subsurface projection of* X to α, $\pi_\alpha(X)$, is the collection of all geodesic arcs in the annular cover corresponding to α which are orthogonal to the geodesic representative of α (all with respect to the pull-back of the X-metric on S to the cover). Distance in α between points of $\mathrm{Teich}(S)$ and curves/markings is defined as the diameter of the union of their projections (as with the case of two curves or markings). This is often called the *relative twisting*, and for $\alpha, \delta \in \mathcal{C}_0(S)$ and $X \in \mathrm{Teich}(S)$, we write

$$\mathrm{tw}_\alpha(\delta, X) = d_\alpha(\delta, X) := d_\alpha(\delta, \pi_\alpha(X)).$$

If α has bounded length and μ is a bounded length marking for X, then

$$\mathrm{tw}_\alpha(\delta, X) \stackrel{+}{\asymp} d_\alpha(\delta, \mu), \tag{2.3}$$

where the additive error depends on the bounds on the length of α and the lengths of those curves in μ (including those defining transversals of μ) which intersect α, but not on the length of δ. To see this, note that the bounds on all the lengths of curves mentioned imply a lower bound on the length of α by Lemma 2.2 and a lower bound on the angle of intersection between the geodesic representatives of any curve from μ and the geodesic representative of α, and these easily imply an upper bound $d_\alpha(\mu, X)$. Coarse Equation (2.3) then follows from the triangle inequality.

The next theorem is a consequence of [20, Lemma 3.1] (see also [10, Lemmas 7.2 and 7.3]), and provides an estimate on length of a curve γ with respect to $X \in \mathrm{Teich}(S)$ in terms of contributions from certain other curves which γ intersects. To describe it, suppose $X \in \mathrm{Teich}(S)$ and γ, δ are two curves on S, and define

$$\ell_\delta(\gamma, X) = \mathrm{i}(\delta, \gamma)(w_\gamma(X) + \ell_\gamma(X)\,\mathrm{tw}_\gamma(\delta, X)). \tag{2.4}$$

Also, for a pants decomposition P, define

$$\mathrm{i}(\delta, P) = \sum_{\gamma \in P} \mathrm{i}(\delta, \gamma).$$

Theorem 2.3. *For any $L > 0$, there exists $K > 0$ so that the following holds. Let $X \in \mathrm{Teich}(S)$ and P is a pants decomposition of S with $\ell_\gamma(X) \le L$ for all $\gamma \in P$. Then for any curve $\delta \in \mathcal{C}_0(S)$, $\delta \notin P$, we have*

$$\left| \ell_\delta(X) - \sum_{\gamma \in P} \ell_\delta(\gamma, X) \right| \stackrel{*}{\prec}_K i(\delta, P).$$

Proof. For every $\gamma \in P$, let $N(\gamma) = N_{w_\gamma(X)/2}(\gamma)$ be the standard collar around γ, where $w_\gamma(X)$ is the width as in (2.1). By Lemma 2.2, these collars are embedded and pairwise disjoint. Every complementary component Q of this set of standard collars is topologically a pair of pants but does not have a geodesic boundary. The decomposition of X into standard collars and complementary components decomposes δ into segments. Then [20, Lemma 3.1(b)] implies that for any segment u that is associated to a standard collar $N(\gamma)$ we have

$$\ell_u(X) \stackrel{+}{\asymp}_C w_\gamma(X) + \ell_\gamma(X)\,\mathrm{tw}_\gamma(\delta, X)$$

for some constant C depending on L. (The language in [20, Lemma 3.1] is slightly different because it also applies to segments in possibly infinite geodesics.) That is, for some constant K_1, we have

$$\left| \sum_u \ell_u(X) - \sum_{\gamma \in P} \ell_\delta(\gamma, X) \right| \overset{*}{\prec}_{K_1} \mathrm{i}(\delta, P).$$

But the difference between $\ell_\delta(X)$ and $\sum_u \ell_u(X)$ is the sum of the lengths of segments in complementary pieces. Now, we note that [20, Lemma 3.1(a)] states that the length of each such segment is uniformly bounded. Also, the number of such segments is $\mathrm{i}(\delta, P)$. Thus, for some K_2,

$$\left| \ell_\delta(X) - \sum_u \ell_u(X) \right| \overset{*}{\prec}_{K_2} \mathrm{i}(\delta, P).$$

Now, setting $K = K_1 + K_2$, the theorem follows from above two inequalities and the triangle inequality. □

The Thurston compactification. The *Thurston boundary of the Teichmüller space* is the space of projective classes of measured laminations $\mathcal{PML}(S)$; see [12]. A sequence of points $\{X_k\} \subset \mathrm{Teich}(S)$ exiting every compact set of $\mathrm{Teich}(S)$ converges to $[\bar{\lambda}]$, the projective class of a measured lamination $\bar{\lambda} \in \mathcal{ML}(S)$, if there exists a sequence of positive real numbers $\{u_k\}_k$ so that

$$\lim_{k\to\infty} u_k \ell_\delta(X_k) = \mathrm{i}(\delta, \bar{\lambda}), \tag{2.5}$$

for every $\delta \in \mathcal{C}_0(S)$. We call $\{u_k\}_k$ a *scaling sequence* for $\{X_k\}_k$, and note that $u_k \to 0$. In fact, a finite set of curves $\delta_1, \ldots, \delta_n$ can be chosen so that for any sequence $\{X_k\}$ exiting every compact subset of $\mathrm{Teich}(S)$, we have $X_k \to [\bar{\lambda}]$ if and only if (2.5) holds for some scaling sequence $\{u_k\}_k$ and the curves $\delta = \delta_i$, for each $i = 1, \ldots, n$. To see this, we let $\delta_1, \ldots, \delta_n$ consist of a pants decomposition together with a pair of transverse curves for each pants curve. Then any measured foliation/lamination is determined by these intersection numbers (indeed, the intersection numbers with the transverse curves suffice to determine the twisting parameters with for the foliation, and hence the foliation; see [12, Exposé 6]). Therefore, if (2.5) holds for some $\bar{\lambda}$, some $\{u_k\}$, and $\delta = \delta_i$, for all $i = 1, \ldots, n$, then all accumulation points of $\{X_k\}$ agree (as they are determined by these intersection numbers), and hence $\{X_k\}$ converges to $[\bar{\lambda}]$. In particular, for any curve α, we can choose the curves $\delta_1, \ldots, \delta_n$ to all have nonzero intersection number with α.

Ending lamination. Suppose $r\colon [0, \infty) \to \mathrm{Teich}(S)$ is an infinite WP geodesic ray. A *pinching curve* for r is a curve γ with $\lim_{t\to\infty} \ell_\gamma(r(t)) = 0$. The *(forward) ending lamination of* r, denoted by $\nu^+ = \nu^+(r)$, is the union of the pinching curves together with the supports of any accumulation points in $\mathcal{PML}(S)$ of an infinite sequence of distinct Bers curves for hyperbolic metrics along $r([0, \infty)$; see [5, Definition 2.7] for more details.

2.1. *Bounded length WP geodesic segments*

Because of the non-completeness of the Weil–Petersson metric and the non-local-compactness of its completion, the usual compactness theorems for geodesic segments of fixed length based at a point is more subtle than in the complete case. Wolpert carried out an initial analysis [30, Proposition 23] that captured how such segments can limit at the completion, but further analysis in [27, Theorem 4.2] captures a stronger non-refraction condition.

Given a curve $\gamma \in \mathcal{C}(S)$ we denote the positive Dehn twist about γ by D_γ. For a multicurve σ on a surface S, we denote the subgroup of $\mathrm{Mod}(S)$ generated by positive Dehn twists about the curves in σ by $\mathrm{tw}(\sigma)$.

Theorem 2.4. (Geodesic limit) *Given $T > 0$, let $\zeta_n : [0,T] \to \mathrm{Teich}(S)$ be a sequence of Weil–Petersson geodesic segments parametrized by arclength with $\zeta_n(0) = X \in \mathrm{Teich}(S)$. Then after passing to a subsequence, we may extract a partition of the interval $[0,T]$ by $0 = t_0 < t_1 < \cdots < t_{k+1} = T$, multicurves σ_l, $l = 1,\ldots,k+1$, with $\sigma_l \cap \sigma_{l+1} = \emptyset$ for $l = 1,\ldots,k$ and a piecewise geodesic segment*

$$\hat{\zeta} : [0,T] \to \overline{\mathrm{Teich}(S)}$$

with $\hat{\zeta}(t_l) \in \mathcal{S}(\sigma_l)$ for $l = 1,\ldots,k+1$, and $\hat{\zeta}((t_l,t_{l+1})) \subset \mathrm{Teich}(S)$ for $l = 0,\ldots,k$, such that the following holds:

(1) $\lim_{n\to\infty} \zeta_n(t) = \hat{\zeta}(t)$ *for all $t \in [t_0,t_1]$*,
(2) *there exist elements $\mathcal{T}_{l,n} \in \mathrm{tw}(\sigma_l)$ for each $l = 1,\ldots,k$ and $n \in \mathbb{N}$, so that letting $\varphi_{l,n} = \mathcal{T}_{l,n} \circ \cdots \circ \mathcal{T}_{1,n}$ we have*

$$\lim_{n\to\infty} \varphi_{l,n}(\zeta_n(t)) = \hat{\zeta}(t)$$

for all $t \in [t_l,t_{l+1}]$.

Remark 2.5. In this theorem, σ_{k+1} may be empty (in which case we have $\hat{\zeta}(t_{k+1}) \in \mathrm{Teich}(S)$). A key feature of this theorem is that $\sigma_l \cap \sigma_{l+1} = \emptyset$, meaning that these two multicurves have no common components. This is responsible for the non-refraction behavior ensuring that $\hat{\zeta}((t_l,t_{l+1}))$ is contained in $\mathrm{Teich}(S)$ as opposed to $\overline{\mathrm{Teich}(S)}$.

We also need the following, which is [27, Corollary 4.10]. Denote a Bers marking at a point $X \in \mathrm{Teich}(S)$ by $\mu(X)$.

Theorem 2.6. *Given ϵ_0, T positive and $\epsilon \in (0,\epsilon_0]$, there is an $N \in \mathbb{N}$ with the following property. Suppose that $\zeta\colon [a,b] \to \mathrm{Teich}(S)$ is a WP geodesic segment of length at most T such that $\sup_{t\in[a,b]} \ell_\alpha(\zeta(t)) \geq \epsilon_0$ and $d_\alpha(\mu(\zeta(a)),\mu(\zeta(b))) > N$. Then, we have*

$$\inf_{t\in[a,b]} \ell_\alpha(\zeta(t)) \leq \epsilon.$$

3. Geodesics with Nonminimal Ending Laminations

In this section, we prove the main result of the paper (see also Theorem 3.17 for a more precise statement).

Theorem 3.1. *There exist Weil–Petersson geodesic rays with nonminimal, nonuniquely ergodic ending laminations whose limit set in the Thurston compactification of Teichmüller space is* 1*-dimensional.*

First, let us briefly sketch our construction of such geodesic rays. The basic idea is similar to Lenzhen's construction for Teichmüller geodesics in [17]. Let S be the closed, genus 2 surface and let $\alpha \subset S$ be a separating simple closed curve cutting S into two one-holed tori that we denote by S_0 and S_1. The stratum $\mathcal{S}(\alpha)$ is isometric to a product of Teichmüller spaces of once-punctured tori, i.e. $\mathcal{S}(\alpha) \cong \mathrm{Teich}(S_0) \times \mathrm{Teich}(S_1)$.

We carefully choose sequences of curves $\{\gamma_i^h\}_i \subset \mathcal{C}(S_h),\ h = 0, 1$ which form quasi-geodesics and limit to minimal filling laminations $\lambda_h,\ h = 0, 1$. Using the fact that $\mathrm{Teich}(S_h)$ with the WP metric is quasi-isometric to $\mathcal{C}(S_h)$, and that it has negative curvature bounded away from 0 we construct geodesic rays $\hat{r}^h$ in $\mathrm{Teich}(S_h)$ which have forward ending laminations λ_h, $h = 0, 1$.

Next, we consider the geodesic $\hat{r} = (\hat{r}^0, \hat{r}^1)$ in $\overline{\mathrm{Teich}(S)}$, and construct a geodesic ray r which fellow travels $\hat{r}$. We estimate the length of an arbitrary curve along r using estimates from Theorem 2.3. From the conditions, we imposed on our sequences of curves, we will see that most of the length of the curve comes from its intersection with curves γ_i^0 and γ_i^1, and so lengths are eventually well-approximated by intersection numbers with linear combinations of measure $\bar{\lambda}_0$ and $\bar{\lambda}_1$ on λ_0 and λ_1, respectively. Consequently, this geodesic ray accumulates on a 1-simplex with vertices $[\bar{\lambda}_0]$ and $[\bar{\lambda}_1]$ in the Thurston boundary. Analyzing a pair of particular sequences of times, we see that the endpoints of the simplex are in the limit set, and so by connectivity, the limit set consists of the entire 1-simplex.

3.1. *Continued fraction expansions and geodesics in* $\mathrm{Teich}(S_{1,1})$

Let λ_h be a minimal, irrational lamination on S_h. This lamination is the straightening of a foliation of the flat square torus, and we assume for convenience that the slope of the leaves of this foliation is greater than 1. The reciprocal of this slope is an irrational number less than 1 which we denote by x_h, and we write its continued fraction expansion as

$$x_h = [0; e_0^h, e_1^h, \ldots], \tag{3.1}$$

(the first coefficient is zero since $x_h < 1$). We assume in all that follows that $e_i^h \geq 4$ for all i and for $h = 0, 1$.

Next, let $\frac{p_i^h}{q_i^h} = [0; e_0^h, e_1^h, \ldots, e_{i-1}^h]$ be the ith *convergent* with finite continued fraction expansion as shown, obtained by truncating that of x_h. Let γ_i^h be the

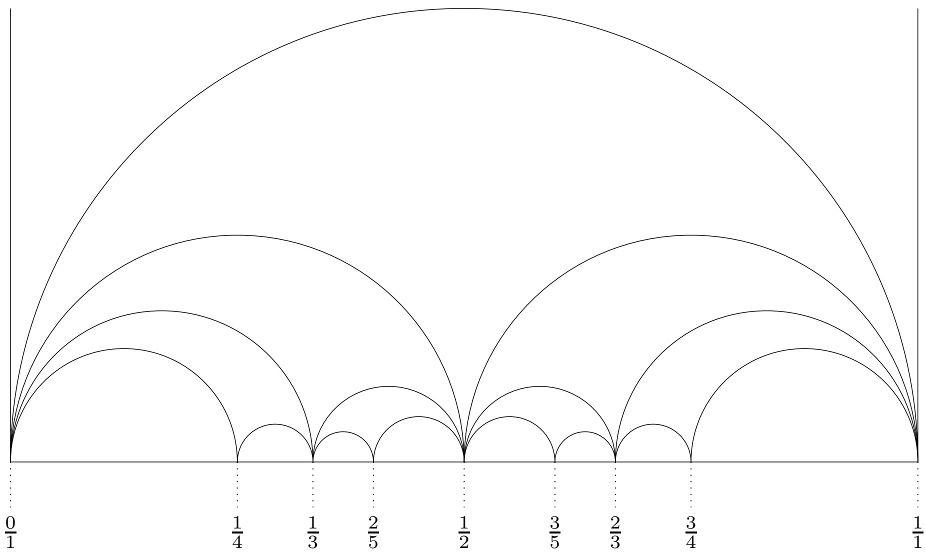

Fig. 1. A piece of the curve graph $\mathcal{C}(S_h)$ visualized as the Farey graph.

simple closed curve on the torus whose slope is the reciprocal, $\frac{q_i^h}{p_i^h}$. Note that γ_0^h is the curve whose reciprocal slope is 0 (that is, γ_0^h is the vertical curve) and we let γ_{-1}^h denote the horizontal curve, by convention.

The *Farey graph* is the graph with vertices corresponding to $\mathbb{Q} \cup \{\infty\}$ and edges between $\frac{p}{q}$ and $\frac{r}{s}$ whenever $|ps - rq| = 1$ [29]; see Fig. 1. Identifying a simple closed curve on the (flat, square) torus with the reciprocal of its slope identifies the curve graph $\mathcal{C}(S_h)$ with Farey graph [26], and we use these two graphs interchangeably depending on our purposes. Our assumption that $e_i^h \geq 4$ ensures that the sequence of curves $\{\gamma_i^h\}_i$ (or equivalently, the sequence of convergents $\{\frac{p_i^h}{q_i^h}\}_i$) is a geodesic; see e.g. [26, Sec. 3]. Our index convention leads to

$$\gamma_{i+1}^h = D_{\gamma_i^h}^{\pm e_i^h}(\gamma_{i-1}^h), \tag{3.2}$$

with the sign determined by the parity of i; see Fig. 2.

The curve graph $\mathcal{C}(S_h)$ — or equivalently the Farey graph — naturally embeds into the Weil–Petersson completion of $\mathrm{Teich}(S_h)$ in such a way that the vertex corresponding to the curve γ is sent to the point in which γ has been pinched, and so that edges between adjacent vertices are sent to WP geodesics. Furthermore, the pants graph and the curve graph of a once punctured torus coincide, and according to [1, Theorem 3.2] this embedding is a quasi-isometry. The usual identification of $\overline{\mathrm{Teich}(S_h)}$ with a subset of the compactified upper half-plane provides the standard embedding of the Farey graph into $\overline{\mathbb{H}^2}$, with vertex set $\mathbb{Q} \cup \{\infty\} \subset \mathbb{R} \cup \{\infty\} = S^1_\infty$. We further note that all maps and identifications are equivariant with respect to the actions of $\mathrm{Mod}(S_h) \cong \mathrm{SL}_2(\mathbb{Z})$ on the various graphs/spaces.

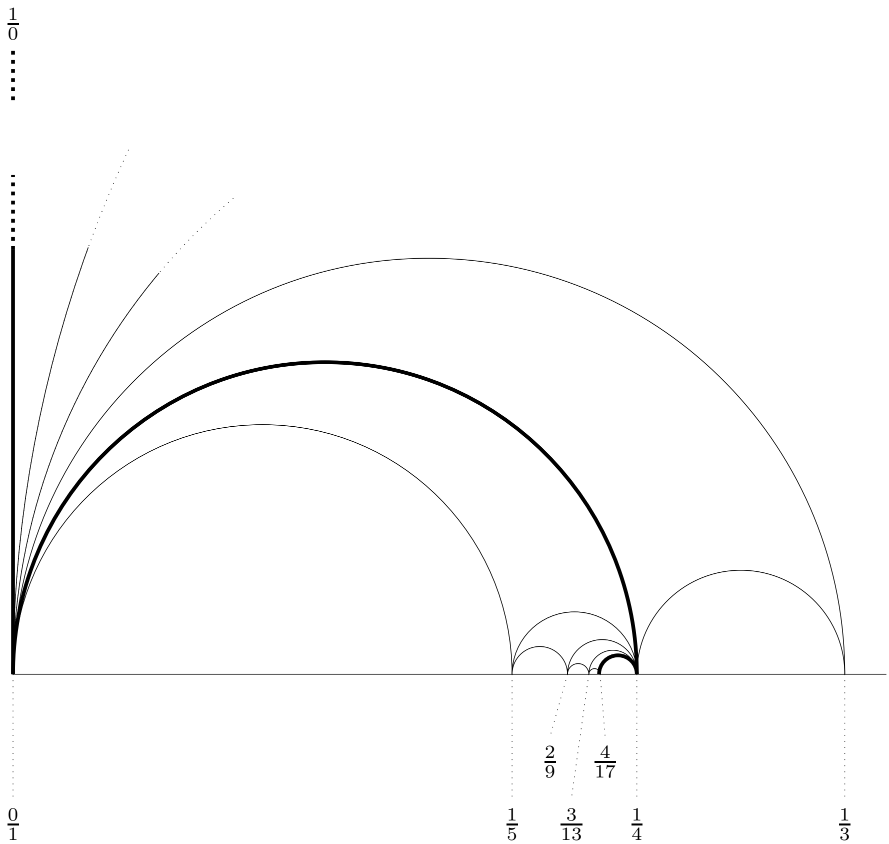

Fig. 2. A picture of the initial segment of the geodesic in the curve graph $\mathcal{C}(S_h)$ (again visualized as the Farey graph) defined by the continued fraction $[0; 4, 4, 4, \ldots]$. The edges of the geodesic are drawn as thicker lines, and the first few vertices, $\gamma_{-1}^h = \frac{1}{0}, \gamma_0^h = \frac{0}{1}, \gamma_1^h = \frac{1}{4}, \gamma_2^h = \frac{4}{17}, \ldots,$ are indicated. As one can see from the first few segments of the geodesic, it "pivots" on opposite sides (c.f. [26]), reflecting the fact that the sign in front of $e_i^h = 4 > 0$ in the Dehn twisting alternates. It follows that the segment $[\gamma_i^h, \gamma_{i+1}^h]$ is separated from the segment $[\gamma_0^h, \gamma_1^h]$ by the segment $[\gamma_{i-1}^h, \gamma_i^h]$ in $\mathcal{C}(S_h)$.

For each $i \geq 0$, let $X_i^h \in \overline{\mathrm{Teich}(S_h)}$ denote the point at which γ_i^h is pinched and $[X_i^h, X_i^{h+1}]$ the geodesic in $\overline{\mathrm{Teich}(S)}$ between points X_i^h and X_{i+1}^h. These geodesic segments are the images of the geodesics $[\gamma_i^h, \gamma_{i+1}^h]$ in $\mathcal{C}(S_h)$ we described above, and since the concatenation of the latter set of segments is a geodesic in $\mathcal{C}(S_h)$, the image is a quasi-geodesic in $\overline{\mathrm{Teich}(S_h)}$. Then since the action of $\mathrm{Mod}(S_h)$ on γ_i^h is transitive, it is clearly transitive on the geodesics segments $[X_i^h, X_{i+1}^h]$. Moreover, $\mathrm{Mod}(S_h)$ acts isometrically on $\mathrm{Teich}(S_h)$, so all geodesics $[X_i^h, X_{i+1}^h]$ have the same lengths, we denote the length by

$$D = d_{\mathrm{WP}}(X_i^h, X_{i+1}^h) > 0. \tag{3.3}$$

Note that $\gamma_0^0 = \gamma_0^1$ is the curve corresponding to the rational number 0, and for convenience we let X_{-1}^h denote the midpoint of the geodesic segment between (the image of) γ_0^h and γ_{-1}^h which has distance $\frac{D}{2}$ to X_0^h (note that $X_{-1}^h = \sqrt{-1}$ in the upper half plane).

Let $\hat{r}_c^h$ be the unit speed parameterization of the concatenation of segments $[X_i^h, X_{i+1}^h]$, $i \in \mathbb{N}$, for $h = 0, 1$. The set of (not necessarily infinite) geodesic rays starting at X_{-1}^h and passing through a point on the geodesic segment $[X_i^h, X_{i+1}^h]$ forms a nested sequence, indexed by i. To see this, note that by the change of the sign of the power of $D_{\gamma_i^h}$ in (3.2), the geodesic $[\gamma_i^h, \gamma_{i+1}^h]$ is separated from γ_{-1}^h by the geodesic $[\gamma_{i-1}^h, \gamma_i^h]$ in $\mathcal{C}(S_h)$; see Fig. 2. This implies that the geodesic $[X_i^h, X_{i+1}^h]$ is separated from X_{-1}^h by the geodesic $[X_{i-1}^h, X_i^h]$ in $\overline{\mathrm{Teich}(S_h)}$, and hence any geodesic starting from X_{-1}^h that passes through $[X_i^h, X_{i+1}^h]$ must also pass through $[X_{i-1}^h, X_i^h]$.

Now, note that $\hat{r}_c^h$ is an infinite quasi-geodesic in $\overline{\mathrm{Teich}(S_h)}$, so the distance between the segments $[X_i^h, X_{i+1}^h]$ and X_{-1}^h go to infinity. Then the negative curvature of the WP metric on $\mathrm{Teich}(S_h)$ [32, Corollary 7.6] implies that the maximum of the smaller angles at X_{-1}^h between any two geodesics in the nested sequence of geodesic segments tends to 0 as $i \to \infty$. This guarantees the existence of a unique ray in the intersection of all these sets. We denote the ray by $\hat{r}^h$ and note that it fellow travels $\hat{r}_c^h$.

Lemma 3.2. *There exists a sequence $\{K_i\}_{i=1}^{\infty}$ so that if $e_i^h > K_i$ for all $i \geq 0$, and if $\{t_i^h\}$ is the sequence of times for which $d_{\mathrm{WP}}(\hat{r}^h(t_i^h), X_i^h)$ is minimized, then for D from* (3.3) *we have*

(1) $d_{\mathrm{WP}}(\hat{r}_c^h(s), \hat{r}^h(s)) \leq \frac{D}{4}$ *for all $s > 0$,*
(2) $d_{\mathrm{WP}}(\hat{r}^h(t_i^h), X_i^h) \leq \frac{D}{2^{i+6}}$ *(which tends to 0 as $i \to \infty$), and*
(3) $|t_i^h - (\frac{D}{2} + iD)| < \frac{D}{8}$ *(in particular, $\{t_i^h\}_i$ is increasing).*

Proof. First, we will show that for all $i \geq 0$, we can choose $K_i > 0$ such that if $e_i^h > K_i$, then we have

$$d_{\mathrm{WP}}(\hat{r}^h(t_i^h), X_i^h) < \frac{D}{2^{i+6}}.$$

To prove this, first let δ_i^h denote the segment of $\hat{r}^h$ with one endpoint on $[X_{i-2}^h, X_{i-1}^h]$ and the other on $[X_{i+1}^h, X_{i+2}^h]$ (recall that $\hat{r}^h$ pass through these geodesics). Since the piecewise geodesic segment $\hat{r}_c^h$ contains a segment of length $4D$ containing $[X_{i-2}^h, X_{i-1}^h]$ and $[X_{i+1}^h, X_{i+2}^h]$, the length of δ_i^h is at most $4D$. Given any $\eta > 0$, we claim that there exists $C(\eta) > 0$ so that if $\hat{r}^h$ stays outside of the η-neighborhood of X_i^h, then the length of δ_i^h is at least $C(\eta)e_i^h$. If we prove this claim, then taking $\eta_i = \frac{D}{2^{i+6}}$, we can set $K_i > \frac{4D}{C(\eta_i)}$ and observe that if $e_i^h > K_i$, then δ_i^h (and hence $\hat{r}^h$) must enter the η_i-neighborhood, as required.

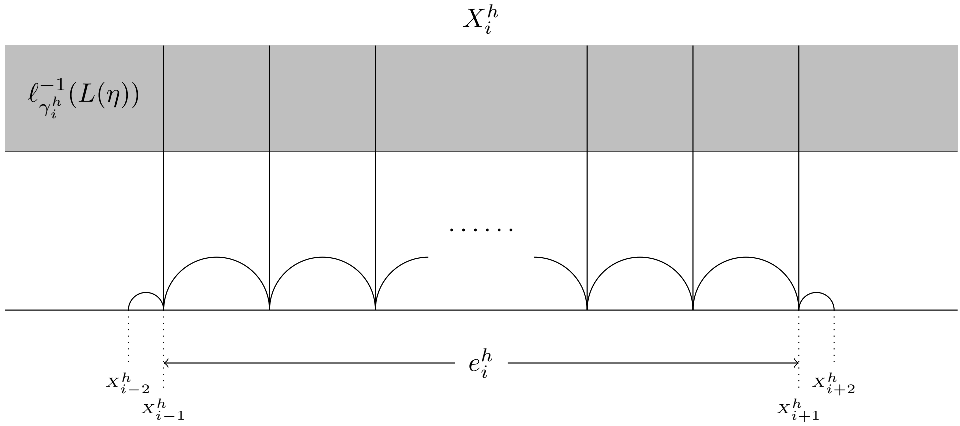

Fig. 3. Applying an element of $\mathrm{Mod}(S_h)$, we can assume that X_i^h is the point at infinity in the upper half-plane model of $\overline{\mathrm{Teich}(S_h)}$, as shown. Then for $L(\eta) > 0$ small, $\ell_{\gamma_i^h}^{-1}((0, L(\eta)])$ is approximately a horoball, since a horoball is the sublevel set of the extremal length function and since hyperbolic lengths and extremal lengths are nearly proportional for small values (see [21]).

To prove the claim, recall that distance from a point $X \in \mathrm{Teich}(S_h)$ to X_i^h is $(2\pi\ell_{\gamma_i^h})^{\frac{1}{2}} + O(\ell^2_{\gamma_i^h})$ (see [31, Corollary 4.10]). In particular, there exists $L(\eta) > 0$ so that the sublevel set $\ell_{\gamma_i^h}^{-1}((0, L(\eta)])$ is contained in the the ball of radius η about X_i^h. By convexity of length functions, [32, Sec. 3.3], the set $\ell_{\gamma_i^h}^{-1}((0, L(\eta)])$ is convex and hence the closest point projection of δ_i^h to it is no longer than δ_i^h. The length of each arc of the boundary of $\ell_{\gamma_i^h}^{-1}((0, L(\eta)])$ intersected with a triangle of the Farey tessellation is some constant $C(\eta) > 0$, and since the projection of δ_i^h to the sublevel set has to cross at least e_i^h of these arcs, its length is at least $C(\eta)e_i^h$, as required; see Fig. 3.

We now assume (as we will for the remainder of the proof) that $e_i^h > K_i$ for all $i \geq 0$, and observe that part (2) of the lemma holds.

By the triangle inequality, it follows that for all $i \geq 0$

$$\begin{aligned} ||t_i^h - t_{i+1}^h| - D| &= |d_{\mathrm{WP}}(\hat{r}^h(t_i^h), \hat{r}^h(t_{i+1}^h)) - d_{\mathrm{WP}}(X_i^h, X_{i+1}^h)| \\ &\leq \frac{D}{2^{i+6}} + \frac{D}{2^{i+7}} < \frac{D}{2^{i+5}}. \end{aligned} \tag{3.4}$$

A similar (simpler) argument proves $|t_0^h - \frac{D}{2}| < \frac{D}{2^6}$.

We claim that for all $i \geq 1$, t_{i+1}^h must lie between t_i^h and t_{i+2}^h. If not, and for example $t_{i+1}^h > \max\{t_i^h, t_{i+2}^h\}$, then $t_{i+1}^h - t_i^h > 0$ and $t_{i+1}^h - t_{i+2}^h > 0$, and applying inequality (3.4) to i and $i+1$, we see that

$$|t_i^h - t_{i+2}^h| = |t_{i+1}^h - t_{i+2}^h - D + (D - (t_{i+1}^h - t_i^h))| \leq \frac{D}{2^{i+5}} + \frac{D}{2^{i+6}}.$$

Hence by the triangle inequality (as above)

$$\begin{aligned} d_{\mathrm{WP}}(X_i^h, X_{i+2}^h) &\leq d_{\mathrm{WP}}(\hat{r}^h(t_i^h), \hat{r}^h(t_{i+2}^h)) + \frac{D}{2^{i+6}} + \frac{D}{2^{i+8}} \\ &\leq |t_i^h - t_{i+2}^h| + \frac{D}{2^{i+6}} + \frac{D}{2^{i+8}} < \frac{D}{4}. \end{aligned}$$

On the other hand, [4, Lemma 3.2] implies that since γ_i^h and γ_{i+2}^h are not adjacent in $\mathcal{C}(S_h)$, we must have $d_{\mathrm{WP}}(X_i^h, X_{i+2}^h) > D$, which is a contradiction. A similar argument produces a contradiction if $t_{i+1}^h < \min\{t_i^h, t_{i+2}^h\}$, hence as we claimed that t_{i+1}^h is between t_i^h and t_{i+2}^h, and thus $\{t_i^h\}_i$ is an increasing sequence.

From (3.4) (and the inequality $|t_0^h - \frac{D}{2}| < \frac{D}{2^6}$), we have

$$\begin{aligned} \left|t_i^h - \left(\frac{D}{2} + iD\right)\right| &= \left|t_0^h + \sum_{j=1}^{i} t_j^h - t_{j-1}^h - \frac{D}{2} - iD\right| \\ &\leq \left|t_0^h - \frac{D}{2}\right| + \sum_{j=1}^{i} |t_j^h - t_{j-1}^h - D| \leq \frac{D}{2^6} + \sum_{j=1}^{i} \frac{D}{2^{j+4}} < \frac{D}{8}. \end{aligned}$$

This proves part (3).

Finally, we note that part (1) follows from (3.4), parts (2) and (3), and convexity of distance between two geodesics in a CAT(0) space. To see this, first note that for all $i \geq 0$

$$\begin{aligned} &d_{\mathrm{WP}}\left(\hat{r}^h\left(\frac{D}{2} + iD\right), \hat{r}_c^h\left(\frac{D}{2} + iD\right)\right) \\ &\quad\leq d_{\mathrm{WP}}\left(\hat{r}^h\left(\frac{D}{2} + iD\right), \hat{r}^h(t_i^h)\right) + d_{\mathrm{WP}}(\hat{r}^h(t_i^h), X_i^h) \\ &\quad\leq \left|t_i^h - \left(\frac{D}{2} + iD\right)\right| + \frac{D}{2^{i+6}} < \frac{D}{8} + \frac{D}{2^{i+6}} < \frac{D}{4}. \end{aligned}$$

Thus, for all $i \geq 0$, convexity of the distance between geodesics implies

$$d_{\mathrm{WP}}(\hat{r}^h(s), \hat{r}_c^h(s)) < \frac{D}{4}, \quad \text{for all } s \in \left[\frac{D}{2} + iD, \frac{D}{2} + (i+1)D\right]$$

(and for $s \in [0, \frac{D}{2}]$). This proves (1) for all $s \geq 0$, completing the proof. □

3.2. *Sequences of times*

Throughout the following, we will always assume that for each $h = 0, 1$, the sequence $\{e_i^h\}_i$ is chosen so that $e_i^h > K_i$ from Lemma 3.2, and we write $\hat{r}^h, \hat{r}_c^h$ to denote the associated geodesics/quasi-geodesics. We keep the same parameterization for $\hat{r}^0$ and $\hat{r}_c^0$ as above, but adjust the parameterization of $\hat{r}^1$ and $\hat{r}_c^1$ by precomposing with the maps $t \mapsto t - \frac{D}{2}$. This does not make sense for $t \in [0, \frac{D}{2})$, so we define $\hat{r}^1$ and $\hat{r}_c^1$ to be constant on this interval.

With this new parameterization, the sequences $\{t_i^1\}$ must be shifted by $\frac{D}{2}$, so that parts (1) and (2) of Lemma 3.2 remain valid. The conclusion in part (3) of the lemma then becomes

$$\left|t_i^0 - \left(\frac{D}{2} + iD\right)\right| < \frac{D}{8} \quad \text{and} \quad |t_i^1 - (i+1)D| < \frac{D}{8}. \tag{3.5}$$

Identifying $\mathcal{S}(\alpha) = \mathrm{Teich}(S_0) \times \mathrm{Teich}(S_1)$, we set

$$\hat{r} = (\hat{r}^0, \hat{r}^1)\colon [0, \infty) \to \mathcal{S}(\alpha) \subset \overline{\mathrm{Teich}(S)}.$$

Notation 3.3. (Relabeling sequences) To simplify some statements and avoid duplication in some of the arguments that follow, we make the following notational convention. For $h = 0, 1$ and $i \geq 0$, set

$$e_{2i+h} = e_i^h,$$

$$\gamma_{2i+h} = \gamma_i^h,$$

$$t_{2i+h} = t_i^h,$$

$$X_{2i+h} = X_i^h.$$

We will use the index k for these sequences, and write $\{e_k\}$, $\{\gamma_k\}$, $\{t_k\}$, and $\{X_k\}$. We also let $\bar{k} \in \{0, 1\}$ denote the residue of k modulo 2, and $i = i(k)$ for the floor of $k/2$. Thus, when we need it, can write $e_k = e_i^{\bar{k}}$, etc. As an abuse of notation, we say things like "$\hat{r}(t_k)$ is close to X_k", though what we really mean is that $\hat{r}^{\bar{k}}(t_k)$ is close to X_k. We also view γ_k as a curve on both S and $S\backslash\alpha$, rather than just a curve on $S_{\bar{k}} \subset S\backslash\alpha \subset S$. Finally, the following sequence of times will also be useful for us

$$t'_k = \frac{t_k + t_{k+1}}{2}.$$

Proposition 3.4. *For all $k \geq 0$, $t_{k+1} - t_k > \frac{D}{4}$. In particular, $\{t_k\}_k$ is increasing. Consequently, $t'_k - t_k > \frac{D}{8}$ and $t_{k+1} - t'_k > \frac{D}{8}$.*

Proof. For $k = 2i$ (even), (3.5) implies

$$t_{k+1} - t_k = t_i^1 - t_i^0 > \left((i+1)D - \frac{D}{8}\right) - \left(\frac{D}{2} + iD + \frac{D}{8}\right) = \frac{D}{2} - \frac{D}{4} = \frac{D}{4}.$$

A similar computation verifies the claim for k odd.

The last sentence follows from the first, and the fact that t'_k is the average of t_k and t_{k+1}. □

Figure 4 provides a useful illustration of the relationship between $\{t_k\}$, $\{t'_k\}$, $\{\gamma_k\}$, and $\{X_k\}$.

Lemma 3.5. *There exists $C > 0$, so that for all $k \geq 2$, we have*

$$\ell_{\gamma_k}(\hat{r}(t'_{k-2})) \leq C \quad \text{and} \quad \ell_{\gamma_k}(\hat{r}(t'_{k+1})) \leq C.$$

Consequently, $\ell_{\gamma_k}(\hat{r}(t)) \leq C$ for all $t \in [t'_{k-2}, t'_{k+1}]$.

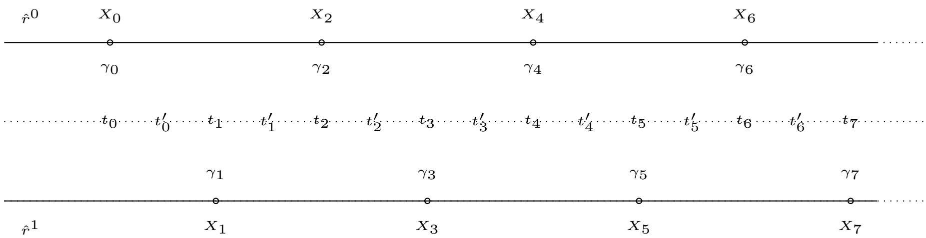

Fig. 4. The times t_k and t_k' are "spaced out" by at least $\frac{D}{8}$. The former are the times when $\hat{r} = (\hat{r}^0, \hat{r}^1)$ is closest to the points $\{X_k\}$: the curve γ_k is very short at time t_k.

Proof. We prove the bound on $\ell_{\gamma_k}(\hat{r}(t'_{k-2}))$. The proof of the other bound is similar.

According to part (2) of Lemma 3.2, for $k \geq 2$

$$d_{\mathrm{WP}}(\hat{r}^{\bar{k}}(t_{k-2}), X_{k-2}) \leq \frac{D}{2^6} \quad \text{and} \quad d_{\mathrm{WP}}(\hat{r}^{\bar{k}}(t_k), X_k) \leq \frac{D}{2^6}.$$

By convexity of distance between geodesics, it follows that there is a point $Y_k \in [X_{k-2}, X_k]$ such that

$$d_{\mathrm{WP}}(\hat{r}^{\bar{k}}(t'_{k-2}), Y_k) \leq \frac{D}{2^6}.$$

On the other hand, by Proposition 3.4, we have

$$\begin{aligned} t_{k-2} + \frac{D}{8} < t'_{k-2} &= (t'_{k-2} - t_{k-1}) + (t_{k-1} - t'_{k-1}) + (t'_{k-1} - t_k) + t_k \\ &< t_k - \frac{3D}{8}. \end{aligned}$$

Therefore, by the triangle inequality, we see that

$$\begin{aligned} d_{\mathrm{WP}}(Y_k, X_{k-2}) &\geq (t'_{k-2} - t_{k-2}) - d_{\mathrm{WP}}(\hat{r}^{\bar{k}}(t'_{k-2}), Y_k) - d_{\mathrm{WP}}(X_{k-2}, \hat{r}^{\bar{k}}(t_{k-2})) \\ &> \frac{D}{8} - \frac{D}{2^6} - \frac{D}{2^6} > \frac{D}{16}. \end{aligned}$$

Similar computations show that

$$d_{\mathrm{WP}}(Y_k, X_k) \geq \frac{D}{4}.$$

So, Y_k is further than $\frac{D}{16}$ from the endpoints of $[X_{k-2}, X_k]$, and so less than $\frac{7D}{16}$ from the midpoint. In particular, the closed ball of radius $\frac{D}{32}$ in $\overline{\mathrm{Teich}(S_{\bar{k}})}$ about Y_k is contained in the closed ball $B_k \subset \overline{\mathrm{Teich}(S_{\bar{k}})}$ of radius $\frac{15D}{32}$ centered at the midpoint M_k of $[X_{k-2}, X_k]$.

We claim that $B_k \subset \mathrm{Teich}(S_{\bar{k}})$ (that is, B_k contains no completion points), and hence B_k is compact. To prove the claim, it suffices to show that the closest point to M_k in $\overline{\mathrm{Teich}(S_{\bar{k}})} \setminus \mathrm{Teich}(S_{\bar{k}})$ is one of the endpoints X_{k-2} or X_k. For this, let

$$X \in \overline{\mathrm{Teich}(S_{\bar{k}})} \setminus (\mathrm{Teich}(S_{\bar{k}}) \cup \{X_{k-2}, X_k\})$$

be any completion point. According to [4, Lemma 3.2], we have that $D = d_{\mathrm{WP}}(X_{k-2}, X_k) \leq d_{\mathrm{WP}}(X_k, X)$. Since triangles in $\overline{\mathrm{Teich}(S_{\bar{k}})}$ are nondegenerate (meaning that edges meet only in a vertex), M_k is not contained in the geodesic segment $[X_k, X]$. Thus, the (strict) triangle inequality implies

$$2d_{\mathrm{WP}}(M_k, X_k) = D \leq d_{\mathrm{WP}}(X_k, X) < d_{\mathrm{WP}}(X_k, M_k) + d_{\mathrm{WP}}(M_k, X).$$

Therefore, $d_{\mathrm{WP}}(M_k, X_k) < d_{\mathrm{WP}}(M_k, X)$.

We now see that the closed ball of radius $\frac{D}{32}$ about Y_k is contained in the $\mathrm{Mod}(S_{\bar{k}})$-orbit of a single compact set in $\mathrm{Teich}(S_{\bar{k}})$, namely the closed ball of radius $\frac{15D}{32}$ about the midpoint of a single Farey edge. Therefore, the length of γ_k (the curve pinched at X_k) is uniformly bounded in the $\frac{D}{32}$-neighborhood of Y_k, independent of k (and independent of the sequence $\{e_k\}$). Since $\hat{r}^{\bar{k}}(t'_{k-2})$ lies in this neighborhood, $\ell_{\gamma_k}(\hat{r}^{\bar{k}}(t'_{k-2}))$ is uniformly bounded, as required.

The proof of the bound on $\ell_{\gamma_k}(\hat{r}^{\bar{k}}(t'_{k+1}))$ is entirely analogous, using the geodesic segment $[X_k, X_{k+2}]$ in place of $[X_{k-2}, X_k]$. The very last statement follows from convexity of length-functions along WP geodesics [32, Sec. 3.3]. $\square$

Corollary 3.6. *For all $k \geq 2$ and $j = k-1, k, k+1, k+2$, we have*

$$\ell_{\gamma_j}(\hat{r}(t'_k)) \leq C.$$

Proof. According to Lemma 3.5, the curve γ_j has length at most C on the interval $[t'_{j-2}, t'_{j+1}]$. The corollary thus follows from the fact that

$$\{t'_k\} = \bigcap_{j=k-1}^{k+2} [t'_{j-2}, t'_{j+1}].$$

$\square$

3.3. *Intersection number estimates*

We will require the following estimate for the intersection number of a curve δ and the curves γ_i^h in terms of the numbers e_i^h, $i \geq 0$.

Lemma 3.7. *Given $\delta \in \mathcal{C}_0(S)$ with $i(\delta, \alpha) \neq 0$, there exists $\kappa = \kappa(\delta) \geq 1$ so that for $h = 0, 1$ and all i sufficiently large we have*

$$\frac{1}{\kappa} e_{i-1}^h \leq i(\delta, \gamma_i^h) \leq \kappa I_h(i),$$

for $I_h(i) = \sum_J \prod_{j \in J} e_j^h$ where J runs over all subsets of $\{0, \ldots, i-1\}$ exactly once.

Proof. Suppose that $\delta \cap S_h$, $h = 0, 1$, consists of n_h geometric arcs with end points on $\alpha = \partial S_h$ (geometric arcs are proper arcs on the surface and homotopic geometric arcs are not identified). Let $[0; e_1^h, e_2^h, \ldots, e_{i-1}^h, \ldots]$ be a continued fraction expansion as in Sec. 3.1 and recall that the curve γ_i^h has slope reciprocal to $\frac{p_i^h}{q_i^h}$ the ith convergent of the continued fraction expansion. Let τ_h be a geometric arc in $\delta \cap S_h$ with the largest intersection number with γ_i^h and let $\frac{a_h}{b_h}$ be the reciprocal of the slope of τ_h. Then, since $\gamma_i^h \subset S_h$, we have that $\mathrm{i}(\tau_h, \gamma_i^h) = |a_h q_i^h - b_h p_i^h|$: to see this, observe that orienting γ_i^h and τ_h, these represent the (relative) homology classes (a_h, b_h) and (p_i^h, q_i^h), respectively, in $H_1(S_h, \partial S_h; \mathbb{Z}) \cong Z^2$, and $\mathrm{i}(\tau_h, \gamma_i^h)$ is the absolute value of the algebraic intersection number, which is the geometric intersection number on a punctured torus.

The standard recursive formula for convergents of continued fraction expansions gives us $q_i^h = e_{i-1}^h q_{i-1}^h + q_{i-2}^h$ (see e.g. [15, Theorem 1]; recall our index convention in Sec. 3.1), we also have that $q_0^h = 1$ and $q_1^h = e_0^h$. Then we can easily verify by induction on i that

$$q_i^h = \sum_{J \subseteq \{0,\dots,i-1\}} \prod_{j \in J} e_j^h, \tag{3.6}$$

where each subset J appears at most once in the sum. Now, since $\lim_{i\to\infty} \frac{p_i^h}{q_i^h} = x_h$, where the irrational number x_h is the reciprocal of the slope of λ_h, we have

$$\lim_{i\to\infty} \frac{|a_h q_i^h - b_h p_i^h|}{q_i^h} = \lim_{i\to\infty} \left| a_h - \left(\frac{p_i^h}{q_i^h}\right) b_h \right| = |a_h - x_h b_h|.$$

Thus, for i sufficiently large

$$|a_h q_i^h - b_h p_i^h| \le 2|a_h - x_h b_h| q_i^h \le 2|a_h - x_h b_h| I_h(i).$$

Then since τ_h is a geometric arc with the largest intersection number with γ_i^h and since there are n_h geometric arcs in $\delta \cap S_h$ we have

$$\mathrm{i}(\delta, \gamma_i^h) \le 2n_h |a_h - x_h b_h| I_h(i). \tag{3.7}$$

Furthermore, by (3.6), $q_i^h \ge \prod_{j=0}^{i-1} e_j^h \ge e_{i-1}^h$. From this inequality and the above limit, we deduce that the inequality

$$|a_h q_i^h - b_h p_i^h| \ge \frac{1}{2}|a_h - x_h b_h| q_i^h \ge \frac{1}{2}|a_h - x_h b_h| e_{i-1}^h \tag{3.8}$$

holds for all i sufficiently large.

Now from inequalities (3.7) and (3.8), we see that the inequalities of the lemma hold for $\kappa = \max\{2n_h|a_h - x_h b_h|, \frac{2}{|a_h - x_h b_h|} : h = 0, 1\}$. $\square$

For any $k \in \mathbb{N}$, appealing to the Notation 3.3, let $I(k) = I_{\bar{k}}(i)$ where $k = 2i + \bar{k}$. The conclusion of Lemma 3.7 then becomes

$$\frac{1}{\kappa} e_{k-2} \le i(\delta, \gamma_k) \le \kappa I(k). \tag{3.9}$$

For the remainder of the paper, we assume that the sequence $\{e_k\}_k$ satisfies the additional growth condition

$$\lim_{k\to\infty} \frac{I(k)}{e_k} = 0, \quad \text{and} \quad \lim_{k\to\infty} \frac{I(k+1)}{e_k} = 0. \tag{3.10}$$

This is possible since $I(k)$ depends only on $\{e_j\}_{j=0}^{k-2}$.

With this convention, we have the following corollary of Lemma 3.7.

Corollary 3.8. *For any curve $\delta \in \mathcal{C}_0(S)$ with $i(\delta, \alpha) \neq 0$, we have*

$$\lim_{k\to\infty} \frac{i(\delta, \gamma_k)}{e_k} = 0, \quad \textit{and} \quad \lim_{k\to\infty} \frac{i(\delta, \gamma_{k+1})}{e_k} = 0.$$

3.4. *Geodesics in Teich(S) and bounded length curves*

We begin by recalling [27, Corollary 3.5] and the inequality inside its proof, which we will use in some of the estimates in this section.

Lemma 3.9. *Given $c > 0$ let $l, a \in [0, c]$ with $l > a$. Suppose that for a curve $\beta \in \mathcal{C}_0(S)$ and points $X, X' \in \mathrm{Teich}(S)$ we have $\ell_\beta(X) \leq l - a$ and $\ell_\beta(X') \geq l$, then*

$$d_{\mathrm{WP}}(X, X') \geq \frac{a}{\sqrt{\frac{2}{\pi} l + O(l^4)}},$$

where the constant of the O-notation depends only on c.

Lemma 3.10. *There is an $\epsilon_1 > 0$ and a $C' > 0$ so that for all points Y in the ϵ_1-neighborhood of $\hat{r}^{\bar{k}}(t'_k)$ and all $j = k-1, k, k+1, k+2$, we have*

$$\ell_{\gamma_j}(Y) < C'.$$

Proof. Let C be the constant from Corollary 3.6 so that $\ell_{\gamma_j}(\hat{r}^{\bar{k}}(t'_k)) \leq C$ for $j = k-1, \ldots, k+2$. Let $a > 0, C' = a + C$ and $c = C' + a + 1$. Then

$$0 < a < C' < c \quad \text{and} \quad \ell_{\gamma_j}(\hat{r}^{\bar{k}}(t'_k)) \leq C = C' - a,$$

for each $j = k-1, k, k+1, k+2$. Define

$$\epsilon_1 = \frac{a}{\sqrt{\frac{2}{\pi} C' + O(C'^4)}}$$

to be as in Lemma 3.9 where the constant of the O-notation only depends on c. Now, if $\ell_{\gamma_j}(Y) \geq C'$ for a point Y in the ϵ_1-neighborhood of $\hat{r}^{\bar{k}}(t'_k)$ and a curve γ_j with $j = k-1, k, k+1, k+2$, then applying Lemma 3.9 we have $d_{\mathrm{WP}}(Y, \hat{r}^{\bar{k}}(t'_k)) \geq \epsilon_1$ which contradicts the fact that Y is in the ϵ_1-neighborhood of $\hat{r}^{\bar{k}}(t'_k)$. Therefore, $\ell_{\gamma_j}(Y) < C'$, for $j = k-1, k, k+1, k+2$, proving the lemma. □

Decreasing ϵ_1 if necessary, we may further assume that for any point $X \in \mathrm{Teich}(S)$ in the $2\epsilon_1$-neighborhood of $\mathcal{S}(\alpha)$ and any curve γ essentially intersecting α, we have that $\ell_\gamma(X)$ is uniformly bounded below. This follows from Lemma 2.2 and the fact that the distance to $\mathcal{S}(\alpha)$ is $(2\pi\ell_\alpha)^{\frac{1}{2}} + O(\ell_\alpha^2)$ (see [31, Corollary 4.10]). In particular, any point in $\overline{\mathrm{Teich}(S)}$ in the $2\epsilon_1$-neighborhood may only lie on a stratum corresponding to a (possibly empty) multicurve having zero intersection number with α.

Now, let $Z \in \operatorname{Teich}(S)$ be a point in the ϵ_1-neighborhood of $\hat{r}(0)$. Let then $[Z, \hat{r}(t'_k)]$ be the geodesic segment connecting Z to $\hat{r}(t'_k)$. By Corollary 3.6, the curves γ_k, γ_{k+1} have bounded lengths at $\hat{r}(t'_k)$ and the sequence of curves $\{\gamma_i^{\bar{k}}\}_i$, $\bar{k} = 0, 1$, is a quasi-geodesic in $\mathcal{C}(S_{\bar{k}})$ that converges to a point in the Gromov boundary of $\mathcal{C}(S_{\bar{k}})$. Moreover, $S\backslash\alpha$ is the union of S_0 and S_1. Then as in [27, Lemma 8.1] we can show that after possibly passing to a subsequence $[Z, \hat{r}(t'_k)]$ converges uniformly on compact subsets to an infinite ray

$$r : [0, \infty) \to \operatorname{Teich}(S).$$

Also, note that the construction of r and the CAT(0) property of the WP metric imply the rays r and $\hat{r}$, ϵ_1-fellow travel.

The following are straightforward consequences of the results of this section.

Corollary 3.11. *For all $k \geq 2$ and $j = k-1, k, k+1, k+2$, we have*

$$\ell_{\gamma_j}(r(t'_k)) \leq C',$$

where $C' > 0$ is the constant from Lemma 3.10.

Proof. As noted above, the two geodesics rays r and $\hat{r}$, ϵ_1-fellow travel, and hence $d_{\mathrm{WP}}(\hat{r}(t'_k), r(t'_k)) < \epsilon_1$. Thus, by Lemma 3.10, for $j = k-1, k, k+1, k+2$, we have

$$\ell_{\gamma_j}(r(t'_k)) \leq C',$$

as desired. □

This, in turn, implies the following corollary.

Corollary 3.12. *For all $k \geq 2$, $t \in [t'_k, t'_{k+1}]$, and $j = k, k+1, k+2$,*

$$\ell_{\gamma_j}(r(t)) \leq C'$$

where $C' > 0$ is the constant from Lemma 3.10.

Proof. By Corollary 3.11, we have $\ell_{\gamma_k}(r(t'_{k-2})) \leq C'$ and $\ell_{\gamma_k}(r(t'_{k+1})) \leq C'$. By convexity of length-functions [32, Sec. 3.3], for all $t \in [t'_{k-2}, t'_{k+1}]$, we have

$$\ell_{\gamma_k}(r(t)) \leq C'.$$

Since $[t'_k, t'_{k+1}] \subset [t'_{k-2}, t'_{k+1}] \cap [t'_{k-1}, t'_{k+2}] \cap [t'_k, t'_{k+3}]$, the result follows. □

Proposition 3.13. *The length of α is bounded by $\ell_\alpha(Z)$ along r. Furthermore, the ending lamination of r is the lamination $\lambda_0 \cup \lambda_1$ or $\lambda_0 \cup \alpha \cup \lambda_1$.*

Proof. First note that by convexity of ℓ_α, [32, Sec. 3.3], and the fact that $\ell_\alpha(\hat{r}(t'_k)) = 0$, it follows that ℓ_α is bounded by $\ell_\alpha(Z)$ on $[Z, \hat{r}(t'_k)]$. Since r is a limit of a subsequence of the geodesics $[Z, \hat{r}(t'_k)]$, the first claim of the proposition holds.

By Corollary 3.11, the curves γ_k, γ_{k+1} have bounded length at $r(t'_k)$ for all k, hence by the definition of ending lamination, λ_0 and λ_1 are contained in the ending lamination of r. Note that the only measurable lamination properly containing $\lambda_0 \cup \lambda_1$ is $\lambda_0 \cup \alpha \cup \lambda_1$, and so $\nu^+(r)$ must be one of these two laminations (and it is the latter one if and only if $\ell_\alpha(r(t)) \to 0$ as $t \to \infty$, i.e. if α is a pinching curve). $\square$

We now turn to estimates for twists about bounded length curves at $r(t'_k)$.

Lemma 3.14. *For any $\delta \in \mathcal{C}_0(S)$ with $i(\delta, \alpha) \neq 0$, there exists $c = c(\delta) > 0$ such that for all $t \in [t'_k, t'_{k+1}]$, we have*

$$\mathrm{tw}_{\gamma_k}(\delta, r(t)) \overset{+}{\asymp}_c e_k,$$

and

$$\mathrm{tw}_{\gamma_{k+1}}(\delta, r(t'_k)) \overset{+}{\asymp}_c 1$$

for all but finitely many k (namely, whenever $i(\gamma_k, \delta) \neq 0$ and $i(\gamma_{k+1}, \delta) \neq 0$, respectively).

Proof. By Corollary 3.12, we may choose a bounded length marking μ at $r(t)$ so that γ_k is in the base and γ_{k+2} projects to the transversal to γ_k. Recall that $\bar{k} \in \{0, 1\}$ is the residue of k modulo 2. Avoiding finitely many k, $i(\delta, \gamma_k) \neq 0$, and we may apply the triangle inequality. Doing so we have

$$|d_{\gamma_k}(\delta, \gamma_{k+2}) - d_{\gamma_k}(\gamma_{\bar{k}}, \gamma_{k+2})| \leq d_{\gamma_k}(\delta, \gamma_{\bar{k}}). \tag{3.11}$$

Since $\mu|_{S_{\bar{k}}}$ is a uniformly bounded length marking at $r(t)$, we have uniform errors (independent of k) in the following coarse equations. First, by (2.3), we have

$$\mathrm{tw}_{\gamma_k}(\delta, r(t)) \overset{+}{\asymp} d_{\gamma_k}(\delta, \mu) = d_{\gamma_k}(\delta, \gamma_{k+2}). \tag{3.12}$$

Since $\gamma_{k+2} = D_{\gamma_k}^{\pm e_k}(\gamma_{k-2})$, it follows from [25, Eq. (2.6)]) that

$$d_{\gamma_k}(\gamma_{k-2}, \gamma_{k+2}) \overset{+}{\asymp} e_k.$$

Furthermore, because $\{\gamma_{\bar{k}+2i}\}_i$ are the vertices of a geodesic in $\mathcal{C}(S_{\bar{k}})$, by [25, Theorem 3.1], we have

$$d_{\gamma_k}(\gamma_{\bar{k}}, \gamma_{k+2}) \overset{+}{\asymp} d_{\gamma_k}(\gamma_{k-2}, \gamma_{k+2}) = e_k. \tag{3.13}$$

Moreover, since we are allowing our error $c = c(\delta)$ to depend on δ, we can combine the coarse equations (3.12) and (3.13) with inequality (3.11) and deduce

$$\mathrm{tw}_{\gamma_k}(\delta, r(t)) \overset{+}{\asymp} e_k.$$

This proves the first coarse equation of the lemma.

To prove the second coarse equation, we note that by Corollary 3.11, we may choose our bounded length marking μ at $r(t'_k)$ so that γ_{k+1} is a base curve and

γ_{k-1} projects to a transversal for γ_{k+1}. Thus, similar to Eq. (3.12), we see that (2.3) implies $\mathrm{tw}_{\gamma_{k+1}}(\delta, r(t'_k)) \overset{+}{\asymp} d_{\gamma_{k+1}}(\delta, \gamma_{k-1})$. Furthermore, similar to (3.11), for k sufficiently large we have

$$|d_{\gamma_{k+1}}(\delta, \gamma_{k-1}) - d_{\gamma_{k+1}}(\gamma_{\overline{k+1}}, \gamma_{k-1})| \leq d_{\gamma_{k+1}}(\delta, \gamma_{\overline{k+1}}),$$

Since $\gamma_{\overline{k+1}}$ and γ_{k-1} precede γ_{k+1} in the $\mathcal{C}(S_{\overline{k+1}})$-geodesic, appealing to [25, Theorem 3.1] again we have

$$d_{\gamma_{k+1}}(\gamma_{\overline{k+1}}, \gamma_{k-1}) \overset{+}{\asymp} 1.$$

Combining these facts just as in the previous paragraph and increasing $c = c(\delta)$ if necessary, we have

$$\mathrm{tw}_{\gamma_{k+1}}(\delta, r(t'_k)) \overset{+}{\asymp}_c 1,$$

which completes the proof of the lemma. □

3.5. *Estimates for the separating curve*

We will eventually impose additional growth conditions on our sequence $\{e_k\}$ to control the length and twisting about the separating curve α. The next two lemmas are used to determine those conditions.

Lemma 3.15. *There exists a function $f_1 : [0, \infty) \to \mathbb{R}^+$, so that for any geodesic ray r constructed as above (from sequences $\{e_i^h\}_i$, $h = 0, 1$, beginning at Z) we have $\ell_\alpha(r(T)) \geq f_1(T)$ for all $T \in [0, \infty)$. Moreover, there exists such a function f_1 which is continuous.*

Proof. The proof is by contradiction. If there is no such function f_1 (not necessarily continuous), then there would be a sequence of geodesics $\{r_n\}$ starting at Z, coming from sequences $\{e_i^h(n)\}_i$, as above, and some $T > 0$ so that $\ell_\alpha(r_n(T)) \to 0$ as $n \to \infty$. Now, the idea of the proof is as follows. Appealing to convexity of ℓ_α on r_n, we can deduce that $\ell_\alpha(r_n(t)) \to 0$ as $n \to \infty$ for all $t \geq T$. In particular, choosing any $T' > T$, we can apply Theorem 2.4 to $r_n|_{[0,T']}$. We will see that the curve α is (eventually) present in all the multicurves from the theorem, producing a contradiction to the non-refraction behavior ensured by the Theorem 2.4. We now proceed to the details.

Recall that we have chosen $Z \in \mathrm{Teich}(S)$ and $\epsilon_1 > 0$ from Lemma 3.10 (and the paragraph following its proof) so that the distance to any stratum $\mathcal{S}(\sigma)$ is at least ϵ_1 whenever σ has nonzero intersection number with α. By Proposition 3.13, $\ell_\alpha(r_n(t)) \leq \ell_\alpha(Z)$ for all n and $t \geq 0$, so since $\lim_{n\to\infty} \ell_\alpha(r_n(T)) = 0$, convexity of ℓ_α implies $\ell_\alpha(r_n(t)) \leq \ell_\alpha(r_n(T))$ for all n sufficiently large and all $t \geq T$. In particular, $\lim_{n\to\infty} \ell_\alpha(r_n(t)) = 0$ for all $t \geq T$, while $\ell_\alpha(r_n(0)) = \ell_\alpha(Z) > 0$.

Now fix any $T' > T$ and apply Theorem 2.4 to the sequence of geodesic segments $r_n|_{[0,T']}$. Let the partition $0 = t_0 < t_1 < \cdots < t_{k+1} = T'$, the piecewise geodesic path $\hat{\zeta} \colon [0, T'] \to \overline{\mathrm{Teich}(S)}$, the multicurves $\{\sigma_l\}_{l=1}^{k+1}$, the multitwists $\{\mathcal{T}_{l,n}\}_{l=1}^{k}$, and the mapping classes $\{\varphi_{l,n}\}_{l=1}^{k}$ obtained by composing the multitwists be from the theorem.

For each $1 \leq l \leq k$ and $n \geq 1$, $\mathcal{T}_{l,n}$ is the composition of powers of Dehn twists about curves in σ_l, but since r_n has distance at least ϵ_1 from all completion strata except strata of multicurves having zero intersection number with α, σ_l consists of possibly the curve α and a number of curves disjoint from α. Therefore, $\varphi_{l,n}(\alpha) = \alpha$, and $\ell_\alpha(\varphi_{l,n}(r_n(t))) = \ell_\alpha(r_n(t))$ for all $t \in [t_l, t_{l+1}]$ and all l. According to Theorem 2.4, we have $\hat{\zeta}(t) \in \operatorname{Teich}(S)$ for all $t \in [T, T']$ except possibly the points $\{t_l\}_{l=0}^{k+1} \cap [T, T']$. Therefore, $\ell_\alpha(\hat{\zeta}(t)) > 0$ for all these values of t. Applying part (2) of the theorem to any such value of t, we have

$$\lim_{n\to\infty} \ell_\alpha(r_n(t)) = \lim_{n\to\infty} \ell_\alpha(\varphi_{l,n}(r_n(t))) = \ell_\alpha(\hat{\zeta}(t)) > 0.$$

This contradicts the fact that $\lim_{n\to\infty} \ell_\alpha(r_n(t)) = 0$ for $t \in [T, T']$. Therefore, $\ell_\alpha(r(T))$ is bounded below by a positive number, depending on T, but independent of the ray r. Thus, we have a function f_1, not necessarily continuous, so that $\ell_\alpha(r(T)) \geq f_1(T)$ for all $T \geq 0$. Since $\ell_\alpha(r(t))$ is decreasing, it is easy to construct a continuous function f_1 which also has this property. □

Lemma 3.16. *There exists a function $f_2 : [0, \infty) \to \mathbb{R}^+$ such that for any geodesic ray r as above $d_\alpha(r(0), r(t)) \leq f_2(t)$ for all $t \in [0, \infty)$. Moreover, there exists such a function f_2 which is continuous.*

Proof. Suppose that such a function does not exist (not necessarily continuous). Then there is a sequence of geodesic rays r_n constructed as above and a $T > 0$, so that $\lim_{n\to\infty} d_\alpha(r_n(0), r_n(T)) = \infty$. Then, since $r_n(0) = Z$ for all $n \geq 1$, we have that $\sup_t \ell_\alpha(r_n(t)) \geq \operatorname{inj}(Z) > 0$. Then, by Theorem 2.6, we have that $\inf_{t\in[0,T]} \ell_\alpha(r_n(t)) \to 0$ as $n \to \infty$. But this contradicts the fact that $\inf_{t\in[0,T]} \ell_\alpha(r_n(t)) \geq \inf_{t\in[0,T]} f_1(t) > 0$ for all $n \geq 1$ by Lemma 3.16. Existence of f_2 now follows from this contradiction. Restricting the argument to a subinterval $[0, T'] \subset [0, T]$, we see that we can replace f_2 by an increasing function, and then by a continuous function, retaining the required property. □

With these two lemmas in place, we now impose our final growth conditions on $\{e_k\}_k$. Let f_1, f_2 be the functions from Lemmas 3.15 and 3.16, and for $k \in \mathbb{N}$ let

$$F_{1,k} = \min\{f_1(s) \,|\, s \in [t_k, t_{k+2}]\}, \tag{3.14}$$

$$F_{2,k} = \max\{f_2(s) \,|\, s \in [t_k, t_{k+2}]\}. \tag{3.15}$$

As our last growth requirement for $\{e_k\}_k$, we assume that e_k grows fast enough that

$$\lim_{k\to\infty} \frac{F_{2,k} - 2\log F_{1,k}}{e_k} = 0. \tag{3.16}$$

3.6. *Limit sets*

For the remainder of the paper, we let $\{e_i^h\}_{i=0}^{\infty}$ be a sequence such that $e_i^h \geq K_i$, for $h = 0, 1$ and all $i \in \mathbb{N}$, where K_i is from Lemma 3.2. Let $\{e_k\}_k$, $\{\gamma_k\}_k$, $\{t_k\}_k$, $\{t'_k\}_k$, $\{X_k\}_k$ be as in Notation 3.3, and assume that $\{e_k\}_k$ satisfies (3.10) and (3.16). The following immediately implies Theorem 3.1.

Theorem 3.17. *The limit set of r in the Thurston compactification of* $\mathrm{Teich}(S)$ *is the* 1*-simplex* $[[\bar{\lambda}_0], [\bar{\lambda}_1]]$ *of projective classes of measures supported on* $\lambda_0 \cup \lambda_1$.

For curves $\delta, \gamma \in \mathcal{C}_0(S)$ and any time $s \in [0, \infty)$, as in (2.4), let

$$\ell_\delta(\gamma, s) = \ell_\delta(\gamma, r(s)) = \mathrm{i}(\delta, \gamma)(w_\gamma(r(s)) + \ell_\gamma(r(s))\,\mathrm{tw}_\gamma(\delta, r(s))). \tag{3.17}$$

Now, suppose that $\{s_k\}_k$ is a sequence such that $s_k \in [t'_k, t'_{k+1}]$. Pass to a subsequence $\{s_k\}_{k\in\mathcal{K}}$ so that $r(s_k) \to [\bar{\nu}]$ in the Thurston compactification (to avoid cluttering the notation with additional subscripts, we have chosen to index a subsequence using a subset $\mathcal{K} \subset \mathbb{N}$). Let $\{u_k\}_{k\in\mathcal{K}}$ be a scaling sequence, so that

$$\lim_{k\to\infty} u_k \ell_\delta(r(s_k)) = i(\delta, \bar{\nu}),$$

for all curves δ.

By Corollary 3.12 and Proposition 3.13, the curves $\gamma_k, \gamma_{k+1}, \alpha$ form a uniformly bounded length pants decomposition on $r(s_k)$. Consequently, by Theorem 2.3, we obtain the following expansion for the length of the curve $\delta \in \mathcal{C}_0(S)$ at $r(s_k)$,

$$\begin{aligned} \ell_\delta(r(s_k)) &= \ell_\delta(\gamma_k, s_k) + \ell_\delta(\gamma_{k+1}, s_k) + \ell_\delta(\alpha, s_k) \\ &\quad + O(\mathrm{i}(\delta, \gamma_k)) + O(\mathrm{i}(\delta, \gamma_{k+1})) + O(\mathrm{i}(\delta, \alpha)), \end{aligned} \tag{3.18}$$

where the constant of the O notation depends only on the uniform upper bounds for the lengths of γ_k, γ_{k+1}, and α.

The next proposition shows that only two of the terms in (3.18) are actually relevant.

Proposition 3.18. *With notation as above, and $\delta \in \mathcal{C}_0(S)$ with $i(\delta, \alpha) \neq 0$, we have*

$$i(\delta, \bar{\nu}) = \lim_{k\to\infty} u_k \ell_\delta(r(s_k)) = \lim_{k\to\infty} u_k(\ell_\delta(\gamma_k, s_k) + \ell_\delta(\gamma_{k+1}, s_k)).$$

For this, we will need the following lemma.

Lemma 3.19. *With notation as above,*

$$w_{\gamma_k}(r(s_k)) + \ell_{\gamma_k}(r(s_k))\,\mathrm{tw}_{\gamma_k}(\delta, r(s_k)) \stackrel{+}{\asymp} \ell_{\gamma_k}(r(s_k)) e_k,$$

where the constant in the coarse equation depends on δ, but not on k.

Proof. By Corollary 3.12,

$$\ell_{\gamma_k}(r(s_k)) \le C' \quad \text{and} \quad \ell_{\gamma_{k+2}}(r(s_k)) \le C'.$$

Then since $\mathrm{i}(\gamma_k, \gamma_{k+2}) = 1$, $\ell_{\gamma_k}(r(s_k))$ is also uniformly bounded below, for otherwise, by Lemma 2.2, $\ell_{\gamma_{k+2}}(r(s_k))$ would be unbounded. So we have $\ell_{\gamma_k}(r(s_k)) \stackrel{*}{\asymp} 1$ and hence again by Lemma 2.2 we have $w_{\gamma_k}(r(s_k)) \stackrel{*}{\asymp} 1$. Moreover, by Lemma 3.14, we have $\mathrm{tw}_{\gamma_k}(\delta, r(s_k)) \stackrel{+}{\asymp} e_k$, and so the lemma follows. ◻

Proof of Proposition 3.18. First, observe that by Corollary 3.8 (and since α is a fixed curve and $e_k \to \infty$), we have

$$\lim_{k\to\infty} \frac{\mathrm{i}(\delta, \gamma_k)}{e_k} = 0, \quad \lim_{k\to\infty} \frac{\mathrm{i}(\delta, \gamma_{k+1})}{e_k} = 0, \quad \text{and} \quad \lim_{k\to\infty} \frac{\mathrm{i}(\delta, \alpha)}{e_k} = 0. \tag{3.19}$$

As in the proof of Lemma 3.19, $\ell_{\gamma_k}(r(s_k)) \stackrel{*}{\asymp} 1$, and so $\ell_{\gamma_k}(r(s_k))e_k \to \infty$ and $\mathrm{i}(\delta, \gamma_k) \to \infty$. From Lemma 3.19 and (3.17), we have $\ell_\delta(\gamma_k, s_k) \stackrel{*}{\succ} e_k$. Moreover, $u_k \ell_\delta(r(s_k)) \to \mathrm{i}(\delta, \bar{\nu}) > 0$, then by (3.18), $u_k \stackrel{*}{\prec} \frac{1}{e_k}$. Combining this with (3.19) and appealing to (3.18) again, we see that

$$i(\delta, \bar{\nu}) = \lim_{k\to\infty} u_k \ell_\delta(r(s_k)) = \lim_{k\to\infty} u_k(\ell_\delta(\gamma_k, s_k) + \ell_\delta(\gamma_{k+1}, s_k) + \ell_\delta(\alpha, s_k)).$$

By similar reasoning, to eliminate the last term (and thus prove the proposition), it suffices to prove

$$\lim_{k\to\infty} \frac{\ell_\delta(\alpha, s_k)}{e_k} = 0. \tag{3.20}$$

To do this, first note that by Lemma 3.15, $\ell_\alpha(r(s_k)) \ge f_1(s_k) \ge F_{1,k}$, and so by Lemma 2.2 we have

$$w_\alpha(r(s_k)) \stackrel{+}{\asymp} -2\log(\ell_\alpha(r(s_k))) \le -2\log(F_{1,k}).$$

By Lemma 3.16, we also have

$$\mathrm{tw}_\alpha(\delta, r(s_k)) \stackrel{+}{\asymp} d_\alpha(r(0), r(s_k)) \le f_2(s_k) \le F_{2,k},$$

where the additive constant depends on δ. Therefore, since $F_{1,k} \stackrel{+}{\prec} 1$, and since $\ell_\alpha(r(s_k))$ is uniformly bounded by Proposition 3.13, we have

$$\ell_\delta(\alpha, s_k) \stackrel{+}{\prec} \mathrm{i}(\delta, \alpha)(-2\log(F_{1,k}) + F_{2,k}),$$

with additive error that again depends on δ. By our growth condition (3.16), since $\mathrm{i}(\delta, \alpha)$ does not depend on k, there is a constant $c' > 0$ so that

$$\lim_{k\to\infty} \frac{\ell_\delta(\alpha, s_k)}{e_k} \le \lim_{k\to\infty} \frac{\mathrm{i}(\delta, \alpha)(-2\log(F_{1,k}) + F_{2,k}) + c'}{e_k} = 0.$$

This proves (3.20), and hence the proposition. ◻

Continue to let $\{s_k\}_{k\in\mathcal{K}}$ be a sequence with $s_k \in [t'_k, t'_{k+1}]$ as above, and suppose that $r(s_k) \to [\bar{\nu}]$ as $k \to \infty$ in the Thurston compactification and that $\{u_k\}_k$ is a scaling sequence. We set

$$x(s_k) = w_{\gamma_k}(r(s_k)) + \ell_{\gamma_k}(r(s_k))\,\mathrm{tw}_{\gamma_k}(\gamma_{\bar{k}}, r(s_k)),$$

and

$$y(s_k) = w_{\gamma_{k+1}}(r(s_k)) + \ell_{\gamma_{k+1}}(r(s_k))\,\mathrm{tw}_{\gamma_{k+1}}(\gamma_{\overline{k+1}}, r(s_k)).$$

Lemma 3.20. *For any $\delta \in \mathcal{C}_0(S)$ with $i(\delta,\alpha) \neq 0$, we have*

$$\lim_{k\to\infty} \frac{x(s_k)i(\delta,\gamma_k) + y(s_k)i(\delta,\gamma_{k+1})}{\ell_{\gamma_k}(\delta,s_k) + \ell_{\gamma_{k+1}}(\delta,s_k)} = 1.$$

For $s_k = t'_k$, we have

$$\lim_{k\to\infty} \frac{x(t'_k)i(\delta,\gamma_k)}{\ell_{\gamma_k}(\delta,t'_k) + \ell_{\gamma_{k+1}}(\delta,t'_k)} = 1.$$

Proof. As in the proof of Lemma 3.14

$$\mathrm{tw}_{\gamma_k}(\delta, r(s_k)) \stackrel{+}{\asymp} \mathrm{tw}_{\gamma_k}(\gamma_{\bar{k}}, r(s_k))$$

and

$$\mathrm{tw}_{\gamma_{k+1}}(\delta, r(s_k)) \stackrel{+}{\asymp} \mathrm{tw}_{\gamma_k}(\gamma_{\overline{k+1}}, r(s_k))$$

where the implicit constant in these coarse equations depends on δ.

According to Corollary 3.12, $\ell_{\gamma_k}(r(s_k)) \leq C'$. From the preceding coarse equations and Lemma 3.19, we have

$$x(s_k) \stackrel{+}{\asymp} w_{\gamma_k}(r(s_k)) + \ell_{\gamma_k}(r(s_k))\,\mathrm{tw}_{\gamma_k}(\delta, r(s_k)) \stackrel{+}{\asymp} \ell_{\gamma_k}(r(s_k))e_k. \tag{3.21}$$

Since $e_k \to \infty$ as $k \to \infty$, the following is immediate:

$$\lim_{k\to\infty} \frac{x(s_k)\mathrm{i}(\delta,\gamma_k)}{\ell_\delta(\gamma_k, s_k)} = \lim_{k\to\infty} \frac{x(s_k)}{w_{\gamma_k}(r(s_k)) + \ell_{\gamma_k}(r(s_k))\,\mathrm{tw}_{\gamma_k}(\delta, r(s_k))} = 1. \tag{3.22}$$

Similar to (3.21), we have

$$y(s_k) \stackrel{+}{\asymp} w_{\gamma_{k+1}}(r(s_k)) + \ell_{\gamma_{k+1}}(r(s_k))\,\mathrm{tw}_{\gamma_{k+1}}(\delta, r(s_k)) = \frac{\ell_{\gamma_{k+1}}(\delta, s_k)}{\mathrm{i}(\delta,\gamma_{k+1})}. \tag{3.23}$$

By (3.9) and the growth condition (3.10), we have

$$\lim_{k\to\infty} \frac{\mathrm{i}(\delta,\gamma_{k+1})}{e_k} = 0.$$

After passing to a subsequence, there are two cases to consider:

Case 1. There exists $R > 0$ so that $y(s_k) \leq R$ for all k.

In this case, appealing to (3.19) and (3.23), we have

$$0 = \lim_{k\to\infty} \frac{y(s_k)\mathrm{i}(\delta, \gamma_{k+1})}{e_k} = \lim_{k\to\infty} \frac{\ell_{\gamma_{k+1}}(\delta, s_k)}{e_k},$$

and thus

$$\lim_{k\to\infty} \frac{x(s_k)\mathrm{i}(\delta,\gamma_k) + y(s_k)\mathrm{i}(\delta,\gamma_{k+1})}{\ell_{\gamma_k}(\delta, s_k) + \ell_{\gamma_{k+1}}(\delta, s_k)} = \lim_{k\to\infty} \frac{\frac{x(s_k)\mathrm{i}(\delta,\gamma_k)}{e_k} + \frac{y(s_k)\mathrm{i}(\delta,\gamma_{k+1})}{e_k}}{\frac{\ell_{\gamma_k}(\delta,s_k)}{e_k} + \frac{\ell_{\gamma_{k+1}}(\delta,s_k)}{e_k}}$$

$$= \lim_{k\to\infty} \frac{x(s_k)\mathrm{i}(\delta,\gamma_k)}{\ell_{\gamma_k}(\delta, s_k)} = 1.$$

Case 2. $\lim_{k\to\infty} y(s_k) = \infty$.

Here, we can argue as for $x(s_k)$, appealing to (3.23) to deduce that

$$\lim_{k\to\infty} \frac{y(s_k)\mathrm{i}(\delta,\gamma_{k+1})}{\ell_{\gamma_{k+1}}(\delta, s_k)} = 1.$$

Combined with (3.22), we have

$$\lim_{k\to\infty} \frac{x(s_k)\mathrm{i}(\delta,\gamma_k) + y(s_k)\mathrm{i}(\delta,\gamma_{k+1})}{\ell_{\gamma_k}(\delta, s_k) + \ell_{\gamma_{k+1}}(\delta, s_k)} = 1.$$

These two cases prove the first claim of the lemma. For the second claim, when $s_k = t'_k$, we note that by Corollary 3.11 we have

$$\ell_{\gamma_{k-1}}(r(t'_k)), \quad \ell_{\gamma_{k+1}}(r(t'_k)) \le C',$$

and so by Lemma 2.2 (as in the proof of Lemma 3.19) we have

$$w_{\gamma_{k+1}}(r(t'_k)) \overset{*}{\asymp} 1 \quad \text{and} \quad \ell_{\gamma_{k+1}}(r(t'_k)) \overset{*}{\asymp} 1$$

so since $\mathrm{tw}_{\gamma_{k+1}}(\delta, r(t'_k)) \overset{+}{\asymp} 1$ by Lemma 3.14, it follows that $y(t'_k)$ is uniformly bounded, and thus as in Case 1, we deduce

$$\lim_{k\to\infty} \frac{x(t'_k)\mathrm{i}(\delta,\gamma_k)}{\ell_{\gamma_k}(\delta, t'_k) + \ell_{\gamma_{k+1}}(\delta, t'_k)} = 1,$$

completing the proof. □

We are now ready for the following Proof of Theorem 3.17.

Proof of Theorem 3.17. First, we show that $[\bar{\lambda}_0]$ and $[\bar{\lambda}_1]$ are in the limit set Λ of r. Consider the sequence of times $\{t'_{2k}\}$ and pass to a subsequence so that $r(t'_{2k}) \to [\bar{\nu}]$ in the Thurston compactification and let $\{u_k\}$ be a scaling sequence for $r(t'_{2k})$. Let δ be any curve with $\mathrm{i}(\delta, \bar{\nu}) \neq 0$ and $\mathrm{i}(\delta, \alpha) \neq 0$. By the second part of Lemma 3.20, together with Proposition 3.18, we have

$$\begin{aligned} 1 &= \lim_{k\to\infty} \frac{x(t'_{2k})\mathrm{i}(\delta,\gamma_{2k})}{\ell_{\gamma_{2k}}(\delta,t'_{2k}) + \ell_{\gamma_{2k+1}}(\delta,t'_{2k})} \\ &= \lim_{k\to\infty} \frac{u_k x(t'_{2k})\mathrm{i}(\delta,\gamma_{2k})}{u_k(\ell_{\gamma_{2k}}(\delta,t'_{2k}) + \ell_{\gamma_{2k+1}}(\delta,t'_{2k}))} \\ &= \frac{\lim_{k\to\infty} \mathrm{i}(\delta, u_k x(t'_{2k})\gamma_{2k})}{\mathrm{i}(\delta,\bar{\nu})}. \end{aligned}$$

Therefore, $\lim_{k\to\infty} \mathrm{i}(\delta, u_k x(t'_{2k})\gamma_{2k}) = \mathrm{i}(\delta,\bar{\nu})$. We apply this to a set of curves $\delta_1,\ldots,\delta_N$ sufficient for determining a measured lamination (see Sec. 2), and so deduce that $\lim_{k\to\infty} u_k x(t'_{2k})\gamma_{2k} = \bar{\nu}$.

On the other hand, $[\gamma_{2k}] \to [\bar{\lambda}_0]$, hence $[\bar{\nu}] = [\bar{\lambda}_0]$, and so $[\bar{\lambda}_0]$ is in Λ. A similar argument using the sequence $\{t'_{2k+1}\}$ shows that $[\bar{\lambda}_1] \in \Lambda$.

Now, suppose that $\{s_k\}_k$ is an arbitrary sequence so that $r(s_k) \to [\bar{\nu}]$ and let $\{u_k\}_k$ be a scaling sequence. Adjusting indices and passing to a subsequence we can assume that $s_k \in [t'_k, t'_{k+1}]$ for all $k \in \mathcal{K}$ (some subset $\mathcal{K} \subseteq \mathbb{N}$). Passing to a further subsequence, if necessary, we may assume that $\mathcal{K}$ is either a subsequence of even integers or odd integers. Arguing as above, appealing to the first part of Lemma 3.20 and Proposition 3.18, we have

$$\begin{aligned} 1 &= \lim_{k\to\infty} \frac{x(s_k)\mathrm{i}(\delta,\gamma_k) + y(s_k)\mathrm{i}(\delta,\gamma_{k+1})}{\ell_{\gamma_k}(\delta,s_k) + \ell_{\gamma_{k+1}}(\delta,s_k)} \\ &= \lim_{k\to\infty} \frac{u_k x(s_k)\mathrm{i}(\delta,\gamma_k) + u_k y(s_k)\mathrm{i}(\delta,\gamma_{k+1})}{u_k(\ell_{\gamma_k}(\delta,s_k) + \ell_{\gamma_{k+1}}(\delta,s_k))} \\ &= \frac{\lim_{k\to\infty} \mathrm{i}(\delta, u_k(x(s_k)\gamma_k + y(s_k)\gamma_{k+1}))}{\mathrm{i}(\delta,\bar{\nu})}. \end{aligned}$$

So, $\lim_{k\to\infty} u_k(x(s_k)\gamma_k + y(s_k)\gamma_{k+1}) = \mathrm{i}(\delta,\bar{\nu})$, and as above

$$\bar{\nu} = \lim_{k\to\infty} u_k(x(s_k)\gamma_k + y(s_k)\gamma_{k+1}) = \lim_{k\to\infty} u_k x(s_k)\gamma_k + \lim_{k\to\infty} u_k y(s_k)\gamma_{k+1}.$$

Now, if $\mathcal{K}$ is a subset of even integers, then since the projective classes of the curves with even indices converge to $[\bar{\lambda}_0]$, the first limit on the right-hand side above is a multiple of $\bar{\lambda}_0$, and since the projective classes of the curves with odd indices converge to $[\bar{\lambda}_1]$, the second limit above is a multiple of $\bar{\lambda}_1$, and hence $[\bar{\nu}] \in [[\bar{\lambda}_0],[\bar{\lambda}_1]]$. When $\mathcal{K}$ is a subset of odd integers we have a similar conclusion. This implies that Λ is contained in $[[\bar{\lambda}_0],[\bar{\lambda}_1]]$. Since Λ contains the endpoints and is connected, it is the entire 1-simplex, as was desired. $\square$

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