Journal of Topology 8 (2015) 1085–1118
doi:10.1112/jtopol/jtv031

The shadow of a Thurston geodesic to the curve graph

Anna Lenzhen, Kasra Rafi and Jing Tao

Abstract

We study the geometry of the Thurston metric on Teichmüller space by examining its geodesics and comparing them to Teichmüller geodesics. We show that, similar to a Teichmüller geodesic, the shadow of a Thurston geodesic to the curve graph is a reparameterized quasi-geodesic. However, we show that the set of short curves along the two geodesics are not identical.

1. *Introduction*

In [22], Thurston introduced a metric on Teichmüller space in terms of the least possible value of the global Lipschitz constant between two hyperbolic surfaces of finite volume. Even though this is an asymmetric metric, Thurston constructed geodesics connecting any pair of points in Teichmüller space that are concatenations of *stretch paths*. However, there is no unique geodesic connecting two points in Teichmüller space $\mathcal{T}(\mathcal{S})$. We construct some examples to highlight the extent of non-uniqueness of geodesics:

Theorem 1.1. *For every $D > 0$, there are points $X, Y, Z \in \mathcal{T}(\mathcal{S})$ and Thurston geodesic segments $\mathcal{G}_1$ and $\mathcal{G}_2$ starting from X and ending in Y with the following properties.*

(1) *Geodesics $\mathcal{G}_1$ and $\mathcal{G}_2$ do not fellow travel each other; the point Z lies in path $\mathcal{G}_1$ but is at least D away from any point in $\mathcal{G}_2$.*

(2) *The geodesic $\mathcal{G}_1$ parameterized in any way in the reverse direction is not a geodesic. In fact, the point Z is at least D away from any point in any geodesic connecting Y to X.*

In view of these examples, one may ask whether geodesics connecting X to Y have any common features. There is a mantra that all notions of a straight line in Teichmüller space behave the same way at the level of the curve graph. That is, the shadow of any such line to the curve graph is a *reparameterized quasi-geodesic*. This has already been shown for Teichmüller geodesics [15], lines of minima [8], grafting rays [7], certain geodesics in the Weil–Petersson metric [6], and Kleinian surface groups [18]. (See also [2] of an analogous result in Outer Space.)

In this paper, we show the following theorem.

Theorem 1.2. *The shadow of a Thurston geodesic to the curve graph is a reparameterized quasi-geodesic.*

Since the curve graph is Gromov hyperbolic [15], quasi-geodesics with common endpoints fellow travel. Hence:

Received 5 June 2014; revised 15 June 2015.

2010 *Mathematics Subject Classification* 30F6 (primary).

The authors acknowledge support from U.S. National Science Foundation grants DMS #1107452, #1107263, #1107367 'RNMS: GEometric structures And Representation varieties' (the GEAR Network). The second author is partially supported by NCERC #435885. The third author is partially supported by NSF DMS #1311834.

Corollary 1.3. *The shadow to the curve graph of different Thurston geodesics connecting X to Y fellow travel each other.*

This builds on the analogy established in [**14**] between Teichmüller geodesics and Thurston geodesics. We showed that if the Teichmüller geodesic connecting X and Y stays in the thick part of Teichmüller space, so does any Thurston geodesic connecting X to Y and in fact all these paths fellow travel each other. However, this analogy does not extend much further; we show that the converse of the above statement is not true:

Theorem 1.4. *There is an $\epsilon_0 > 0$ such that, for every $\epsilon > 0$, there are points $X, Y \in \mathcal{T}(\mathcal{S})$ and a Thurston geodesic connecting X to Y that stays in the ϵ_0-thick part of Teichmüller space, whereas the associated Teichmüller geodesic connecting X to Y does not stay in the ϵ-thick part of Teichmüller space.*

In particular, this means that the set of short curves along a Teichmüller geodesic and a Thurston geodesic are not the same.

1.1. *Outline of the proof*

To prove Theorem 1.2, one needs a suitable definition for when a curve is *sufficiently horizontal* along a Thurston geodesic. This is in analogy with both the study of Teichmüller geodesics and geodesics in Outer space after [**2**, **15**]. In the Teichmüller metric, geodesics are described by a quadratic differential, which in turn defines a singular flat structure on a Riemann surface. The flat metric is then deformed by stretching the horizontal foliation and contracting the vertical foliation of the flat surface. If a curve is not completely vertical, then its horizontal length grows exponentially fast along the Teichmüller geodesic. Similarly, for Outer space, the geodesics are described as folding paths associated to train-tracks. If an immersed curve has a sufficiently long *legal segment* at a point along a folding path, then the length of the horizontal segment grows exponentially fast along the folding path. These two facts, respectively, play important roles in the proofs of the hyperbolicity of curve complexes and free factor complexes.

The notion of horizontal foliation in the setting of Teichmüller geodesics is replaced by the *maximally stretched lamination* in the setting of Thurston geodesics (see Subsection 2.9). However, it is possible for a curve α on the surface to fellow travel the maximally stretched lamination λ for a long time only to have its length go down later along the Thurston geodesic. That is, the property of fellow-traveling λ geometrically does not persist (see Example 4.4).

In Section 4, we define the notion of a curve α being *sufficiently horizontal* along a Thurston geodesic to mean that α fellow travels λ for sufficiently long time both topologically and geometrically. We show that if a curve is sufficiently horizontal at a point along a Thurston geodesic, then it remains sufficiently horizontal throughout the geodesic, with exponential growth of the length of its horizontal segment (Theorem 4.2). In Section 5, we define a projection map from the curve complex to the shadow of a Thurston geodesic sending a curve α first to the earliest time in the Thurston geodesic where α is sufficiently horizontal and then to a curve of bounded length at that point. We show in Theorem 5.6 that this map is a coarse Lipschitz retraction. Theorem 1.2 follows from this fact using a standard argument. In Section 6, we construct the examples of Thurston geodesics that illustrate the deviant behaviors of Thurston geodesics from Teichmüller geodesics, as indicated by Theorems 1.1 and 1.4.

The proof is somewhat technical, because all we know about a Thurston geodesic is that the length of the maximally stretched lamination (which may not be a filling lamination) is growing exponentially. Using this and some delicate hyperbolic geometry arguments, we are able to control the geometry of the surface. For the ease of exposition, we have collected several technical lemmas in Section 3. These statements should be intuitively clear to a reader familiar with hyperbolic geometry and the proofs can be skipped in the first reading of the paper. Section 6 is also less technical and can be read independently from the rest of the paper.

2. *Background*

We briefly review some background material needed for this paper. We refer the reader to [**13**, **19**, **22**] and the references therein for background on hyperbolic surfaces and the Thurston metric on Teichmüller space.

2.1. *Notation*

We adopt the following notation to simplify some calculations. Call a constant C *universal* if it depends only on the topological type of a surface, and not on a hyperbolic metric on the surface. Then, given a universal constant C and two quantities a and b, we write

(i) $a \overset{*}{\prec} b$ if $a \leqslant Cb$;
(ii) $a \overset{*}{\asymp} b$ if $a \overset{*}{\prec} b$ and $b \overset{*}{\prec} a$;
(iii) $a \overset{+}{\prec} b$ if $a \leqslant b + C$;
(iv) $a \overset{+}{\asymp} b$ if $a \overset{+}{\prec} b$ and $b \overset{+}{\prec} a$;
(v) $a \prec b$ if $a \leqslant Cb + C$;
(vi) $a \asymp b$ if $a \prec b$ and $b \prec a$.

We will also write $a = O(1)$ to mean $a \overset{*}{\prec} 1$.

2.2. *Coarse maps*

Given two metric spaces $\mathcal{X}$ and $\mathcal{Y}$, a multivalued map $f\colon \mathcal{X} \to \mathcal{Y}$ is called a *coarse map* if the image of every point has uniformly bounded diameter. The map f is (*coarsely*) *Lipschitz* if $d_{\mathcal{Y}}(f(x), f(y)) \prec d_{\mathcal{X}}(x, y)$ for all $x, y \in \mathcal{X}$, where

$$d_{\mathcal{Y}}(f(x), f(y)) = \mathrm{diam}_{\mathcal{Y}}(f(x) \cup f(y)).$$

Given a subset $\mathcal{A} \subset \mathcal{X}$, a coarse Lipschitz map $f : \mathcal{X} \to \mathcal{A}$ is a *coarse retraction* if $d_{\mathcal{X}}(a, f(a)) = O(1)$ for all $a \in \mathcal{A}$.

2.3. *Curve graph*

Let $\mathcal{S}$ be a connected oriented surface of genus g with p punctures with $3g + p - 4 \geqslant 0$. By a curve on $\mathcal{S}$ we will mean an essential simple closed curve up to free homotopy. *Essential* means the curve is not homotopic to a point or a puncture of $\mathcal{S}$. For two curves α and β, let $\mathrm{i}(\alpha, \beta)$ be the minimal intersection number between the representatives of α and β. Two distinct curves are disjoint if their intersection number is 0. A *multicurve* on $\mathcal{S}$ is a collection of pairwise disjoint curves. A *pair of pants* is homeomorphic to a thrice-punctured sphere. A *pants decomposition* on $\mathcal{S}$ is a multicurve whose complement in $\mathcal{S}$ is a disjoint union of pairs of pants.

We define the *curve graph* $\mathcal{C}(\mathcal{S})$ of $\mathcal{S}$ as introduced by Harvey [**11**]. The vertices of $\mathcal{C}(\mathcal{S})$ are curves on $\mathcal{S}$, and two curves span an edge if they intersect minimally on $\mathcal{S}$. For a surface with $3g + p - 4 > 0$, the minimal intersection number is 0; for the once-punctured torus, the minimal intersection number is 1; and for the four-times punctured sphere, the minimal intersection number is 2. The curve graph of a pair of pants is empty since there are no essential curves. By an element or a subset of $\mathcal{C}(\mathcal{S})$, we will always mean a vertex or a subset of the vertices of $\mathcal{C}(\mathcal{S})$.

Assigning each edge of $\mathcal{C}(\mathcal{S})$ to have length 1 endows $\mathcal{C}(\mathcal{S})$ with a metric structure. Let $d_{\mathcal{C}(\mathcal{S})}(\cdot, \cdot)$ be the induced path metric on $\mathcal{C}(\mathcal{S})$. The following fact will be useful for bounding curve graph distances [21]: for any $\alpha, \beta \in \mathcal{C}(\mathcal{S})$,

$$d_{\mathcal{C}(\mathcal{S})}(\alpha, \beta) \leqslant \log_2 \mathrm{i}(\alpha, \beta) + 1. \tag{1}$$

By [15], for any surface $\mathcal{S}$, the graph $\mathcal{C}(\mathcal{S})$ is hyperbolic in the sense of Gromov. More recently, it was shown contemporaneously and independently by [1, 5, 9, 12] that there is a uniform δ such that $\mathcal{C}(\mathcal{S})$ is δ-hyperbolic for all $\mathcal{S}$.

2.4. *Teichmüller space*

A *marked* hyperbolic surface is a complete finite-area hyperbolic surface equipped with a fixed homeomorphism from $\mathcal{S}$. Two marked hyperbolic surfaces X and Y are considered equivalent if there is an isometry from X to Y in the correct homotopy class. The collection of equivalence classes of all marked hyperbolic surfaces is called the Teichmüller space $\mathcal{T}(\mathcal{S})$ of $\mathcal{S}$. This space $\mathcal{T}(\mathcal{S})$ equipped with its natural topology is homeomorphic to $\mathbb{R}^{6g-6+2p}$.

2.5. *Short curves and collars*

Given $X \in \mathcal{T}(\mathcal{S})$ and a simple geodesic ω on X, let $\ell_X(\omega)$ be the arc length of ω. Since X is marked by a homeomorphism to $\mathcal{S}$, its set of curves is identified with the set of curves on $\mathcal{S}$. For a curve α on $\mathcal{S}$, let $\ell_X(\alpha) = \ell_X(\alpha^*)$, where α^* is the geodesic representative of α on X. A curve is called a *systole* of X if its hyperbolic length is minimal among all curves. Given a constant C, a multicurve on X is called C-short if the length of every curve in the set is bounded above by C. The *Bers constant* $\epsilon_B = \epsilon_B(\mathcal{S})$ is the smallest constant such that every hyperbolic surface X admits an ϵ_B-short pants decomposition. In most situations, we will assume a curve or multicurve is realized by geodesics on X.

We state the well-known Collar Lemma with some additional properties (see [13, Subsection 3.8]).

LEMMA 2.1 (Collar Lemma). *Let X be a hyperbolic surface. For any simple closed geodesic α on X, the regular neighborhood about α*

$$U(\alpha) = \left\{ p \in X \mid d_X(p, \alpha) \leqslant \sinh^{-1} \frac{1}{\sinh(0.5\ell_X(\alpha))} \right\}$$

is an embedded annulus. If two simple closed geodesics α and β are disjoint, then $U(\alpha)$ and $U(\beta)$ are disjoint. Moreover, given a simple closed geodesic α and a simple geodesic ω (not necessarily closed, but complete), if ω does not intersect α and does not spiral toward α, then it is disjoint from $U(\alpha)$.

We will refer to $U(\alpha)$ as *the standard collar of* α. There is a universal upper and lower bound on the arc length of the boundary of $U(\alpha)$ provided that α is ϵ_B-short.

Using the convention

$$\log(x) = \begin{cases} \ln(x) & \text{if } x \geqslant e, \\ 1 & \text{if } x \leqslant e, \end{cases} \tag{2}$$

we note that, for $0 \leqslant x \leqslant \epsilon_B$,

$$\sinh^{-1}(1/\sinh(0.5x)) \stackrel{+}{\asymp} \log(1/x).$$

A consequence of the Collar Lemma is the existence of a universal constant δ_B such that if a curve β intersects an ϵ_B-short curve α, then

$$\ell_X(\beta) \geqslant \mathrm{i}(\alpha, \beta)\delta_B.$$

Also, for any geodesic segment ω,

$$\ell_X(\omega) \geqslant (\mathrm{i}(\alpha,\omega) - 1)\delta_B.$$

We will refer to δ_B as the *dual constant* to the Bers constant ϵ_B.

2.6. *Various notions of twisting*

In this section, we will define several notions of *relative twisting* of two objects or structures about a simple closed curve γ. The notation will always be $\mathrm{twist}_\gamma(\cdot,\cdot)$.

First suppose A is a compact annulus and γ is the core curve of A. Given two simple arcs η and ω with endpoints on the boundary of A, we define

$$\mathrm{twist}_\gamma(\eta,\omega) = \mathrm{i}(\eta,\omega),$$

where $\mathrm{i}(\eta,\omega)$ is the minimal number of interior intersections between isotopy classes of η and ω fixing the endpoints pointwise.

Now suppose that γ is a curve in $\mathcal{S}$. The annular cover $\hat{A}$ of $\mathcal{S}$ corresponding to $\langle\gamma\rangle < \pi_1(\mathcal{S})$ can be compactified in an intrinsic way. Let $\hat{\gamma}$ be the core curve of $\hat{A}$. Given two simple geodesics or curves η and ω in $\mathcal{S}$, let $\hat{\eta}$ and $\hat{\omega}$ be any lifts to $\hat{A}$ that join the boundary of $\hat{A}$ (such lifts exist when η and ω intersect γ). The *relative twisting* of η and ω about γ is

$$\mathrm{twist}_\gamma(\eta,\omega) = \mathrm{twist}_{\hat{\gamma}}(\hat{\eta},\hat{\omega}).$$

This definition is well defined up to an additive error of 1 with different choices of $\hat{\eta}$ and $\hat{\omega}$.

Now suppose $X \in \mathcal{T}(\mathcal{S})$ and let ω be a geodesic arc or curve in X. We want to measure the number of times ω twists about γ in X. To do this, represent γ by a geodesic and lift the hyperbolic metric of X to the annular cover $\hat{A}$. Let $\hat{\tau}$ be any geodesic perpendicular to $\hat{\gamma}$ joining the boundary of $\hat{A}$. We define the twist of ω about γ on X to be

$$\mathrm{twist}_\gamma(\omega, X) = \mathrm{i}(\hat{\omega},\hat{\tau}),$$

where $\hat{\omega}$ is any lift of ω joining the boundary of $\hat{A}$. Since there may be other choices of $\hat{\tau}$, this notion is well defined up to an additive error of at most one. Note that if $\mathrm{twist}_\gamma(\omega,X) = 0$ and $\mathrm{twist}_\gamma(\eta,\omega) = n$, then $\mathrm{twist}_\gamma(\eta,X) \stackrel{+}{\asymp} n$.

When γ is ϵ_B-short, fix a perpendicular arc τ to the standard collar $U(\gamma)$, then the quantity $\mathrm{i}(\omega,\tau)$ differs from $\mathrm{twist}_\gamma(\omega,X)$ by at most one [**17**, Lemma 3.1].

Given $X, Y \in \mathcal{T}(\mathcal{S})$, the relative twisting of X and Y about γ is

$$\mathrm{twist}_\gamma(X,Y) = \mathrm{i}(\hat{\tau}_X,\hat{\tau}_Y),$$

where $\hat{\tau}_X$ is an arc perpendicular to $\hat{\gamma}$ in the metric X, and $\hat{\tau}_Y$ is an arc perpendicular to $\hat{\gamma}$ in the metric Y. Again, choosing different perpendicular arcs changes this quantity by at most one.

2.7. *Subsurface projection and bounded combinatorics*

Let $\Sigma \subset \mathcal{S}$ be a compact and connected subsurface such that each boundary component of Σ is an essential simple closed curve. We assume that Σ is not a pair of pants or an annulus. From [**16**], we recall the definition of subsurface projection $\pi_\Sigma\colon \mathcal{C}(\mathcal{S}) \to \mathcal{P}(\mathcal{C}(\Sigma))$ from the curve graph of S to the space of subsets of the curve graph of Σ.

Equip $\mathcal{S}$ with a hyperbolic metric and represent Σ as a convex set with geodesic boundary. (The projection map does not depend on the choice of the hyperbolic metric.) Let $\hat{\Sigma}$ be the Gromov compactification of the cover of $\mathcal{S}$ corresponding to $\pi_1(\Sigma) < \pi_1(\mathcal{S})$. There is a natural homeomorphism from $\hat{\Sigma}$ to Σ, which allows us to identify $\mathcal{C}(\hat{\Sigma})$ with $\mathcal{C}(\Sigma)$. For any curve α on $\mathcal{S}$, let $\hat{\alpha}$ be the closure of the lift of α in $\hat{\Sigma}$. For each component β of $\hat{\alpha}$, let $\mathcal{N}_\beta$ be a regular neighborhood of $\beta \cup \partial\hat{\Sigma}$. The isotopy class of each component β' of $\partial\mathcal{N}_\beta$, with isotopy relative to $\partial\hat{\Sigma}$, can be regarded as an element of $\mathcal{P}(\mathcal{C}(\Sigma))$; β' is the empty set if β' is isotopic into $\partial\hat{\Sigma}$. We define

$$\pi_\Sigma(\alpha) = \bigcup_{\beta\subset\hat{\alpha}} \bigcup_{\beta'\subset\partial\mathcal{N}_\beta} \{\beta'\}.$$

The projection distance between two elements $\alpha, \beta \in \mathcal{C}(\mathcal{S})$ in Σ is

$$d_{\mathcal{C}(\Sigma)}(\alpha,\beta) = \operatorname{diam}_{\mathcal{C}(\Sigma)}(\pi_\Sigma(\alpha)\cup\pi_\Sigma(\beta)).$$

Given a subset $K \subset \mathcal{C}(\mathcal{S})$, we also define $\pi_\Sigma(K) = \bigcup_{\alpha\in K}\pi_\Sigma(\alpha)$, and the projection distance between two subsets of $\mathcal{C}(\mathcal{S})$ in Σ is likewise defined. For any $\Sigma\subset\mathcal{S}$, the projection map π_Σ is a coarse Lipschitz map [**16**].

For any $X \in \mathcal{T}(\mathcal{S})$, a pants decomposition $\mathcal{P}$ on X is called *short* if $\sum_{\alpha\subset\mathcal{P}} \ell_X(\alpha)$ is minimized. Note that a short pants decomposition is always ϵ_B-short, and two short pants decompositions have bounded diameter in $\mathcal{C}(\mathcal{S})$.

Let $X_1, X_2 \in \mathcal{T}(\mathcal{S})$. For $i = 1, 2$, let $\mathcal{P}_i$ be a short pants decomposition on X_i. We will say X_1 and X_2 have *K-bounded combinatorics* if there exists a constant K such that the following two properties hold.

(i) For $\Sigma = \mathcal{S}$, or Σ a subsurface of $\mathcal{S}$,

$$d_{\mathcal{C}(\Sigma)}(\mathcal{P}_1,\mathcal{P}_2) \leqslant K.$$

(ii) For every curve γ in $\mathcal{S}$

$$\operatorname{twist}_\gamma(X_1, X_2) \leqslant K.$$

2.8. *Geodesic lamination*

Let X be a hyperbolic metric on $\mathcal{S}$. A geodesic lamination μ is a closed subset of $\mathcal{S}$ which is a union of disjoint simple complete geodesics in the metric of X. These geodesics are called *leaves* of μ, and we will call their union the *support* of μ. A basic example of a geodesic lamination is a multicurve (realized by its geodesic representative).

Given another hyperbolic metric on $\mathcal{S}$, there is a canonical one-to-one correspondence between the two spaces of geodesic laminations. We therefore will denote the space of geodesic laminations on $\mathcal{S}$ by $\mathcal{GL}(\mathcal{S})$ without referencing to a hyperbolic metric. The set $\mathcal{GL}(\mathcal{S})$ endowed with the Hausdorff distance is compact. A geodesic lamination is said to be *chain-recurrent* if it is in the closure of the set of all multicurves.

A transverse measure on a geodesic lamination μ is a Radon measure on arcs transverse to the leaves of the lamination. The measure is required to be invariant under projections along the leaves of μ. When μ is a simple closed geodesic, the transverse measure is just the counting measure times a positive real number. It is easy to see that an infinite isolated leaf spiraling toward a closed leaf cannot be in the support of a transverse measure.

The *stump* of a geodesic lamination μ is a maximal (with respect to inclusion) compactly supported sublamination of μ which admits a transverse measure of full support.

2.9. *Thurston metric*

In this section, we will give a brief overview of the Thurston metric, sometimes referred to as the Lipschitz metric or Thurston's 'asymmetric' metric in the literature. All facts in this section are due to Thurston and contained in [**22**]. We also refer the reader to [**19**] for additional reference.

Given $X, Y \in \mathcal{T}(\mathcal{S})$, the distance *from* X to Y in the Thurston metric is defined to be

$$d_{\mathrm{Th}}(X,Y) = \log \mathrm{L}(X,Y),$$

where $\mathrm{L}(X, Y)$ is the infimum of Lipschitz constants over all homeomorphisms from X to Y in the correct homotopy class. Since the inverse of a Lipschitz map is not necessarily Lipschitz, there is no reason for the metric to be symmetric. In fact, $\mathrm{L}(X, Y)$ is in general not equal to $\mathrm{L}(Y, X)$, as shown in the example on page 5 of [**22**].

Thurston showed that the quantity $\mathrm{L}(X, Y)$ can be computed using ratios of lengths of curves on $\mathcal{S}$.

THEOREM 2.2 [**22**]. *For any* $X, Y \in \mathcal{T}(\mathcal{S})$,

$$L(X, Y) = \sup_{\alpha} \frac{\ell_Y(\alpha)}{\ell_X(\alpha)},$$

where α *ranges over all curves on* $\mathcal{S}$.

The length function extends continuously to measured laminations, and the space of projectivized measured laminations is compact. Hence there is a measured lamination that realizes the supremum above. It might not be unique, but one can assign to an ordered pair (X, Y) a geodesic lamination $\mu(X, Y)$ admitting a transverse measure that contains the supports of all the measured laminations realizing the supremum.

For any sequence $\{\alpha_i\}$ of curves on $\mathcal{S}$ with $\lim_{i\to\infty} \ell_Y(\alpha_i)/\ell_X(\alpha_i) \to L(X, Y)$, let α_∞ be a limiting geodesic lamination of $\{\alpha_i\}$ in the Hausdorff topology. Set

$$\lambda(X, Y) = \bigcup \{\alpha_\infty\},$$

where the union on the right-hand side ranges over the limits of all such sequences. Thurston showed that $\lambda(X, Y)$ is a geodesic lamination, called the *maximally stretched lamination* from X to Y, which contains $\mu(X, Y)$ as its stump. Moreover, there is a $\mathrm{L}(X, Y)$-Lipschitz homeomorphism from X to Y in the correct homotopy class that stretches $\lambda(X, Y)$ by $\mathrm{L}(X, Y)$ and whose local Lipschitz constant outside $\lambda(X, Y)$ is strictly less than $\mathrm{L}(X, Y)$. In particular, the infimum is realized in the definition of $L(X, Y)$ and $d_{\mathrm{Th}}(X, Y)$. The existence of such a map follows from the fact that one can connect X to Y by a concatenation of finitely many stretch paths (see Subsection 2.10), all of which contain $\lambda(X, Y)$ in its stretch locus. We will call a homeomorphism $f\colon X \to Y$ *optimal* if f is a $\mathrm{L}(X, Y)$-Lipschitz map. Note that our sense of 'optimal' is more in the sense of L^∞ metric than L^1, since we only require the global Lipschitz constant to be minimized.

Thurston showed that $\mathcal{T}(\mathcal{S})$ equipped with the Thurston metric is a (asymmetric) geodesic metric space. That is, for any $X, Y \in \mathcal{T}(\mathcal{S})$, there exists a geodesic from X to Y, that is, a parameterized path $\mathcal{G}\colon [0, d] \to \mathcal{T}(\mathcal{S})$ such that $d = d_{\mathrm{Th}}(X, Y)$, $\mathcal{G}(0) = X$, $\mathcal{G}(d) = Y$, and for any $0 \leqslant s < t \leqslant d$, $d_{\mathrm{Th}}(\mathcal{G}(s), \mathcal{G}(t)) = t - s$. Any geodesic from X to Y is characterized by the property that the maximally stretched lamination $\lambda(X, Y)$ is stretched maximally at all times. Thus, there is only one such geodesic only when $\lambda(X, Y)$ is a maximal lamination (the complement of $\lambda(X, Y)$ are ideal triangles). In general, the set of geodesics from X to Y can have uncountable cardinality: the idea is that one is free to deform any part of the surface that is not forced to be maximally stretched. We refer to the proof of Theorem 1.1 in Section 6 for an example of such deformation.

Given a geodesic segment $\mathcal{G}\colon [a, b] \to \mathcal{T}(\mathcal{S})$, we will often denote by $\lambda_{\mathcal{G}}$ the maximally stretched lamination from $\mathcal{G}(a)$ to $\mathcal{G}(b)$. The maximally stretched lamination is well defined for geodesic rays or bi-infinite geodesics. Suppose $\mathcal{G}\colon \mathbb{R} \to \mathcal{T}(\mathcal{S})$ is a bi-infinite geodesic. Consider two sequences $\{t_n\}_n, \{s_m\}_m \subset \mathbb{R}$ with

$$\lim_{n\to\infty} t_n \longrightarrow \infty \quad \text{and} \quad \lim_{m\to\infty} s_m \longrightarrow -\infty.$$

Set $X_m = \mathcal{G}(s_m)$ and $Y_n = \mathcal{G}(t_n)$. The sequence $\lambda(X_m, Y_n)$ is increasing by inclusion as $m, n \to \infty$, hence $\lambda = \bigcup_{m,n} \lambda(X_m, Y_n)$ is defined. The lamination λ is independent of the sequences X_m

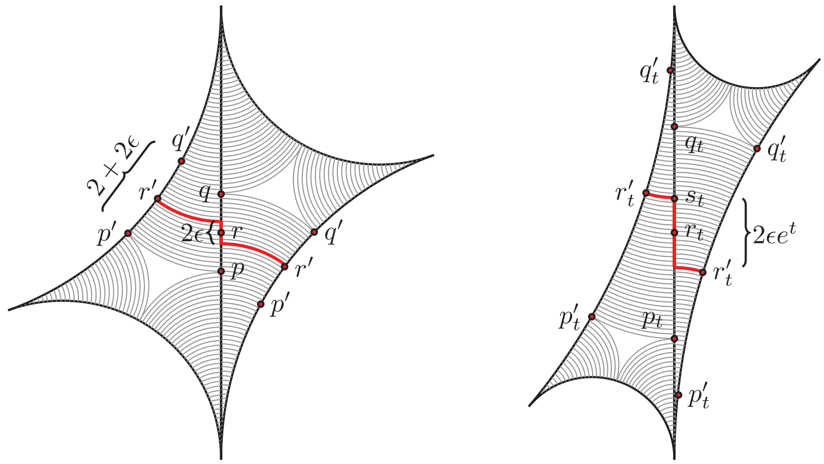

FIGURE 1. *The annuli A_0 and A_t are obtained by gluing two ideal triangles. The shifts for A_t are e^t times those of A_0.*

and Y_n, hence $\lambda = \lambda_{\mathcal{G}}$ is the maximally stretched lamination for $\mathcal{G}$. Similarly, the maximally stretched lamination $\lambda_{\mathcal{G}}$ is defined for a geodesic ray $\mathcal{G}\colon\ [a, \infty) \to \mathcal{T}(\mathcal{S})$.

Throughout this paper, we will always assume that Thurston geodesics are parameterized by arc length.

2.10. *Stretch paths*

To prove $\mathcal{T}(\mathcal{S})$ is a geodesic metric space, Thurston introduced a special family of geodesics called *stretch paths*. Namely, let λ be a maximal geodesic lamination (all complementary components are ideal triangles). Then λ, together with a choice of basis for relative homology, defines *shearing coordinates* on Teichmüller space (see [**3**]). In fact, in this situation, the choice of basis does not matter. For any hyperbolic surface X, and time t, define $\mathrm{stretch}(X, \lambda, t)$ to be the hyperbolic surface where the shearing coordinates are e^t times the shearing coordinates at X. That is, the path $t \mapsto \mathrm{stretch}(X, \lambda, t)$ is a straight line in the shearing coordinate system associated to λ. Thurston showed [**22**] that this path is a geodesic in $\mathcal{T}(\mathcal{S})$.

In the case where λ is a finite union of geodesics, the shearing coordinates are easy to understand. An ideal triangle has an inscribed circle tangent to each edge at a point which we refer to as an *anchor point*. Then the shearing coordinate associated to two adjacent ideal triangles is the distance between the anchor points coming from the two triangles (see Figure 1). To obtain the surface $\mathrm{stretch}(X, \lambda, t)$, one has to *slide* every pair of adjacent triangles against each other such that the distance between the associated pairs of anchor points is increased by a factor of e^t.

2.11. *An example*

We illustrate some possible behaviors along a stretch path in the following basic example.

Fix a small $0 < \epsilon \ll 1$. Let A_0 be an annulus which is glued out of two ideal triangles as follows. One pair of the sides is glued with a shift of 2, and another pair is glued with a shift of $2 + 2\epsilon$, as in Figure 1. That is, if p, p' are the anchor points associated to one triangle and q, q' are anchor points associated to the other triangle, and the sides containing p and q are glued, and same for p' and q', then the segments $[p, q]$ and $[p', q']$ have lengths 2 and $2 + 2\epsilon$, respectively. Note that by adding enough ideal triangles to this construction and gluing them appropriately, one can obtain a hyperbolic surface of arbitrarily large complexity. For example, adding an ideal triangle to the top and to the bottom of A_0 with zero shift and then identifying the top edges with zero shift and the bottom edges with zero shift gives rise to a hyperbolic surface X_0 which is topologically a sphere with four points removed. Also, the sides of the four ideal triangles define a maximal geodesic lamination λ on X_0. When two triangles are glued with zero shift, the associated shearing coordinate remains unchanged under a stretch map. Hence, we concentrate on how the geometry of A_0 changes only.

Let r and r' be the midpoints of $[p, q]$ and $[p', q']$. There is an isometry of A_0 that switches the two triangles and fixes r and r'. Hence, if γ is the core curve of the annulus A_0, then the geodesic representative of γ, which is unique and is fixed by this isometry, passes through points r and r'.

Define $X_t = \text{stretch}(X_0, \lambda, t)$. We give an estimate for the hyperbolic length of γ at X_t for $t \in \mathbb{R}_+$.

Claim. *For $t \in \mathbb{R}_+$, we have $\ell_{X_t}(\gamma) \stackrel{*}{\asymp} \epsilon e^t + e^{-e^t}$.*

Proof. Let A_t be the annulus obtained by gluing two triangles when the shifts are $2e^t$ and $(2+2\epsilon)e^t$ and let p_t, q_t, r_t and r'_t be points that are defined similar to p, q, r, and r' in A_0. The geodesic representative of the core curve of A_t, which we still denote by γ, passes through the points r_t and r'_t. Denote the length of the segment $[r_t, r'_t]$ by $d(r_t, r'_t)$. Then

$$\ell_{X_t}(\gamma) = 2d(r_t, r'_t).$$

To estimate $d(r_t, r'_t)$, we work in one of the ideal triangles. Let $s_t \in [p_t, q_t]$ be the point that is on the same horocycle as r'_t. Consider the triangle $[r_t, r'_t] \cup [r'_t, s_t] \cup [s_t, r_t]$. By the triangle inequality, we have

$$d(r_t, r'_t) \leqslant d(r_t, s_t) + d(s_t, r'_t) \leqslant 2\max\{d(r_t, s_t), d(s_t, r'_t)\}.$$

On the other hand, the angle between the segments $[r'_t, s_t]$ and $[s_t, r_t]$ is at least $\frac{\pi}{2}$, which implies that the side $[r_t, r'_t]$ is the largest of the triangle. Hence, we also have

$$d(r_t, r'_t) \geqslant \max\{d(r_t, s_t), d(s_t, r'_t)\}.$$

That is, up to a multiplicative error of at most 4, the length $\ell_t(\gamma)$ is $d(r_t, s_t) + d(s_t, r'_t)$. The distance $d(s_t, r'_t)$ is asymptotically (as $t \to +\infty$) equal to the length of the horocycle between s_t and r'_t. It is straightforward to see that since $d(p_t, s_t) = e^t(1+\epsilon)$, the length of the horocycle is $e^{-e^t(1+\epsilon)}$. Also $d(r_t, s_t) = \epsilon e^t$, and we have

$$\ell_{X_t}(\gamma) \stackrel{*}{\asymp} \epsilon e^t + e^{-e^t(1+\epsilon)}.$$

The second term in the sum can be replaced with e^{-e^t} without increasing the multiplicative error by much. This is true because the first term ϵe^t in the sum is bigger than the second term when ϵe^t is bigger than 1. This proves the claim. □

We can now approximate the minimum of $\ell_{X_t}(\gamma)$ for $t \in \mathbb{R}_+$. At $t = 0$ and $t = \log \frac{1}{\epsilon}$ the length of γ is basically 1. If ϵ is small enough, there is $t_0 > 0$ such that $\epsilon e^{t_0} = e^{-e^{t_0}}$. Then we have

$$\ell_{X_t}(\gamma) \stackrel{*}{\asymp} \begin{cases} e^{-e^t}, & t < t_0, \\ \epsilon e^t, & t > t_0. \end{cases}$$

This means in particular that the length of γ decreases super-exponentially fast, reaches its minimum, and grows back up exponentially fast. We will not compute the exact value of t_0, but if we take log twice we see that $t_0 \stackrel{+}{\asymp} \log\log(1/\epsilon)$, with additive error at most $\log 2$. Then $\ell_{t_0}(\gamma) \stackrel{*}{\asymp} \epsilon \log(1/\epsilon)$ and this is, up to a multiplicative error, the minimum of $\ell_{X_t}(\gamma)$.

The reason the curve γ gets short and then long again is because it is more efficient to twist around γ when γ is short. To see this, we estimate the relative twisting of X_0 and X_t around γ at $t = \log(1/\epsilon)$, that is when the length of γ grew back to approximately 1. The lamination λ is nearly perpendicular to γ at X_0 and does not twist around it. Hence, we need to compute how many times it twists around γ in X_t.

For a fixed $t > 0$, choose lifts $\tilde{\gamma}$ and $\tilde{\lambda}$ to $\mathbb{H}$ of the geodesic representative of γ and of the leaf of λ containing $[p_t, q_t]$, such that $\tilde{\gamma}$ and $\tilde{\lambda}$ intersect. Let ℓ be the length of the orthogonal projection of $\tilde{\lambda}$ to $\tilde{\gamma}$. Then (see [**17**, Section 3])

$$\mathrm{twist}_\gamma(\lambda, X_t) \stackrel{+}{\asymp} \frac{\ell}{\ell_{X_t}(\gamma)}.$$

To find ℓ, we note that $\cosh \ell/2 = 1/\sin\alpha$, where α is the angle between $\tilde{\gamma}$ and $\tilde{\lambda}$. In the triangle $[r_t, r'_t] \cup [r'_t, s_t] \cup [s_t, r_t]$, α is the angle between segments $[s_t, r_t]$ and $[r_t, r'_t]$. Let β be the angle between $[r'_t, s_t]$ and $[s_t, r_t]$. Since β is asymptotically $\pi/2$, by the hyperbolic sine rule, we have

$$\cosh \ell/2 \stackrel{*}{\asymp} \frac{\sinh d(r'_t, r_t)}{\sinh d(r'_t, s_t)}.$$

Assuming $t = \log(1/\epsilon)$, we have $\sinh d(r'_t, r_t) \stackrel{*}{\asymp} 1$ and $\sinh d(r'_t, s_t) \stackrel{*}{\asymp} e^{-1/\epsilon}$ which implies $\ell \stackrel{+}{\asymp} 2/\epsilon$. Hence

$$\mathrm{twist}_\gamma(\lambda, X_{\log(1/\epsilon)}) \stackrel{*}{\asymp} \frac{1}{\epsilon}.$$

To summarize, the surface $X_{\log(1/\epsilon)}$ is close to $D^n_\gamma(X_0)$, where D_γ is a Dehn twist around γ and $n \stackrel{*}{\asymp} 1/\epsilon$. The stretch path $\mathrm{stretch}(X_0, \lambda, t)$ from X_0 to $X_{\log(1/\epsilon)}$ changes only an annular neighborhood of γ, first decreasing the length of γ super-exponentially fast to $\epsilon \log(1/\epsilon)$ and then increasing it exponentially fast. In fact, further analysis shows that essentially all the twisting is done near the time t_0 when the length of γ is minimum.

2.12. *Shadow map*

For any $X \in \mathcal{T}(\mathcal{S})$, the set of systoles on X has uniformly bounded diameter in $\mathcal{C}(\mathcal{S})$. We will call the coarse map $\pi\colon \mathcal{T}(\mathcal{S}) \to \mathcal{C}(\mathcal{S})$ sending X to the set of systoles on X the *shadow map*. The following lemma shows that the shadow map is Lipschitz.

LEMMA 2.3. *The shadow map $\pi\colon \mathcal{T}(\mathcal{S}) \to \mathcal{C}(\mathcal{S})$ satisfies, for all $X, Y \in \mathcal{T}(\mathcal{S})$,*

$$d_{\mathcal{C}(\mathcal{S})}(\pi(X), \pi(Y)) \prec d_{Th}(X, Y).$$

Proof. Let $X, Y \in \mathcal{T}(\mathcal{S})$ and let $K = L(X, Y)$. Let α be a systole on X and let β be a systole on Y. Recall that ϵ_B is the Bers constant and δ_B its dual constant defined in Subsection 2.5. We have $\ell_X(\alpha) \leqslant \epsilon_B$ and $\ell_Y(\beta) \leqslant \epsilon_B$. Now

$$\mathrm{i}(\alpha, \beta) \leqslant \frac{\ell_Y(\alpha)}{\delta_B} \leqslant \frac{K\ell_X(\alpha)}{\delta_B} \leqslant K\frac{\epsilon_B}{\delta_B}.$$

Therefore, equation (1) implies $d_{\mathcal{C}(\mathcal{S})}(\alpha, \beta) \prec \log K = d_{\mathrm{Th}}(X, Y)$. □

For simplicity, we will often write $d_{\mathcal{C}(\mathcal{S})}(X, Y) := d_{\mathcal{C}(\mathcal{S})}(\pi(X), \pi(Y))$.

3. *Hyperbolic geometry*

In this section, we establish some basic properties of the hyperbolic plane $\mathbb{H}$ and hyperbolic surfaces. Many of these results are known in spirit, but to our knowledge the exact statements do not directly follow from what is written in the literature.

Recall that $\mathbb{H}$ is Gromov hyperbolic, that is, there is a constant $\delta_{\mathbb{H}}$ such that all triangles in $\mathbb{H}$ are $\delta_{\mathbb{H}}$ slim: every edge of a triangle is contained in a $\delta_{\mathbb{H}}$-neighborhood of the union of the other two edges.

3.1. *Geodesic arcs on hyperbolic surfaces*

Let α be a simple closed geodesic on a hyperbolic surface X and let $U(\alpha)$ be the standard collar of α. When ω is a geodesic segment contained in $U(\alpha)$ with endpoints p and p', we denote the distance between p and p' in $U(\alpha)$ by $d_{U(\alpha)}(\omega)$. The following lemma can be read as saying that all the twisting around a curve α takes place in $U(\alpha)$.

LEMMA 3.1. *Let P be a pair of pants in a hyperbolic surface X with geodesic boundary lengths less than ϵ_B. For each connected component $\alpha \subset \partial P$, there is an arc τ_α in $U(\alpha)$ perpendicular to α such that the following holds. Any finite subarc ω of a simple complete geodesic λ that is contained in P can be subdivided into three pieces*

$$\omega = \omega_\alpha \cup \omega_0 \cup \omega_\beta$$

such that

(a) *the interior of ω_0 is disjoint from every $U(\gamma)$, for $\gamma \subset \partial P$, and $\ell_X(\omega_0) = O(1)$;*
(b) *the segment ω_α, $\alpha \subset \partial P$, is contained in $U(\alpha)$ and intersects any curve in $U(\alpha)$ that is equidistant to α at most once. That is, as one travels along ω_α, the distance to α changes monotonically. Furthermore,*

$$\ell_X(\omega_\alpha) \stackrel{+}{\asymp} \mathrm{i}(\tau_\alpha, \omega)\ell(\alpha) + d_{U(\alpha)}(\omega_\alpha),$$

(c) *the same holds for ω_β (α and β may be the same curve).*

Proof. Note that if ω intersects $U(\alpha)$, then by Lemma 2.1, λ either intersects α or spirals toward α. This implies that ω intersects at most two standard collars, say $U(\alpha)$ and $U(\beta)$. We allow the possibility that $\alpha = \beta$. Let $\omega_\alpha = \omega \cap U(\alpha)$, $\omega_\beta = \omega \cap U(\beta)$, and ω_0 be the remaining middle segment. In P, there is a unique geodesic segment η perpendicular to α and β. Set $\tau_\alpha = \eta \cap U(\alpha)$ and $\tau_\beta = \eta \cap U(\beta)$. If $\alpha = \beta$, then η is the unique simple segment intersecting $U(\alpha)$ twice and perpendicular to α, and τ_α and τ_β are the two components of η in $U(\alpha)$. In the universal cover, a lift $\widetilde{\omega}$ of ω is in a $2\delta_{\mathbb{H}}$-neighborhood of the union of a lift $\widetilde{\alpha}$ of α, a lift $\widetilde{\beta}$ of β and a lift $\widetilde{\eta}$ of η.

In fact, a point in $\widetilde{\omega}$ is either in a δ_B-neighborhood of $\widetilde{\alpha} \cup \widetilde{\beta}$, where δ_B is the dual constant to ϵ_B, or is uniformly close to $\widetilde{\eta}$. This fact has two consequences. First, because $U(\alpha)$ and $U(\beta)$ have thicknesses bigger than δ_B, the lift $\widetilde{\omega}'$ of ω' is contained in a uniform bounded neighborhood of $\widetilde{\eta}$ and hence, the length of ω' is comparable to the length of η outside of $U(\alpha)$ and $U(\beta)$, which is uniformly bounded.

Secondly, $\widetilde{\omega}_\alpha$ is in a $\delta_{\mathbb{H}}$-neighborhood of $\tilde{\tau}_\alpha$ and $\widetilde{\alpha}$. The portion that is in the neighborhood of $\tilde{\tau}_\alpha$ has a length that is, up to an additive error, equal to $d_{U(\alpha)}(\omega_\alpha)$. The portion that is in the neighborhood of $\widetilde{\alpha}$ has a length that is (up to an additive error) equal to

$$\mathrm{i}(\tau_\alpha, \omega_\alpha)\ell(\alpha).$$

The formula in part (b) follows from adding these two estimates.

The only remaining point is that, in the above, the choice of τ_α depends on β. However, we observe that the choice of τ_α is not important and for any other segment τ'_α perpendicular to α that spans the width of $U(\alpha)$, we have

$$\mathrm{i}(\tau_\alpha, \omega_\alpha) \stackrel{+}{\asymp} \mathrm{i}(\tau'_\alpha, \omega_\alpha).$$

This completes the proof. ∎

Let $X \in \mathcal{T}(\mathcal{S})$ and P be a pair of pants in X with boundary lengths at most ϵ_B. Roughly speaking, the following technical lemma states that if a subsegment ω of λ intersects a closed (non-geodesic) curve γ enough times and the consecutive intersection points are far enough apart, then ω cannot be contained in P.

LEMMA 3.2. *There exist $D_0 > 0$ and $K_0 > 0$ with the following property. Let γ be a simple closed curve (γ may not be a geodesic) that intersects ∂P and let ω be a simple geodesic in P where the consecutive intersections of ω with γ are at least D_0 apart in ω. Let $\overline{\omega}$ be a subarc of ω with endpoints on γ, and let ω_1 and ω_2 be the connected components of $\omega \smallsetminus \overline{\omega}$. Then at least one of $\overline{\omega}$, ω_1, or ω_2 has length bounded by $K_0\ (\ell_X(\gamma)+1)$.*

Proof. For $i = 1, 2$, let p_i be the endpoint of $\overline{\omega}$ that is also an endpoint of ω_i. Let α_i be the boundary curve of P with $p_i \in U(\alpha_i)$, where $U(\alpha_i)$ is the standard collar of α_i. If p_i does not belong in any of the collar neighborhoods of a curve in ∂P, then we choose α_i arbitrarily. For simplicity, denote $U(\alpha_i)$ by U_i and let $\overline{\omega}_i$ be the component of $\overline{\omega} \cap U_i$ with endpoint p_i ($\overline{\omega}_i$ may be empty). It is enough to show that, for $i = 1, 2$, we have either

$$\ell_X(\overline{\omega}_i) \prec \ell_X(\gamma) \quad \text{or} \quad \ell_X(\omega_i) \prec \ell_X(\gamma). \tag{3}$$

This is because if ω_1 and ω_2 are both very long, then equation (3) implies that $\overline{\omega}_1$ and $\overline{\omega}_2$ both have length bounded by $\ell_X(\gamma)$ up to a small error. Since the middle part of $\overline{\omega}$ has bounded length (Lemma 3.1), this implies the desired upper bound for the length of $\overline{\omega}$.

We now prove equation (3). The point p_i subdivides U_i into two sets V_i and W_i, where V_i and W_i are regular annuli (their boundaries are equidistance curves to α) with disjoint interiors and p_i is on the common boundary of V_i and W_i. By part (c) of Lemma 3.1, one of these annuli contains ω_i and the other contains $\overline{\omega}_i$. Also, since γ passes through p_i, it intersects both boundaries of either V_i or W_i.

Let V be either V_i or W_i such that γ intersects both boundaries of V and let η be either ω_i or $\overline{\omega}_i$ that is contained in V. We want to show

$$\ell_X(\eta) \prec \ell_X(\gamma),$$

which is equivalent to equation (3).

From Lemma 3.1, we have

$$\ell_X(\eta) \stackrel{+}{\asymp} \mathrm{i}(\eta, \tau_\beta)\ell_X(\alpha) + d_V(\eta). \tag{4}$$

Let σ be the geodesic representative of the subarc of γ connecting the boundaries of V. Then

$$d(\eta, V) \stackrel{+}{\prec} \ell_X(\sigma) \leqslant \ell_X(\gamma). \tag{5}$$

The intersection numbers between arcs in an annulus satisfy the triangle inequality up to a small additive error. Hence,

$$\mathrm{i}(\eta, \tau_\alpha) \stackrel{+}{\prec} \mathrm{i}(\eta, \sigma) + \mathrm{i}(\sigma, \tau_\alpha) \leqslant \mathrm{i}(\eta, \gamma) + \mathrm{i}(\sigma, \tau_\alpha). \tag{6}$$

Let $D_0 > \epsilon_B$ be any constant. By assumption, consecutive intersections of ω with γ are at least D_0 apart in ω. We have $\mathrm{i}(\eta, \gamma) \leqslant (1/D_0)\ell_X(\eta)$. Also, $\mathrm{i}(\sigma, \tau_\alpha) \leqslant \ell_X(\sigma)/\ell_X(\alpha) + 1$. These facts and equation (6) imply

$$\mathrm{i}(\eta, \tau_\alpha) \stackrel{+}{\prec} \frac{1}{D_0}\ell_X(\eta) + \frac{\ell_X(\sigma)}{\ell_X(\alpha)}.$$

Combining this with equations (4) and (5), we have

$$\begin{aligned}\ell_X(\eta) &\stackrel{+}{\prec} \left(\frac{1}{D_0}\ell_X(\eta) + \frac{\ell_X(\sigma)}{\ell_X(\alpha)}\right)\ell_X(\alpha) + \ell_X(\gamma)\\ &\leqslant \frac{\epsilon_B}{D_0}\ell_X(\eta) + 2\ell_X(\gamma).\end{aligned}$$

That is,

$$\left(1 - \frac{\epsilon_B}{D_0}\right)\ell_X(\eta) \stackrel{+}{\prec} 2\ell_X(\gamma).$$

The constant $1 - \epsilon_B/D_0$ is positive, since $D_0 > \epsilon_B$. So taking K_0 sufficiently larger than $2(1 - \epsilon_B/D_0)^{-1}$ completes the proof. □

LEMMA 3.3. *Let $X \in \mathcal{T}(\mathcal{S})$ and let α be a simple closed geodesic in the hyperbolic metric of X. Let ω be a simple geodesic arc in X. If*

$$\mathrm{i}(\omega, \alpha) \geqslant 3,$$

then any curve γ in the homotopy class of α that is disjoint from α intersects ω at least once.

REMARK 3.4. Note that the statement is sharp in the sense that if γ is not disjoint from α or if ω intersects α less than three times, γ can be disjoint from ω.

Proof. Curves α and γ as above bound an annulus A in X. Suppose that ω intersects α at points p_1, p_2 and p_3, the points being ordered by their appearances along ω. For $i = 1, 2$, let $[p_i, p_{i+1}]$ be the segment of ω between p_i and p_{i+1}. Since α and ω are geodesics, their segments cannot form bigons. Therefore, one of the segments $[p_1, p_2]$ and $[p_2, p_3]$ intersects the interior, and therefore both boundary components of A. □

3.2. *We now prove several useful facts about geodesics in the hyperbolic plane.*

PROPOSITION 3.5. *Let $\phi\colon \mathbb{H} \to \mathbb{H}$ be a hyperbolic isometry with axis γ and translation length $\ell(\phi) \leqslant \epsilon$ for some $\epsilon > 0$. Suppose that α and β are two geodesic lines such that the following two conditions hold:*

1. *α and β intersect γ at the points a and b, respectively, with $d_{\mathbb{H}}(a, b) \leqslant \epsilon$;*
2. *let c be the point on β which is closest to α (the intersection point between α and β if they intersect). Assume $d_{\mathbb{H}}(c, b) \geqslant 2\epsilon$.*

Fix an endpoint β_+ of β and let α_+ be the endpoint of α that is on the same side of γ as β_+. Then, there exists $k \in \mathbb{Z}$ such that α_+ is between β_+ and $\phi^k(\beta_+)$ and $|k|\ell(\phi) \leqslant 4\epsilon + 3$.

Proof. We assume that β_+ is the endpoint of the ray $\overrightarrow{cb}$ (see Figure 2). The case when β_+ is the endpoint of $\overrightarrow{bc}$ is similar.

By exchanging ϕ with ϕ^{-1} if necessary, then we may assume that α_+ is between β_+ and the attracting fixed point γ_+ of ϕ. Let k be a positive integer with

$$3\epsilon + 3 \leqslant k\ell(\phi) \leqslant 4\epsilon + 3.$$

Such a k exists since $\ell(\phi) \leqslant \epsilon$. Let $\beta' = \phi^k(\beta)$, $b' = \phi^k(b)$ and $\beta'_+ = \phi^k(\beta_+)$.

Since $k\ell(\phi) \geqslant \epsilon$, we know a is between b and b'. To show α_+ is between β_+ and β'_+, we need to show that the rays $\overrightarrow{a\alpha_+}$ and $\overrightarrow{b'\beta'_+}$ do not intersect. Note that

$$d_{\mathbb{H}}(b', a) = d_{\mathbb{H}}(b', b) - d_{\mathbb{H}}(a, b) \geqslant (3\epsilon + 3) - \epsilon = 2\epsilon + 3 > d_{\mathbb{H}}(a, b).$$

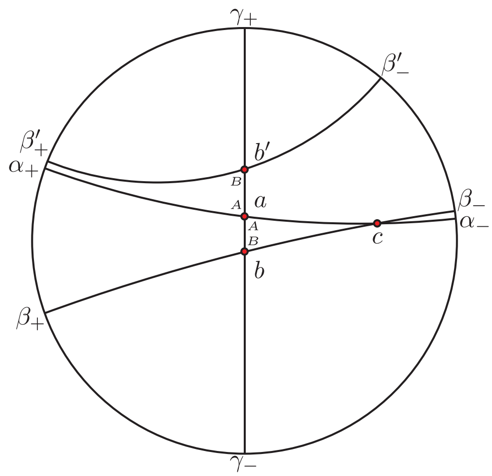

FIGURE 2. *An endpoint of α is sandwiched between β and its close translate.*

Let $A = \angle\alpha_- a\gamma_- = \angle\alpha_+ a\gamma_+$ and $B = \angle\beta_- b\gamma_+ = \angle\beta'_+ b'\gamma_-$. If α and β are disjoint, then $\overrightarrow{a\alpha_+}$ and $\overrightarrow{b'\beta'_+}$ are also disjoint and we are done. Hence, we can assume that α and β intersect at c.

Using the law of cosines, $\overrightarrow{a\alpha_+}$ and $\overrightarrow{b'\beta'_+}$ do not intersect if

$$\sin A \sin B \cosh d_{\mathbb{H}}(a, b') - \cos A \cos B \geqslant \cos 0 = 1. \tag{7}$$

Since $d_{\mathbb{H}}(a, b') \geqslant 2\epsilon + 3$,

$$\cosh d_{\mathbb{H}}(a, b') \geqslant \cosh(2\epsilon + 3) > \frac{e^{2\epsilon+3}}{2}.$$

Hence, by equation (7), it suffices to show that

$$\frac{e^{2\epsilon+3}}{2} \geqslant \frac{1 + \cos A \cos B}{\sin A \sin B}. \tag{8}$$

Before starting the calculations, we make an elementary observation. For any $y > 0$, the function

$$f(x) = \frac{\sinh(x + y)}{\sinh x}$$

is decreasing and $f(x) \leqslant 2e^y$ for all $x \geqslant y$. This is because $f'(x) = -\sinh(y)/\sinh^2(x) < 0$. Hence, for all $x \geqslant y$,

$$f(x) \leqslant f(y) = \frac{\sinh 2y}{\sinh y} = 2\cosh y \leqslant 2e^y.$$

We argue in three cases. Suppose $A > \pi/2$. Since $d_{\mathbb{H}}(c, a) \geqslant d_{\mathbb{H}}(c, b) - d_{\mathbb{H}}(a, b) \geqslant \epsilon$, we have

$$\begin{aligned} \frac{1 + \cos A \cos B}{\sin A \sin B} &= \frac{1 - \cos(\pi - A)\cos B}{\sin A \sin B} \leqslant \frac{\sin^2(\max\{A, B\})}{\sin^2(\min\{A, B\})} \\ &= \frac{\sin^2 A}{\sin^2 B} = \frac{\sinh^2 d_{\mathbb{H}}(c, b)}{\sinh^2 d_{\mathbb{H}}(c, a)} \leqslant \frac{\sinh^2\left(d_{\mathbb{H}}(c, a) + \epsilon\right)}{\sinh^2 d_{\mathbb{H}}(c, a)} \leqslant 4e^{2\epsilon}. \end{aligned}$$

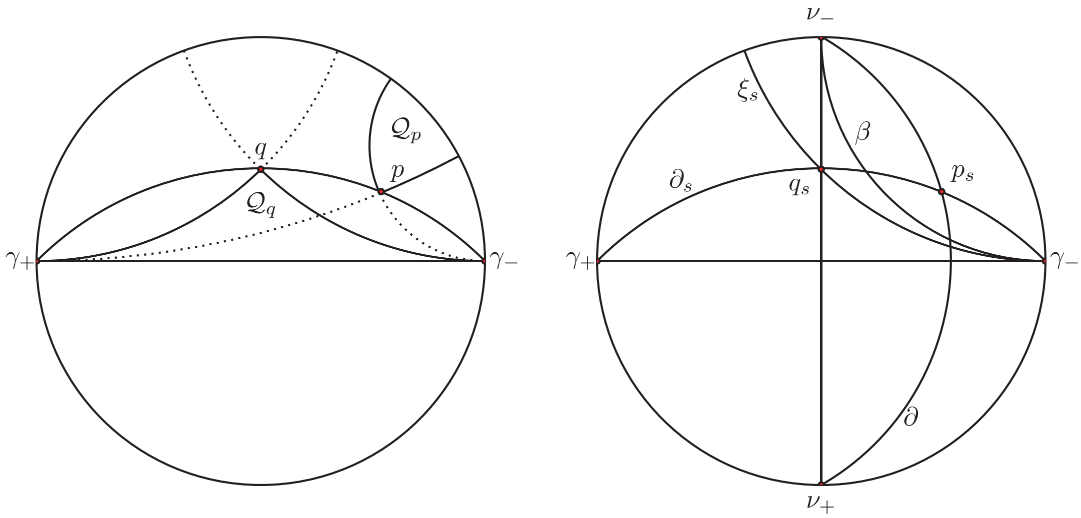

FIGURE 3. *There is a lower bound on the distance between any point in $\mathcal{Q}_p$ and any point in $\mathcal{Q}_q$.*

Similarly, if $B \geqslant \pi/2$ instead, then

$$\frac{1+\cos A \cos B}{\sin A \sin B} \leqslant \frac{\sinh^2(d_{\mathbb{H}}(c,b)+\ell(\phi))}{\sinh^2(d_{\mathbb{H}}(c,b))} \leqslant 4e^{2\epsilon}.$$

In the case where both A and B are at most $\pi/2$, let w be the point on the segment $\overline{ab}$ which is the foot of the perpendicular from c to $\overline{ab}$. We have

$$\begin{aligned}\frac{1+\cos A \cos B}{\sin A \sin B} &\leqslant \frac{2}{\sin A \sin B} = 2\frac{\sinh d_{\mathbb{H}}(c,a)}{\sinh d_{\mathbb{H}}(c,w)}\frac{\sinh d_{\mathbb{H}}(c,b)}{\sinh d_{\mathbb{H}}(c,w)}\\ &\leqslant 2\frac{\sinh^2(d_{\mathbb{H}}(c,w)+\ell(\phi))}{\sinh^2 d_{\mathbb{H}}(c,w)} \leqslant 8e^{2\epsilon}.\end{aligned}$$

But $8e^{2\epsilon} < e^{2\epsilon+3}/2$ and we are done. □

We need some definitions for the next two lemmas. Let γ be a geodesic in $\mathbb{H}$ with endpoints γ_+ and γ_-. Fix a δ-neighborhood U of γ and let p be any point on the boundary of U. The geodesic through p with endpoint γ_+ and the geodesic through p with endpoint γ_- together subdivide $\mathbb{H}$ into four quadrants. The quadrant disjoint from the interior of U will be called the *upper quadrant* at p, and the quadrant diametrically opposite will be called the *lower quadrant* at p.

LEMMA 3.6. *For every $\delta_0 > 0$ and $M > 0$, there is $d_0 > 0$ such that the following holds* (*see the left-hand side of Figure* 3). *Fix a geodesic γ in $\mathbb{H}$ and let U be the δ-neighborhood of γ with $\delta \geqslant \delta_0$. Let p and q be points on the same boundary component of U. Suppose $d_{\mathbb{H}}(p,q) \geqslant M$. Then any point in the upper quadrant $\mathcal{Q}_p$ at p is at least d_0 away from any point in the lower quadrant $\mathcal{Q}_q$ at q.*

Proof. It suffices to find a lower bound for $d_{\mathbb{H}}(p,\xi)$, where ξ is the closest boundary component of $\mathcal{Q}_q$ to p. Assume that ξ has the endpoint γ_-. Note that $d_{\mathbb{H}}(p,\xi)$ increases with $d_{\mathbb{H}}(p,q)$, since any geodesic from γ_- that intersects the boundary of U between p and q separates p from ξ.

Now fix $d_{\mathbb{H}}(p,q) = M$ and see the right-hand side of Figure 3 for the rest of the proof. Fix a perpendicular geodesic ν to γ and let ∂ be a boundary component of the M-neighborhood of ν. Fix an endpoint ν_- of ν and let ∂_s be a boundary curve of the s-neighborhood of γ contained in the complement $\mathbb{H} \smallsetminus \gamma$ determined by ν_-. Let q_s be the intersection of ν and ∂_s and let p_s be the intersection of ∂ and ∂_s. Let ξ_s be the geodesic through q_s with endpoint γ_- and let β be the geodesic connecting γ_- and ν_-. Since β is asymptotic to ν, $d_{\mathbb{H}}(p_s, \beta)$ is increasing as a function of s, and for s big enough, β separates p_s from ξ_s. Therefore, there exist $d_1 > 0$ and $s_1 \geqslant \delta_0$ such that $d_{\mathbb{H}}(p_s, \xi_s) > d_{\mathbb{H}}(p_s, \beta) > d_1 > 0$ for all $s \geqslant s_1$. Finally, since $d_{\mathbb{H}}(p_s, \xi_s)$ is continuous as a function of s and is only zero when $s = 0$, it is bounded away from zero on the segment $[\delta_0, s_1]$. This completes the proof. □

Lemma 3.7. *Given $\delta_0 > 0$ and $M > 0$, the constant d_0 of Lemma 3.6 also satisfies the following property. Let γ be a geodesic in $\mathbb{H}$. Let U be the δ-neighborhood of γ with $\delta \geqslant \delta_0$. Let ω_1 be a geodesic intersecting γ and let ω_2 be the image of ω_1 under an isometry fixing γ. Let p_i be the intersection of ω_i with a fixed boundary component of U. Let ω_1^+ be the component of ω_1 contained in the upper quadrant at p_1. Suppose $d_{\mathbb{H}}(p_1, p_2) \geqslant M$. Then $d_{\mathbb{H}}(\omega_1^+, \omega_2) \geqslant d_0$.*

Proof. Let ω_2^+ be the component of ω_2 contained in the upper quadrant at p_2. By Lemma 3.6, it is enough to show $d_{\mathbb{H}}(\omega_1^+, \omega_2^+) \geqslant d_0$. Let r_i be the point on ω_i such that $d_{\mathbb{H}}(r_1, r_2)$ realizes the distance between ω_1 and ω_2. Because of the symmetry of ω_1 and ω_2 relative to a rotation fixing γ, either both r_1 and r_2 are contained in U, and hence each r_i is contained in the lower quadrant at p_i, or one is in the lower quadrant and the other is in the upper quadrant.

Identify the space of pairs of (x, y), where $x \in \omega_1$ and $y \in \omega_2$, with $\mathbb{R}^2$. Then the function $\mathbb{R}^2 \to \mathbb{R}$ sending (x, y) to $d_{\mathbb{H}}(x, y)$ is a convex function realizing its minimum at $(r_1, r_2) \in \mathbb{R}^2$. In the first case, since $d_{\mathbb{H}}(p_i, \omega_{3-i}^+) \geqslant d_0$ by Lemma 3.6 and the distance between ω_1^+ and ω_2^+ increases from p_1 and p_2 on, and we can conclude $d_{\mathbb{H}}(\omega_1^+, \omega_2^+) \geqslant d_0$. In the second case, invoking Lemma 3.6 again implies $d_{\mathbb{H}}(r_1, r_2) \geqslant d_0$, hence $d_{\mathbb{H}}(\omega_1^+, \omega_2^+) \geqslant d_0$ by minimality of $d_{\mathbb{H}}(r_1, r_2)$. □

4. *A notion of being sufficiently horizontal*

Let I be a closed connected subset of $\mathbb{R}$, let $\mathcal{G}\colon I \to \mathcal{T}(\mathcal{S})$ be a Thurston geodesic and let $\lambda_{\mathcal{G}}$ be its maximally stretched lamination. The main purpose of this section is to develop a notion of a closed curve α being sufficiently horizontal along $\mathcal{G}$, such that if α is horizontal, then it remains horizontal and its horizontal length grows exponentially along $\mathcal{G}$.

Definition 4.1. Given a curve α, we will say that α is *(n, L)-horizontal* at $t \in I$ if there exist an ϵ_B-short curve γ on $X_t = \mathcal{G}(t)$ and a leaf λ in $\lambda_{\mathcal{G}}$ such that the following statements hold (see Figure 4).

(H1) In the universal cover $\widetilde{X}_t \cong \mathbb{H}$, there exists a collection of lifts $\{\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_n\}$ of γ and a lift $\widetilde{\lambda}$ of λ intersecting each $\widetilde{\gamma}_i$ at a point p_i (the points p_i are indexed by the order of their appearances along $\widetilde{\lambda}$) such that $d_{\mathbb{H}}(p_i, p_{i+1}) \geqslant L$ for all $i = 1, \ldots, n-1$.

(H2) There exists a lift $\widetilde{\alpha}$ of α such that $\widetilde{\alpha}$ intersects $\widetilde{\gamma}_i$ at a point q_i with $d_{\mathbb{H}}(p_i, q_i) \leqslant \epsilon_B$ for each i.

We will call γ an *anchor curve* for α and $\widetilde{\alpha}$ an *(n, L)-horizontal lift* of α.

Set $\mathcal{G}(t) = X_t$. The main result of this section is the following.

Theorem 4.2. *There are constants n_0, L_0 and s_0 such that the following holds. Suppose that a curve α is (n_s, L_s)-horizontal at $s \in I$ with $n_s \geqslant n_0$ and $L_s \geqslant L_0$. Then we have the*

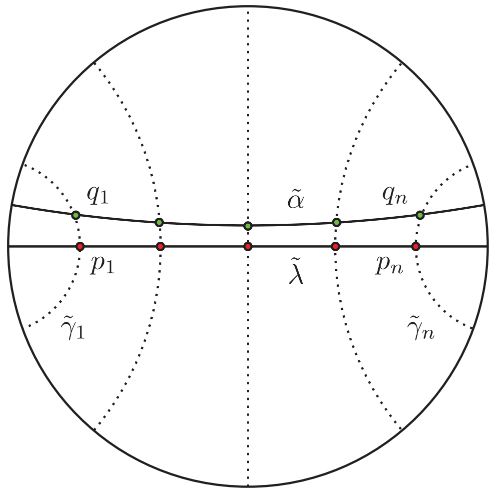

FIGURE 4. *The curve α is (n, L)-horizontal.*

following propositions.

(I) *For any $t \geqslant s + s_0$, α is (n_t, L_t)-horizontal at t, with*

$$n_t \overset{*}{\succ} n_s, \quad \text{and} \quad L_t \geqslant L_s.$$

(II) *Furthermore, for any A, if $d_{\mathcal{C}(\mathcal{S})}(X_s, X_t) \geqslant A$, then*

$$\log \frac{n_t}{n_s} \succ A \quad \text{and} \quad L_t n_t \overset{*}{\succ} e^{t-s} L_s n_s.$$

DEFINITION 4.3 (Sufficiently Horizontal). Let (n_0, L_0) be the constants given by Theorem 4.2. A curve α will be said to be *sufficiently horizontal* at $t \in I$ if it is (n, L)-horizontal for some $n \geqslant n_0$ and $L \geqslant L_0$.

EXAMPLE 4.4. Definition 4.1 is a bit technical and warrants some justification. To stay sufficiently horizontal along a Thurston geodesic $\mathcal{G}$, we require the curve α to fellow-travel λ both *geometrically* and *topologically* for a long time. The following example illustrates why these requirements are necessary. Namely, we will show that the weaker version of geometric fellow-traveling does not always persist along a Thurston geodesic.

Referring to the example in Subsection 2.11, for any ϵ, there exists a Thurston geodesic $\mathcal{G}(t) = X_t$ and a curve γ such that a leaf λ of $\lambda_{\mathcal{G}}$ intersects γ and, for $t > 0$, $\ell_{X_t}(\gamma) \overset{*}{\asymp} \epsilon e^t + 2e^{-e^t}$. Consider the following two points along $\mathcal{G}(t)$:

$$X = \mathcal{G}(\log\log(1/\epsilon)) \quad \text{and} \quad Y = \mathcal{G}(\log 1/\epsilon).$$

On Y, let α be the shortest curve that intersects γ with $\operatorname{twist}_\gamma(\alpha, Y) = 0$. It was shown in Subsection 2.11 that

$$\operatorname{twist}_\gamma(\lambda, Y) \overset{+}{\asymp} \frac{1}{\epsilon}.$$

This implies $\operatorname{twist}_\gamma(\alpha, \lambda) \overset{+}{\asymp} 1/\epsilon$, so α intersects λ at an angle close to $\pi/2$ in Y. Furthermore, since $\ell_Y(\gamma) \overset{*}{\asymp} 1$, we have $\ell_Y(\alpha) = O(1)$. That is, every large enough segment of any lift of α to the universal cover $\widetilde{Y}$ intersects a lift of λ at a near right angle. Therefore, in $\widetilde{Y}$, no lift of α will fellow-travel any lift of λ.

On the other hand, it was also shown in Subsection 2.11 that

$$\ell_X(\gamma) \stackrel{*}{\asymp} \epsilon \log(1/\epsilon).$$

For a given L, we can choose ϵ small enough such that the collar neighborhood of γ in X has width at least L. Since α and λ both pass through this collar, in the universal cover $\widetilde{X}$, there exists a lift $\widetilde{\alpha}$ and a lift $\widetilde{\lambda}$ that are $O(1)$-close for L-length. In other words, α and λ fellow-travel in $\widetilde{X}$ but they do not in $\widetilde{Y}$.

REMARK 4.5. For a given $t \in I$, it is possible that there are no sufficiently horizontal curves at t. For instance, when the stump of $\lambda_{\mathcal{G}}$ is a curve that does not intersect any ϵ_B-short curve. But this is the only problem, since if the stump of $\lambda_{\mathcal{G}}$ intersects an ϵ_B-short curve γ, then any sequence of curves converging to the stump will eventually be sufficiently horizontal, with anchor curve γ. In Section 5, we will show that one can always find a sufficiently horizontal curve after moving a bounded distance in $\mathcal{C}(\mathcal{S})$ (Proposition 5.10).

The next proposition will show that the condition (H2) of Definition 4.1 can be obtained by just assuming that $\widetilde{\alpha}$ stays ϵ_B-close to the segment $[p_1, p_n]$ in $\widetilde{\lambda}$. A priori, even if $\widetilde{\alpha}$ is ϵ_B-close to $[p_1, p_n]$, then the distance between p_i and q_i may still be large if $\widetilde{\gamma}_i$ is nearly parallel to $\widetilde{\lambda}$ or $\widetilde{\alpha}$.

PROPOSITION 4.6. *There are constants n_0 and L_0 such that, for any hyperbolic surface X and constants $n \geqslant n_0$ and $L \geqslant L_0$, the following statement holds. Suppose that γ is an ϵ_B-short curve in X, λ is a complete simple geodesic in X, and n lifts $\{\widetilde{\gamma}_i\}$ of $\widetilde{\gamma}$ and a lift $\widetilde{\lambda}$ are chosen to satisfy $(H1)$. If α is a curve in X that has a lift $\widetilde{\alpha}$ which stays, up to a bounded multiplicative error, ϵ_B-close to the segment $[p_1, p_n]$ in $\widetilde{\gamma}$, then there exist indices l and r with $r - l \stackrel{+}{\succ} n$ such that $\widetilde{\alpha}$ intersects $\widetilde{\gamma}_i$ at a point q_i and $d_{\mathbb{H}}(p_i, q_i) \leqslant \epsilon_B$ for all $i = l, \ldots, r$.*

Proof. Recall the standard collar $U(\gamma)$ is a regular neighborhood of γ in X that is an embedded annulus with boundary length $\stackrel{*}{\asymp} 1$. Let δ be the distance between γ and the boundary of $U(\gamma)$. We have (see Section 3)

$$\delta \stackrel{+}{\asymp} \log(1/\ell_X(\gamma)).$$

Let L_0 satisfy inequality

$$\epsilon_B e^{-L_0} < \delta.$$

The distance between $\widetilde{\alpha}$ and $\widetilde{\lambda}$ is a convex function that essentially either increases or decreases exponentially fast. Hence, we can choose a segment $\overline{\alpha}$ of α and index i_0 such that $\overline{\alpha}$ is within $\epsilon_B e^{-L}$-Hausdorff distance of $\overline{\lambda} = [p_{i_0}, p_{n-i_0}]$. The index i_0 can be chosen independent of n or L because the distance between p_i and p_{i+1} is at least L. Let $n_0 \geqslant 2i_0 + 2$.

Let U_i be the δ-neighborhood of $\widetilde{\gamma}_i$. By the choice of δ, U_i and U_j are disjoint for $i \neq j$. Since $\epsilon_B e^{-L} < \delta$, endpoints of $\overline{\alpha}$ are contained in U_{i_0} and U_{n-i_0}. Also, for $i_0 < i < n - i_0$, U_i separates U_{i_0} and U_{n-i_0} in $\mathbb{H}^2$. Hence $\overline{\alpha}$ intersects every $\widetilde{\gamma}_i$. Consider such an i and, for simplicity, set $\widetilde{\gamma} = \widetilde{\gamma}_i$, $U = U_i$, $p = p_i$ and $q = q_i$. To prove the Proposition, we need to show that $d_{\mathbb{H}}(p, q) \leqslant \epsilon_B$. We refer to Figure 5 in the following.

Assume, for contradiction, that $d_{\mathbb{H}}(p, q) > \epsilon_B$. Let ϕ be a hyperbolic isometry with axis $\widetilde{\gamma}$ and translation length $\ell_X(\gamma)$. Then, since γ is ϵ_B-short, up to replacing ϕ with ϕ^{-1}, the point $o = \phi(p)$ is strictly between p and q.

Since $\widetilde{\lambda}$ and $\phi(\widetilde{\lambda})$ are disjoint, for some boundary component of U, that we will denote by ∂U, the following holds. For $p' = \widetilde{\lambda} \cap \partial U$ and $q' = \widetilde{\alpha} \cap \partial U$, the point $o' = \phi(p')$ of intersection of ∂U and $\phi(\widetilde{\lambda})$ is between p' and q'. The curve ∂U is equidistant to $\widetilde{\gamma}$ and the distance function

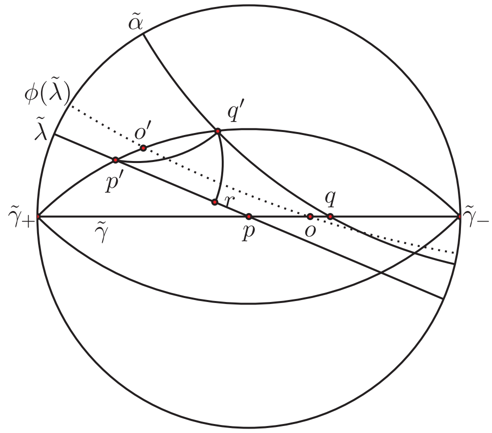

FIGURE 5. *If $d_{\mathbb{H}}(p,q) \geqslant \epsilon_B$, then q is far from $\widetilde{\lambda}$, which is a contradiction.*

$d_{\mathbb{H}}$ is convex along this curve, therefore

$$d_{\mathbb{H}}(q',p') \geqslant d_{\mathbb{H}}(o',p') = \ell_X(\partial U) \stackrel{*}{\asymp} 1. \tag{9}$$

Let r be the closest point on $\widetilde{\lambda}$ to q'. The point r is contained in one of the quadrants at p', hence $d_{\mathbb{H}}(q',r) \stackrel{*}{\succ} 1$ by Lemma 3.6. But for sufficiently large L_0, this will contradict that q' is $\epsilon_B e^{-L_0}$ close to $\widetilde{\lambda}$. Hence $d_{\mathbb{H}}(p,q) \leqslant \epsilon_B$ and we are done. $\square$

Proof of Theorem 4.2. Let $n_s \geqslant n_0$ and $L_s \geqslant L_0$, n_0, L_0 to be determined later. Let α be an (n_s, L_s)-horizontal curve at X_s. As in the definition, we have an anchor curve γ, a lift $\widetilde{\alpha}$ of α, a lift $\widetilde{\lambda}$ of a leaf of $\lambda_{\mathcal{G}}$ and n_s-lifts $\{\widetilde{\gamma}_i\}$ of γ, such that $d_{\mathbb{H}}(p_i,q_i) \leqslant \epsilon_B$ and $d_{\mathbb{H}}(p_i,p_{i+1}) \geqslant L_s$, where p_i is the intersection of $\widetilde{\gamma}_i$ with $\widetilde{\lambda}$, and q_i the intersection of $\widetilde{\gamma}_i$ with $\widetilde{\alpha}$.

Throughout the proof, we will add several conditions on n_0, s_0 and L_0. Let n_0 and L_0 be at least as big as the corresponding constants obtained in Proposition 4.6.

Let c be the point on $\widetilde{\lambda}$ to which $\widetilde{\alpha}$ is closest. To be able to apply Proposition 3.5 to the curve γ_i, we need c to have a distance of at least $2\epsilon_B$ from p_i. Assuming $L_0 > 4\epsilon_B$, we have c is $2\epsilon_B$-close to at most one p_i. That is, we can choose indices l and r, with $(l,r) = (1, n_s - 1)$ or $(l,r) = (2, n_s)$, such that c has a distance at least $2\epsilon_B$ from both p_l and p_r.

See Figure 6 for the following. Applying a Möbius transformation if necessary, we can assume that the center of the disk o is the midpoint between p_l and p_r. Let $\widetilde{\lambda}_+$ and $\widetilde{\lambda}_-$ be, respectively, the endpoints of $\widetilde{\lambda}$ determined by the rays $\overrightarrow{op_l}$ and $\overrightarrow{op_r}$. Let $\widetilde{\alpha}_+$ be the endpoint of $\widetilde{\alpha}$ closest to $\widetilde{\lambda}_+$. Let ϕ be the hyperbolic isometry with axis $\widetilde{\gamma}_l$ and translation length $\ell(\phi) = \ell_s(\gamma)$. Let k be the constant of Proposition 3.5 and let

$$\widetilde{\lambda}' = \phi^k(\widetilde{\lambda}), \quad \widetilde{\lambda}'_+ = \phi^k(\widetilde{\lambda}_+) \quad \text{and} \quad p'_l = \phi^k(p_l).$$

We have that $\widetilde{\alpha}_+$ is sandwiched between $\widetilde{\lambda}_+$ and $\widetilde{\lambda}'_+$ and $d_{\mathbb{H}}(p_l,p'_l) \stackrel{*}{\prec} \epsilon_B$. Similarly, by considering the hyperbolic isometry ψ with axis $\widetilde{\gamma}_r$, we can sandwich $\widetilde{\alpha}_-$ between $\widetilde{\lambda}_-$ and $\widetilde{\lambda}''_-$ with $d_{\mathbb{H}}(p_r,p'_r) \stackrel{*}{\prec} \epsilon_B$, where

$$\widetilde{\lambda}'' = \psi^k(\widetilde{\lambda}), \quad \widetilde{\lambda}''_- = \psi^k(\widetilde{\lambda}_-) \quad \text{and} \quad p'_r = \psi^k(p_r).$$

Let $s_0 \geqslant 0$, $t \geqslant s + s_0$ and $f\colon X_s \to X_t$ be an optimal map, that is, an e^{t-s}-Lipschitz map. Since $\lambda_{\mathcal{G}}$ is in the stretch locus of f, there is a lift $\widetilde{f}\colon \widetilde{X}_s \to \widetilde{X}_t$ of f such that $\widetilde{f}(\widetilde{\lambda}) = \widetilde{\lambda}$ and $\widetilde{f}(\widetilde{\lambda}_\pm) = \widetilde{\lambda}_\pm$.

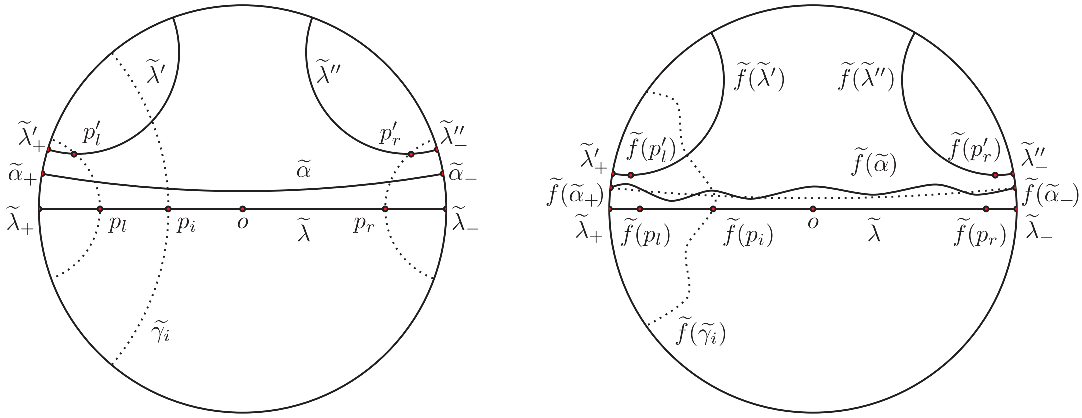

FIGURE 6. *The endpoints of $\widetilde{f}(\widetilde{\alpha})$ are sandwiched between the endpoints of geodesics $\widetilde{f}(\widetilde{\lambda})$, $\widetilde{f}(\widetilde{\lambda}')$ and $\widetilde{f}(\widetilde{\lambda}'')$.*

CLAIM 4.7. *The geodesic representative α' of $\widetilde{f}(\widetilde{\alpha})$ stays $O(\epsilon_B)$-close to $\widetilde{\lambda}$ from $\widetilde{f}(p_{l+1})$ and $\widetilde{f}(p_{r-1})$.*

Proof. Composing with a Möbius transformation if necessary, we may assume $\widetilde{f}(o) = o$. Note that $\widetilde{f}(\widetilde{\lambda}')$ is a geodesic and $\widetilde{f}(\widetilde{\alpha}_+)$ is sandwiched between $\widetilde{\lambda}_+$ and $\widetilde{f}(\widetilde{\lambda}'_+)$. Similarly, $\widetilde{f}(\widetilde{\alpha}_-)$ is sandwiched between $\widetilde{\lambda}_-$ and $\widetilde{f}(\widetilde{\lambda}'_-)$. Consider the sector V_+ between the rays $\overrightarrow{o\,\widetilde{\lambda}_+}$ and $\overrightarrow{o\widetilde{f}(p'_l)}$ and the sector V_- between the rays $\overrightarrow{o\,\widetilde{\lambda}_-}$ and $\overrightarrow{o\,\widetilde{f}(p'_r)}$. The geodesic α' connecting $\widetilde{f}(\widetilde{\alpha}_+)$ and $\widetilde{f}(\widetilde{\alpha}_-)$ stays in a bounded neighborhood of the union V_+ and V_-.

Note that

$$d_{\mathbb{H}}(\widetilde{f}(p_l), \widetilde{f}(p_{l+1}) \geqslant L_0 e^{t-s} \quad \text{and} \quad d_{\mathbb{H}}(\widetilde{f}(p_l), \widetilde{f}(p'_l)) \leqslant e^{t-s}\epsilon_B.$$

Also, the distance between intersecting geodesics increases exponentially fast. Hence

$$\begin{aligned} d_{\mathbb{H}}(\alpha', \widetilde{f}(p_{l+1})) &\overset{+}{\prec} e^{-d_{\mathbb{H}}(\widetilde{f}(p_l),\widetilde{f}(p_{l+1}))} d_{\mathbb{H}}(\widetilde{f}(p_l), \widetilde{f}(p'_l)) \\ &\leqslant e^{-L_0 e^{t-s}} e^{t-s}\epsilon_B \leqslant \epsilon_B. \end{aligned}$$

The last inequality holds as long as $L_0 \geqslant 1$. Similarly, $d_{\mathbb{H}}(\alpha', \widetilde{f}(p_{r-1})) \leqslant \epsilon_B$. □

We next show that the projection to X_t of any long enough piece of the segment of $\widetilde{\lambda}$ between $\widetilde{f}(p_l)$ and $\widetilde{f}(p_r)$ intersects a lift of an ϵ_B-short curve.

CLAIM 4.8. *For any $l \leqslant i \leqslant r-3$, let $\widetilde{\omega}$ be the arc connecting $\widetilde{f}(p_i)$ and $\widetilde{f}(p_{i+3})$, and let ω be the projection of $\widetilde{\omega}$ to X_t. Then ω intersects an ϵ_B-short curve.*

Proof. Recall the dual constant $\delta_B > 0$ to ϵ_B, which is a lower bound for the length of any curve that intersects an ϵ_B-short curve.

If ω is not simple, then $\widetilde{\lambda}$ is a lift of a closed curve λ and ω wraps around λ. By definition λ intersects an ϵ_B-short curve in X_s which implies that $\ell_{X_s}(\lambda) \geqslant \delta_B$ and $\ell_{X_t}(\lambda) \geqslant \delta_B e^{t-s}$. If $t - s \geqslant s_0 > \log(\epsilon_B/\delta_B)$, then $\ell_{X_t}(\lambda) \geqslant \epsilon_B$ and λ has to intersect an ϵ_B-short curve in X_t. Thus, ω will also intersect an ϵ_B-short curve.

Now assume that ω is simple. Let $\gamma' = f(\gamma)$. Note that γ' is not necessarily a geodesic in the metric X_t. If ω misses all the curves of length at most ϵ_B, then it is contained in a pair of pants P in X_t with boundary lengths at most ϵ_B. Since endpoints of ω lie on γ', $\gamma' \cap P$ is non-empty.

First, we assume that γ' does not intersect ∂P. Then $\gamma' \subset P$ and it is homotopic to a boundary component of P. That is, the geodesic representative γ^* of γ' in X_s is ϵ_B-short. The arc $f^{-1}(\omega)$ is a geodesic in X_s and intersects γ (which is a geodesic in X_s) at least three times, and so, by Lemma 3.3, $f^{-1}(\omega)$ intersects $f^{-1}(\gamma^*)$ at least once. This implies ω intersects γ^* which proves the claim.

Now assume that γ' intersects ∂P. For $j = 0, 1, 2$, let ω_j be the subarc of ω coming from projecting $[\widetilde{f}(p_{i+j}), \widetilde{f}(p_{i+j+1})]$ to X_t. Let D_0 and K_0 be the constants of Lemma 3.2. Note that if a subarc of λ intersects γ', then the arc length between two consecutive intersections is at least $\delta_B e^{t-s} \geqslant \delta_B e^{s_0}$. Assuming $s_0 \geqslant \log \frac{D_0}{\delta_B}$, we have, for each j, $\ell_t(\omega_j)$ is at least $L_0 e^{t-s}$, while $\ell_t(\gamma')$ is at most $\epsilon_B e^{t-s}$. Let L_0 be bigger than $K_0(\epsilon_B + 1)$. Then, by Lemma 3.2, at least one of w_j has

$$\ell_t(\omega_j) \leqslant K_0 \cdot (\ell_t(\gamma') + 1) < L_0 e^{t-s}$$

which is impossible and hence ω intersects one of the pants curves. □

Claim 4.8 implies that, for some ϵ_B-short curve (call it γ_t) and some $n_t \overset{*}{\succ} n_s$, the projection of $[\widetilde{f}(p_{l+1}), \widetilde{f}(p_{r-1})]$ to X_t intersects γ_t at least n_t times where the arc length between every two intersection points is at least $L_t = L_s e^{t-s}$. And Claim 4.7 implies that there is a large segment of α that remains $O(\epsilon_B)$-close to the segment $[\widetilde{f}(p_{l+1}), \widetilde{f}(p_{r-1})]$. Applying Proposition 4.6, we conclude that α is (n_t, L_t)-horizontal, which proves part (I).

We now prove part (II) of the Theorem 4.2. Suppose

$$d_{\mathcal{C}(\mathcal{S})}(X_s, X_t) \geqslant A. \tag{10}$$

We need to show that n_t lifts of an ϵ_B-short curve intersect the segment $[\widetilde{f}(p_{l+1}), \widetilde{f}(p_{r-1})]$, where $\log(n_t/n_s) \succ A$ such that any two consecutive intersections are at least $L_t \overset{*}{\succ} L_s n_s e^{t-s}/n_t$ away.

Let $\mathcal{P}$ be an ϵ_B-short pants decomposition on X_t and $m = \min_{\beta \in \mathcal{P}} \mathrm{i}(\beta, \gamma)$. From equation (1), we have

$$A \overset{*}{\prec} \log m.$$

Note that, for small values of A, part (II) follows from Part (I). Hence, we assume that A is large, which implies in particular that γ intersects every curve in $\mathcal{P}$. Even though part (II) seems to be more general, this last condition is used in an essential way in the proof of Part (II) and the proof does not naturally extend to prove part (I).

We also have

$$m \overset{*}{\prec} \frac{\ell_{X_t}(\gamma)}{\delta_B} \leqslant e^{t-s} \frac{\epsilon_B}{\delta_B} \overset{*}{\prec} e^{t-s}.$$

Cut the segment $[\widetilde{f}(p_l), \widetilde{f}(p_r)]$ into mn_s equal pieces and let $\widetilde{\omega}$ be one of them. Denote the projection of $\widetilde{\omega}$ to X_t by ω. We would like to show that ω intersects a curve in $\mathcal{P}$. As in the proof of Claim 4.8, if ω is not simple, then it wraps around a simple closed $\lambda \in \lambda_G$ and assuming $s_0 > \log(\epsilon_B/\delta_B)$, it has to intersect a curve in $\mathcal{P}$. Hence we assume that ω is simple.

Assume for a contradiction that ω is disjoint from $\mathcal{P}$. Then $\omega \subset P$ for some pair of pants P with ϵ_B-short boundaries. It follows from Lemma 3.1 that there is $\beta \subset \partial P$ such that $U(\beta)$ contains an endpoint of ω and such that

$$\ell_{X_t}(\omega) \overset{+}{\prec} 2(\mathrm{i}(\omega, \tau_\beta)\ell_{X_t}(\beta) + \ell_{X_t}(\tau_\beta)). \tag{11}$$

Then γ intersects β at least m times. Pick any of the subarcs σ of γ that connect both boundary components of $U(\beta)$. Then,

$$\mathrm{i}(\omega, \tau_\beta) \stackrel{+}{\prec} \mathrm{i}(\omega, \sigma) + \mathrm{i}(\sigma, \tau_\beta) \stackrel{+}{\prec} \frac{1}{m}\mathrm{i}(\omega, \gamma) + \mathrm{i}(\sigma, \tau_\beta). \tag{12}$$

The last inequality holds because ω intersects every component of $\gamma^\star \cap U(\beta)$ essentially the same number of times. Also,

$$\mathrm{i}(\sigma, \tau_\beta)\, \ell_t(\beta) + \ell_{X_t}(\tau_\beta) \stackrel{+}{\prec} \ell_{X_t}(\sigma) \stackrel{+}{\prec} \frac{1}{m}\ell_{X_t}(\gamma). \tag{13}$$

and

$$I(\omega, \gamma) \stackrel{+}{\prec} \frac{\ell_{X_t}(\omega)}{\delta_B\, e^{t-s}}. \tag{14}$$

From the last four equations and using $\ell_t(\gamma) \leqslant \epsilon_B e^{t-s}$ and $\ell_{X_t}(\beta) \leqslant \epsilon_B$, we have

$$\ell_{X_t}(\omega) \stackrel{+}{\prec} \ell_{X_t}(\omega)\frac{2\epsilon_B}{m\delta_B e^{t-s}} + \frac{2\epsilon_B e^{t-s}}{m}.$$

But $e^{t-s} \stackrel{*}{\succ} m$. Hence, for some uniform constant C

$$\ell_{X_t}(\omega)\left(1 - \frac{2\epsilon_B}{m\delta_B e^{t-s}}\right) \leqslant \frac{Ce^{t-s}}{m}. \tag{15}$$

The expression in parentheses on the left-hand side is strictly positive since we have assumed $s_0 > \log(\epsilon_B/\delta_B)$. Finally, if we choose L_0 such that

$$L_0\left(1 - \frac{\epsilon_B}{\delta_B e^{s_0}}\right) > C,$$

then equation (15) contradicts

$$\ell_{X_t}(\omega) \geqslant \frac{L_s e^{t-s}}{m}.$$

Contradiction proves that ω intersects some curve in $\mathcal{P}$.

There are at least $m(n_s - 4)$ such subsegments in $[\widetilde{f}(p_{l+1}), \widetilde{f}(p_{r-1})]$ and each intersects a lift of a curve in $\mathcal{P}$. If we choose every other segment, then we can guarantee that the distance along λ between these intersection points is larger than $L_t = e^{t-s}L_s$. Color these segments according to which curve in $\mathcal{P}$ their projection to X_t intersects and let β be the curve used most often. Then the number n_t of segments intersecting a lift of β satisfies $n_t \stackrel{*}{\succ} mn_s$. Applying Proposition 4.6 finishes the proof. □

5. *Shadow to the curve graph*

To show that the shadow of a Thurston geodesic to the curve graph is a quasi-geodesic, we construct a retraction from the curve graph to the image of the shadow sending a curve α to the shadow of the point in the Thurston geodesic where α is *balanced*.

5.1. *Balanced time for curves*

Let n_0 and L_0 be the constants of Theorem 4.2.

Definition 5.1 (Balanced time). Let $\mathcal{G}\colon [a, b] \to \mathcal{T}(\mathcal{S})$ be a Thurston geodesic segment. For any curve α, let

$$t_\alpha = \inf\{t \in [a, b] \mid \ \alpha \text{ is } (n_0, L_0)\text{-horizontal at } \mathcal{G}(t)\}$$

Let $t_\alpha = b$ if the above set is empty. We refer to t_α as the *balanced time* of α along $\mathcal{G}$.

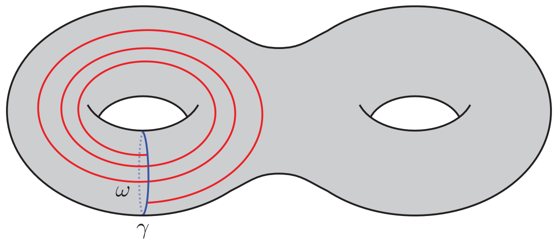

FIGURE 7. *Corridor generated by γ and ω.*

Recall the shadow map $\pi\colon \mathcal{T}(\mathcal{S}) \to \mathcal{C}(\mathcal{S})$ from Section 2.12. The following theorem asserts that the shadow map is a coarse Lipschitz map.

THEOREM 5.2. *Let $\mathcal{G}\colon [a,b] \to \mathcal{T}(\mathcal{S})$ be a Thurston geodesic and let $\pi\colon \mathcal{T}(\mathcal{S}) \to \mathcal{C}(\mathcal{S})$ be the shadow map. Suppose α and β are disjoint curves with $t_\beta \geqslant t_\alpha$. Then $\pi \circ \mathcal{G}([t_\alpha, t_\beta])$ has uniformly bounded diameter in $\mathcal{C}(\mathcal{S})$.*

In the following, we develop some notions that will be used to prove Theorem 5.2.

Let X be a hyperbolic surface. A *rectangle* R in X is the image of a continuous map $\phi\colon [0,a] \times [0,b] \to X$ such that ϕ is a homeomorphism on the interior of $[0,a] \times [0,b]$ and the image of each boundary segment of $[0,a] \times [0,b]$ is a geodesic arc in X.

DEFINITION 5.3. Let γ be a simple closed geodesic on X, ω be a geodesic arc and R be a rectangle given by $\phi\colon [0,a] \times [0,b] \to X$. We say R is an (n,L)-corridor generated by γ and ω (Figure 7) if the following conditions are satisfied.

(i) Edges $\{0\} \times [0,b]$ and $\{a\} \times [0,b]$ are mapped to subarcs of ω.
(ii) There are $0 = t_1 < \cdots < t_n = b$ such that each $[0,a] \times \{t_i\}$ is mapped to a subarc of γ, for all $i = 1, \ldots, n$.
(iii) Arcs $\phi([t_i, t_{i+1}] \times \{0\})$ and $\phi([t_i, t_{i+1}] \times \{a\})$ have lengths at least L.

LEMMA 5.4. *Let X be a hyperbolic surface and γ be an ϵ_B-short curve on X. For any constants n and L, let ω be a simple geodesic arc (possibly closed) with endpoints on γ such that*

$$\mathrm{i}(\gamma, \omega) \geqslant C(n,L) = (6|\chi(X)| + 1)n \left\lceil \frac{L}{\delta_B} \right\rceil + 3|\chi(X)| + 1.$$

Then there exists an (n,L)-corridor generated by γ and ω.

Proof. Fix n and L, and let $C = C(n,L)$. Let $\mathrm{i}(\omega, \gamma) = N \geqslant C$. The closure of each connected component of $X \smallsetminus \{\gamma \cup \omega\}$ is a surface with a piecewise geodesic boundary. Let Q be a complementary component. We refer to points in the boundary of Q where two geodesic pieces meet as an angle. Define the total combinatorial curvature of Q to be

$$\kappa(Q) = \chi(Q) - \frac{\#\text{ of angles in } \partial Q}{4}.$$

We can represent X combinatorially with all angles having value $\pi/2$ to obtain

$$\sum_Q \kappa(Q) = \chi(X).$$

Note that, for every component Q that is not a rectangle, $\kappa(Q) < -\frac{1}{2}$ and hence, the number of components that are not rectangles is bounded by $2|\chi(X)|$. In fact, the number of angles that appear in non-rectangle components is at most $12|\chi(X)|$, because the ratio of the number of angles to Euler characteristic is maximum in the case of a hexagon. Since the total number of angles is $4N-4$ (there are only two angles at the first and the last intersection points), the number of rectangles is at least

$$\frac{(4N-4)-12|\chi(X)|}{4} = N - 3|\chi(X)| - 1 \geqslant (6|\chi(X)|+1)n\left\lceil \frac{L}{\delta_B} \right\rceil.$$

We will say two rectangle components can be joined if they share an arc of γ. A *maximal sequence of joined rectangles* is a sequence $\{Y_1, \dots Y_s\}$ of rectangles in $X \smallsetminus \{\omega \cup \gamma\}$ such that Y_i and Y_{i+1} can be joined for $i = 1, \dots, (s-1)$ and such that Y_1 and Y_s share a boundary with a non-rectangle component. The number of edges of rectangles that share with a non-rectangle component is at most two more than the number of angles of non-rectangle components (again coming from the first and last intersection points of γ and ω). That is, the number of maximal sequences of joined rectangles is at most

$$\frac{2 \cdot (\#\text{ of rectangles})}{12|\chi(X)|+2}.$$

Therefore, there must be at least one maximal sequence of joined rectangles $\{Y_1, Y_2, \dots, Y_M\}$ where $M \geqslant n\lceil L/\delta_B \rceil$. For each $i = 1, \dots, M$, the sides of Y_i coming from arcs of ω have endpoints on γ and thus are at least δ_B long. Therefore, the union $\bigcup_{j=1}^{M} Y_i$ is an (n, L)-corridor for γ after letting t_i be the point that maps to the intersection number $i \cdot \lceil L/\delta_B \rceil$. ☐

PROPOSITION 5.5. *Let n_0 and L_0 be the constants from Theorem 4.2. There exists a constant n_1 such that, for any $n \geqslant n_1$ and $L \geqslant L_0$, if α is (n, L)-horizontal at $\mathcal{G}(t) = X_t$, then any curve β disjoint from α is either (n_0, L_0)-horizontal at X_t or $d_{\mathcal{C}(\mathcal{S})}(X_t, \beta) = O(1)$.*

Proof. Let $n_1 = 3C(n_0, L_0)$ (see Lemma 5.4). Also let $n \geqslant n_1$ and $L \geqslant L_0$.

Suppose that α is (n, L)-horizontal at X_t. Let γ be an ϵ_B-short curve on X_t, $\widetilde{\lambda}$ be a lift of a leaf of $\lambda_{\mathcal{G}}$, $\widetilde{\alpha}$ be a lift of α and $\{\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n\}$ be n-lifts of $\widetilde{\gamma}$ together satisfying Definition 4.1. Choose a most central segment $\widetilde{\omega} \subset \widetilde{\alpha}$ between $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_n$ such that $\widetilde{\omega}$ intersects $C(n_0, L_0)$ lifts of γ, including two intersections coming from the endpoints of $\widetilde{\omega}$. Let ω be the projection of $\widetilde{\omega}$ to X_t. If $\mathrm{i}(\omega, \gamma) < C(n_0, L_0)$, then $\omega = \alpha$ and we are done since

$$d_{\mathcal{C}(\mathcal{S})}(X_t, \beta) \overset{+}{\prec} d_{\mathcal{C}(\mathcal{S})}(\gamma, \alpha) \overset{*}{\prec} \log C(n_0, L_0) = O(1).$$

Otherwise, by Lemma 5.4, there exists (n_0, L_0)-corridor R generated by γ and ω. Let $\phi\colon [0, a] \times [0, b] \to X_t$ be the map whose image is R satisfying the conditions of Definition 5.3.

Lift ϕ to the map

$$\tilde{\phi}\colon [0, a] \times [0, b] \longrightarrow \widetilde{X}_t \quad \text{with} \quad \tilde{\phi}(\{0\} \times [0, b]) = \widetilde{\omega}.$$

Note that $\tilde{\phi}(\{a\} \times [0, b])$ is a translate of a subarc $\widetilde{\omega}'$ of $\widetilde{\alpha}$ by an isometry of $\mathbb{H}^2$ fixing $\widetilde{\lambda}$. Also, $\widetilde{\omega}'$ intersects the same number of $\widetilde{\gamma}_i$, and $\widetilde{\omega}'$ and $\widetilde{\omega}$ intersect. But $n \geqslant 3C(n_0, L_0)$ and $\widetilde{\omega}$ was central. Thus $\widetilde{\omega}'$ is still between $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_n$ and hence is ϵ_B-close to $\widetilde{\lambda}$.

Since β is disjoint from α, if β intersects R it has to enter from the edge $\phi([0, a] \times \{0\})$, travel through the corridor and exit from the edge $\phi([0, a] \times \{b\})$. Therefore, there must exist a lift $\widetilde{\beta}$ of β passing through $\tilde{\phi}((0, a) \times [0, b])$ intersecting every $\tilde{\phi}([0, a] \times \{t_i\})$. That is, $\widetilde{\beta}$ intersects n_0 of γ_i at a distance at most ϵ_B from $\widetilde{\lambda}$. Thus, by Proposition 4.6 β is (n_0, L_0)-horizontal at X_t if β intersects R.

Now suppose that β is disjoint from R. Fix a parameterization $\psi\colon [0, c] \to X_t$ for ω. Let $\omega_1 = \phi(\{0\} \times [0, b])$ and $\omega_2 = \phi(\{a\} \times [0, b])$. The parameterization ψ traverses ω_1 or ω_2 either in the same or opposite direction as ϕ and either traverses ω_1 before ω_2 or vice versa. We assume that ψ traverses ω_1 in the same direction and speed as ϕ and ψ traverses ω_1 before ω_2 (the proofs in the other cases are similar). Let

$$0 \leqslant s < t < s+b < t+b \leqslant c$$

be such that $\omega_1 = \psi([s, s+b])$ and $\omega_2 = \psi([t, t+b])$. Let $\omega' = \psi([s,t])$ and $\gamma' = \phi([0,a] \times \{0\})$. Consider the curve η that is a concatenation of ω' and γ'. Topologically, η is a non-trivial simple closed curve with $\mathrm{i}(\eta, \gamma) \leqslant \mathrm{i}(\omega, \gamma) = C(n_0, L_0)$. Since β is disjoint from R, it is disjoint from η. Therefore,

$$d_{\mathcal{C}(\mathcal{S})}(X_t, \beta) \overset{+}{\prec} d_{\mathcal{C}(\mathcal{S})}(\gamma, \eta) = O(1).$$

This concludes the proof of the proposition. □

Proof of Theorem 5.2. The proof now follows from Proposition 5.5 and Theorem 4.2.

Let n_1 be as in Proposition 5.5 and L_0 be as in Theorem 4.2. Let $s > t_\alpha$ be the first time in $[a,b]$ that α is (n_1, L_0)-horizontal at s (let $s = b$ if this never happens). By Theorem 4.2, $\pi \circ \mathcal{G}([t_\alpha, s])$ has uniformly bounded diameter in $\mathcal{C}(\mathcal{S})$. If $s \geqslant t_\beta$, then we are done. Otherwise, $s < t_\beta$ and, for any $t \in [s, t_\beta)$, by Proposition 5.5, $d_{\mathcal{C}(\mathcal{S})}(X_t, \beta) = O(1)$. Therefore, $\pi \circ \mathcal{G}([s, t_\beta])$ also has uniformly bounded diameter in $\mathcal{C}(\mathcal{S})$. □

5.2. *Retraction*

THEOREM 5.6. *Given a Thurston geodesic $\mathcal{G}\colon [a,b] \to \mathcal{T}(\mathcal{S})$, the map $\mathcal{C}(\mathcal{S}) \to \pi \circ \mathcal{G}([a,b]) \subset \mathcal{C}(\mathcal{S})$ taking a curve α to $\pi(X_{t_\alpha})$ is a coarse Lipschitz retraction.*

Before proving Theorem 5.6, we show how to derive Theorem 1.2. First we give a precise definition of *reparameterized quasi-geodesic*.

Fix a constant $K > 0$. We will call a path $\phi\colon [a,b] \to \mathcal{X}$ in a metric space $\mathcal{X}$ a K-quasi-geodesic if, for all $a \leqslant s \leqslant t \leqslant b$,

$$\frac{1}{K}(t-s) - K \leqslant d_{\mathcal{X}}(\phi(s), \phi(t)) \leqslant K(t-s) + K.$$

We will say ϕ is a *reparameterized* K-quasi-geodesic if there is an increasing function $h\colon [0,n] \to [a,b]$ such that $\phi \circ h$ is a K-quasi-geodesic. Furthermore, for all $i \in [0, n-1]$, we have $\mathrm{diam}_{\mathcal{X}}([\phi(h(i)), \phi(h(i+1))]) \leqslant K$. In the case that h is not onto, we also require that $\mathrm{diam}_{\mathcal{X}}([\phi(a), \phi(h(0))]) \leqslant K$ and $\mathrm{diam}_{\mathcal{X}}([\phi(h(n)), \phi(b)]) \leqslant K$. A collection $\{\phi_i\}_{i \in I}$ of reparameterized quasi-geodesics is uniform if there is a constant K that works for the collection.

The following is a restatement of Theorem 1.2.

THEOREM 5.7. *The collection of $\{\pi \circ \mathcal{G}\colon [a,b] \to \mathcal{C}(\mathcal{S})\}$ ranging over Thurston geodesics $\mathcal{G}\colon [a,b] \to \mathcal{T}(\mathcal{S})$ is a uniform family of reparameterized quasi-geodesics in $\mathcal{C}(\mathcal{S})$.*

Proof. This argument is standard [**4**] and follows easily from Theorem 5.6.

Let $\mathcal{G}\colon [a,b] \to \mathcal{T}(\mathcal{S})$ be a Thurston geodesic. Let $\alpha \in \pi \circ \mathcal{G}(a)$ and $\alpha' \in \pi \circ \mathcal{G}(b)$ be two curves. Choose a geodesic $\alpha = \alpha_0, \ldots, \alpha_n = \alpha'$ in $\mathcal{C}(\mathcal{S})$. Let t_i be the balanced time of α_i along $\mathcal{G}$, and let $t_{n+1} = b$. By Theorem 5.6, $\mathrm{diam}_{\mathcal{C}(\mathcal{S})}([\mathcal{G}(t_i), \mathcal{G}(t_{i+1})])$ is uniformly bounded. The times t_i may not occur monotonically along $[a,b]$, but, for each $0 \leqslant i \leqslant n$, there exists $j \geqslant i$, such that $t_j \leqslant t_i \leqslant t_{j+1}$, and $\mathrm{diam}_{\mathcal{C}(\mathcal{S})}([\mathcal{G}(t_i), \mathcal{G}(t_{j+1})]) \leqslant \mathrm{diam}_{\mathcal{C}(\mathcal{S})}([\mathcal{G}(t_j), \mathcal{G}(t_{j+1})]) = O(1)$. Thus, there is a sequence $0 = i_0 < i_1 < \cdots < i_k = n$, such that $t_{i_{j+1}} > t_{i_j}$ and $\mathrm{diam}_{\mathcal{C}(\mathcal{S})}([\mathcal{G}(t_{i_j}), \mathcal{G}(t_{i_{j+1}})])$ is uniformly bounded. We will call such a sequence admissible and choose one with minimal length k. For simplicity, we will relabel each t_{i_j} by t_j.

Now let $h\colon [0,k] \to [a,b]$ be defined by sending each subinterval $[j, j+1]$ to $[t_j, t_{j+1}]$ by a linear map for all $j = 0, \ldots, k-1$. By Theorem 5.6, $\operatorname{diam}_{\mathcal{C}(\mathcal{S})}([\mathcal{G}(a), \mathcal{G}(t_0)]) = O(1)$ and $\operatorname{diam}_{\mathcal{C}(\mathcal{S})}([\mathcal{G}(t_k), \mathcal{G}(b)]) = O(1)$. Set $\mathcal{G}_i = \mathcal{G} \circ h(i)$. By construction, $\operatorname{diam}_{\mathcal{C}(\mathcal{S})}([\mathcal{G}_i, \mathcal{G}_{i+1}]) = O(1)$ for all $i \in [0, k-1]$ and $k \leqslant n = d_{\mathcal{C}(\mathcal{S})}(\alpha, \beta)$. Therefore, for all $0 \leqslant i \leqslant i' \leqslant k$, we have

$$d_{\mathcal{C}(\mathcal{S})}(\mathcal{G}_i, \mathcal{G}_{i'}) \prec i' - i.$$

The only thing remaining to check is that the lower bound for the definition of quasi-geodesic, that is, for all $0 \leqslant i \leqslant i' \leqslant k$, we will show

$$i' - i \leqslant d_{\mathcal{C}(\mathcal{S})}(\mathcal{G}_i, \mathcal{G}_{i'}) + 2.$$

It is enough to prove this for $i, i' \in \{0, \ldots, k\}$. For $0 \leqslant i < i' \leqslant k$, let $\beta \in \pi \circ \mathcal{G}_i = \pi \circ \mathcal{G}(t_i)$ and $\beta' \in \pi \circ \mathcal{G}_{i'} = \pi \circ \mathcal{G}(t_{i'})$. Let $m = d_{\mathcal{C}(\mathcal{S})}(\beta, \beta')$ and choose a geodesic $\beta = \beta_0, \ldots, \beta_m = \beta'$ in $\mathcal{C}(\mathcal{S})$. Let s_i be the balance time of β_i along $\mathcal{G}$. After choosing a subsequence, we may assume that s_i appear monotonically along $[a, b]$ and that $t_i < s_0$ and $s_m < t_{i'}$. We can modify the admissible sequence

$$t_0 < \cdots < t_i < \cdots < t_{i'} < \cdots < t_k$$

by

$$t_0 < \cdots < t_i < s_0 < \cdots < s_m < t_{i'} < \cdots < t_k,$$

which is still admissible. By minimality of k, we must have $i' - i \leqslant m + 2$ which proves what we want. $\square$

The proof of Theorem 5.6 requires some technical results about hyperbolic surfaces.

Given a hyperbolic surface X, consider a simple closed geodesic γ that is ϵ_B-short and a simple geodesic λ on X. Assuming that λ intersects γ many times, we would like to find a simple closed curve α with a uniformly bounded intersection number with γ that is (n_0, L_0)-horizontal. The argument here is somewhat delicate since there are essentially two possible situations; either λ twists around a relatively short curve α or α is somewhat longer and a long subsegment of it stays close to λ in the universal cover.

We find the appropriate curve α by applying surgery between λ and γ such that α contains a long subsegment of λ. But we also need to have some control such that after pulling α tight, it still stays close to λ. The following two lemmas will give the needed control.

In the following, orient the curve γ so that the left-hand side and the right-hand side of γ are defined. We will say an arc ω with endpoints on γ hits γ on opposite sides if the two endpoints of ω are on different sides of γ; otherwise, ω hits γ on the same side.

LEMMA 5.8. *Let γ and λ be as above and let $\alpha = \eta \cup \omega$ be a closed curve in X that is obtained from concatenation of a subarc η of γ and a subarc ω of λ. Also, assume that η hits ω on opposite sides and $L = \ell_X(\omega) \geqslant 4\epsilon_B$. Then, α^*, the geodesic representative of α in X, stays in an $O(\epsilon_B)$-neighborhood of α.*

Proof. This is a well-known fact in hyperbolic geometry. We sketch the proof here. Consider the lift of α to $\mathbb{H}^2$ as a concatenation of segments $\overline{\eta}_i$ and $\overline{\omega}_i$ that are lifts of η and ω, respectively. Since ω is a subarc of a complete simple geodesic in X, the segments $\overline{\omega}_i$ lie on complete geodesics $\widetilde{\omega}_i$ in $\mathbb{H}^2$ that are disjoint. The condition that η hits ω on opposite sides means that $\overline{\omega}_{i+1}$ does not backtrack along ω_i.

Fixing o, the center of segment ω_0, as the center of the Poincaré disk, the Euclidean distance between and endpoints of geodesics $\widetilde{\omega}_i$ in $\partial\mathbb{H}^2$ form a Cauchy sequence (in fact, they decrease exponentially fast) and hence they converge. Namely, the visual angle at o of the endpoint of $\widetilde{\omega}_1$ is at most $O(\epsilon_B e^{-L/2})$ and the visual angle at o between the endpoints of $\widetilde{\omega}_i$ and $\widetilde{\omega}_{i+1}$

decrease exponentially with $|i|$. Hence the lift of γ^* starts and ends near the endpoints of $\widetilde{\omega}_0$ with a visual angle of $O(\epsilon_B e^{-L/2})$. Therefore, an $O(\epsilon_B)$-neighborhood of lift of γ^* contains $\overline{\omega}_0$. This completes the proof. □

LEMMA 5.9. *Let β and β' be simple closed curves in X (possibly $\beta = \beta'$) with lengths longer than δ_B and let η be a geodesic segment that is disjoint from both. Let γ and γ' be two segments of length $O(1)$ connecting the endpoints of η to β and β', respectively, so the curve α obtained by the concatenation*

$$\beta \cup \gamma \cup \eta \cup \gamma' \cup \beta' \cup \gamma' \cup \eta \cup \gamma$$

is simple. Let α^* be the geodesic representative of α. Then, in the universal cover, any lift of η is contained in a bounded neighborhood of the union of a lift of α^*, a lift of β and a lift of β'.

Proof. The lemma is non-trivial because two copies of η are used and they may backtrack each other. It is essential that there is a lower bound on the lengths of β and β' and the lemma essentially follows from Lemma 3.7.

Consider a lift of α to the universal cover. Ignoring the lifts of γ and γ' which have bounded length, we consider the segments $\beta_i \cup \eta_i \cup \beta'_i \cup \overline{\eta}_i$, where η_i and $\overline{\eta}_i$ are lifts of η, β_i are lifts of β, β'_i are lifts of β' and endpoints of every segment are in a uniformly bounded neighborhood of an endpoint of the next segment.

The segments η_i and $\overline{\eta}_i$ lie on geodesics λ_i and $\overline{\lambda}_i$ that are disjoint. In fact, there is an isometry of $\mathbb{H}^2$, associated to the curve β', whose axis contains β'_i and sends η_i to $\overline{\eta}_i$. Let δ_0 be large enough such that the δ_0-neighborhood of β' contains the standard collar $U(\beta')$. Then δ_0 is a universal constant since there is a lower bound on the length of β'. Let $U_{\delta_0}(\beta')$ be the δ_0-neighborhood of β' and U_i be the lift that contains β'_i. There is a universal lower bound on the length of the boundary of $U_{\delta_0}(\beta')$, which means there is a lower bound on the distance between the intersection points of η_i and $\overline{\eta}_i$ with U_i. It now follows from Lemma 3.7 that there is a lower bound of d_0 for the distance between the subsegments of η_i and $\overline{\eta}_i$ that are outside of U_i. That is, if η_i is not near β_i, it cannot be too close to $\overline{\eta}_i$ and hence η_i and $\overline{\eta}_i$ do not fellow-travel for a long time outside of a uniform neighborhood of β'_i. A similar statement is true for $\overline{\eta}_i$, η_{i+1} and β_i.

Since $\mathbb{H}$ is Gromov hyperbolic, the lift of α^* is contained in a uniform neighborhood of segments $\beta_i \cup \eta_i \cup \beta'_i \cup \overline{\eta}_i$. In fact, each point in any of these segments is either close to the lift of α^* or close to a point in some other segment. We have shown that η_i and $\overline{\eta}_i$ do not fellow-travel for a large subsegment. This means that any point in η_i is close to either β_i, β'_i, or to the lift of α^*. □

We would like to show that, at every time t, there is a curve α which has balanced time $t_\alpha = t$ and whose distance in $\mathcal{C}(\mathcal{S})$ from the shadow of X_t is uniformly bounded. The next proposition shows that a coarse version of this statement holds.

PROPOSITION 5.10. *Let $\mathcal{G}\colon [a,b] \to \mathcal{T}(\mathcal{S})$ be a Thurston geodesic segment. For any $t \in [a,b]$, if $\lambda_{\mathcal{G}}$ intersects some short curve on $X_t = \mathcal{G}(t)$, then there exists an (n_0, L_0)-horizontal curve α on X_t such that $\mathrm{i}(\alpha, \gamma) = O(1)$ for any ϵ_B-short curve γ. Furthermore, $d_{\mathcal{C}(\mathcal{S})}(X_{t_\alpha}, X_t) = O(1)$.*

Proof. First, we will construct α. Let

$$N_0 = n_0 \left\lceil \frac{L_0}{\delta_B} \right\rceil + K,$$

where K is the additive error coming from Lemma 5.9.

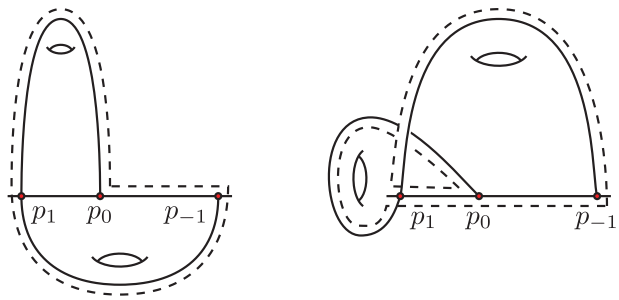

FIGURE 8. *Case* (1) *and* (2) *of Proposition* 5.10.

If $\lambda_{\mathcal{G}}$ has a closed leaf λ that intersects every short curve on X_t at most $5N_0$ times, then $\alpha = \lambda$ has the desired properties. Otherwise, we can fix a leaf λ in the stump of $\lambda_{\mathcal{G}}$ that intersects some ϵ_B-short curve more than $5N_0$ times.

Fix a segment ω of λ such that ω has endpoints on a short curve γ with $\mathrm{i}(\gamma, \omega) = 5N_0$, and ω intersects all other short curves at most $5N_0$ times. Orient γ such that we can talk about the two sides of γ. We will show that, applying a surgery between ω and γ, we can obtain a simple closed curve α that still intersects γ and stays close to λ for a long time. Unfortunately, this process is delicate and depending on the intersection pattern of γ and ω, we may have to apply a different surgery. The conclusion will follow if either of the following two cases occur.

Case (1): There is a subarc η of ω that hits γ on opposite sides with $\mathrm{i}(\eta, \gamma) \geqslant N_0$, and the endpoints of η can be joined by a segment of γ that is disjoint from the interior of η.

In this case, the geodesic representative α of the concatenation of η and a segment of γ is (n_0, L_0)-horizontal by Lemma 5.8 (see the left-hand side of Figure 8).

Case (2): There is a subarc η of ω and a closed curve β disjoint from η, such that $\mathrm{i}(\eta, \gamma) \geqslant N_0$ and $\ell_X(\beta) \geqslant \delta_B$, and the endpoints of η are close to the same point on β. Furthermore, each endpoint of η can be joined to a nearby point on β by a segment of γ that is disjoint from β and the interior of η.

In this case (see the right-hand side of Figure 8), let α be the curve obtained by closing up η with β and one or two subarcs of γ. If η twists around β N_0 times, then β is (n_0, L_0)-horizontal by Proposition 4.6; otherwise, by Lemma 5.9, α has a segment that intersects γ N_0 times and stays close to η, in which case α is (n_0, L_0)-horizontal by Proposition 4.6.

Case (3): There is a subarc η of ω and there are two closed curves β and β' that are disjoint from η, such that $\mathrm{i}(\eta, \gamma) \geqslant N_0$, $\ell_X(\beta) \geqslant \delta_B$ and $\ell_X(\beta') \geqslant \delta_B$, and the two endpoints of η are close to β and β'. Furthermore, there exists a segment of γ joining one endpoint of η to β and a segment of γ joining the other endpoint of η to β', such that both segments are disjoint from β, β' and the interior of η.

In this case (see Figure 9), let α be the curve obtained by gluing two copies of η, β, β' and a few sub-arcs of γ. If η twists around either β or β' N_0 times, then either β or β' is (n_0, L_0)-horizontal by Proposition 4.6; otherwise, by Lemma 5.9, α has a segment that intersects γ N_0 times and stays close to η, in which case α is (n_0, L_0)-horizontal by Proposition 4.6.

We now show that at least one of these three cases happens.

Let p_0 and q_0 be the endpoints of ω. Let p_{-1} and p_1 be the adjacent intersection points along γ to p_0 and q_{-1} and q_1 be the adjacent intersection points along γ to q_0. By relabeling if necessary, we may assume that ω passes from p_0 to p_1 and then to p_{-1}. Assume that ω passes from q_0 to q_1 and then to q_{-1}. We allow the possibility $p_{-1} = q_0$ and $q_{-1} = p_0$, or $p_{-1} = q_{-1}$.

Let ω_0 be the subarc of ω from p_0 to p_1 and let ω_1 be the subarc from p_1 to p_{-1}. Suppose $\omega_0 \cup \omega_1$ intersects γ at least $2N_0$ times. In this situation, we have several possibilities that will yield case (1) or (2). If $\omega_0 \cup \omega_1$ hits γ on opposite sides, as in the left-hand side of Figure 8,

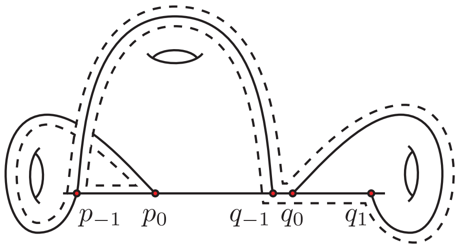

FIGURE 9. *Case* (3) *of Proposition* 5.10.

then we are in case (1) with $\eta = \omega_0 \cup \omega_1$. Otherwise, one of ω_i hits γ on opposite sides and the other one hits γ on the same side. Assume that ω_0 hits γ on opposite sides, as in the right-hand side of Figure 8. We are again in case (1) if ω_0 intersects γ at least N_0 times. If not, let β be the closed curve obtained from closing up ω_0 with an arc of γ. Since ω_0 has endpoints on γ and β stays close to ω_0 by Lemma 5.8, $\ell_X(\beta) \geqslant \delta_B$. We are now in case (2) with $\eta = \omega_1$. The dotted line in the right-hand side of Figure 8 represents the closed curve obtained from this surgery.

Similarly, let ω_0' be the subarc of ω from q_0 to q_1 and let ω_1' be the arc from q_1 to q_{-1}. As above, we are done if $\omega_0' \cup \omega_1'$ intersects γ at least $2N_0$ times.

Since ω intersects γ $5N_0$ times, if neither $\omega_0 \cup \omega_1$ nor $\omega_0' \cup \omega_1'$ intersects γ at least $2N_0$ times, then the arc η from p_{-1} to q_{-1} must have at least N_0 intersections with γ. If η hits γ on opposite sides, then we are in case (1). Otherwise, at least one of ω_0, ω_1, or $\omega_0 \cup \omega_1$ hits γ on opposite sides. Close this arc to obtain a closed curve β which has length at least δ_B. Similarly, let β' be a closed curve obtained from closing up either ω_0', ω_1' or $\omega_0' \cup \omega_1'$. We are now in case (3). In Figure 9, we have illustrated the situation when $\omega_0 \cup \omega_1$ forms β and ω_0' forms β'.

It remains to show $d_{\mathcal{C}(\mathcal{S})}(X_{t_\alpha}, X_t) = O(1)$. By definition, $t_\alpha \leqslant t$ and α is (n_0, L_0)-horizontal on X_{t_α}. Assume $t - t_\alpha \geqslant s_0$, where s_0 is the constant of Theorem 4.2. Let γ_α be an anchor curve for α at time t_α. The assumption implies $d_{\mathcal{C}(\mathcal{S})}(X_t, \alpha) = O(1)$, so it is enough to show $d_{\mathcal{C}(\mathcal{S})}(\gamma_\alpha, \alpha) = O(1)$. Define $D = d_{\mathcal{C}(\mathcal{S})}(\gamma_\alpha, \alpha)$. Recall that $D \leqslant \log_2 \mathrm{i}(\alpha, \gamma_\alpha) + 1$.

Let $\widetilde{\lambda}$ and $\widetilde{\alpha}$ be as in Definition 4.1, let ω and τ be the segments of $\widetilde{\lambda}$ and $\widetilde{\alpha}$ which are at most ϵ_B Hausdorff distance apart and which intersect n_0 lifts of γ_α. We may assume $\mathrm{i}(\alpha, \gamma_\alpha)$ is large enough such that τ projects to a proper subarc of α, that is, the length of τ is smaller than the length of α. Let $\widetilde{f}$ be the lift of an optimal map $f\colon X_{t_\alpha} \to X_t$. By the proof of Theorem 4.2, up to a multiplicative error $\widetilde{f}(\omega)$ intersects $n_0 2^D$ lifts of a short curve γ' on X_t. Moreover, a segment τ' of the geodesic representative of $\widetilde{f}(\widetilde{\alpha})$ is ϵ_B-close to $\widetilde{f}(\widetilde{\lambda})$ and also intersects the $n_0 2^D$ lifts of γ' up to a bounded error. The length of α on X_t is bigger than the length of τ', hence $\mathrm{i}(\alpha, \gamma') \overset{*}{\succ} n_0 2^D$. But $\mathrm{i}(\alpha, \gamma') = O(1)$ by assumption, therefore D must be bounded. □

Proof of Theorem 5.6. By Theorem 5.2, π is a coarse Lipschitz map. Let α be any short curve on X_t and λ_G be the maximally stretched lamination. We will show $\mathrm{diam}_{\mathcal{C}(\mathcal{S})}([X_t, X_{t_\alpha}]) = O(1)$.

If (the stump of) λ_G is a short curve, it may not intersect any other short curve at X_t. Let s be the first time λ_G intersects some short curve γ in X_s. Since λ_G is a short curve in the interval $[t, s]$, we have $\mathrm{diam}_{\mathcal{C}(\mathcal{S})}([X_t, X_s]) \overset{+}{\asymp} d_{\mathcal{C}(\mathcal{S})}(\alpha, \gamma) = O(1)$.

We now show $\mathrm{diam}_{\mathcal{C}(\mathcal{S})}([X_s, X_{t_\alpha}]) = O(1)$. Since $\lambda_{\mathcal{G}}$ intersects a short curve γ on X_s, by Proposition 5.10, there exists a curve β on X_s with $\mathrm{i}(\beta, \gamma) = O(1)$ and $\mathrm{diam}_{\mathcal{C}(\mathcal{S})}([X_{t_\beta}, X_s]) = O(1)$. We have $d_{\mathcal{C}(\mathcal{S})}(\alpha, \beta) \leqslant d_{\mathcal{C}(\mathcal{S})}(\alpha, \gamma) + d_{\mathcal{C}(\mathcal{S})}(\gamma, \beta) = O(1)$, which implies by Theorem 5.2 that $\mathrm{diam}_{\mathcal{C}(\mathcal{S})}([X_{t_\alpha}, X_{t_\beta}]) = O(1)$. The conclusion follows by the triangle inequality. □

6. *Examples of geodesics*

In this section, we construct several examples of geodesics in the Thurston metric demonstrating various possible behaviors, proving Theorems 1.1 and 1.4 from the introduction. The main idea in all these examples is that it is possible for the maximally stretched lamination associated to some Thurston geodesic to be contained in some subsurface W where the lengths of all curves disjoint from W (including ∂W) stay constant along the geodesic. This contrasts the behavior of a Teichmüller geodesic, where in a similar situation the length of ∂W would get short along the geodesic [**20**]. Our construction can be made to be very general. However, in the interest of simplicity, we make an explicit construction when W is a torus with one boundary component.

For a constant ℓ, let $\mathcal{T}(S_{1,1}, \ell)$ be the space of hyperbolic structures on a torus with one geodesic boundary where the length of the boundary curve is ℓ. Note that we can equip $\mathcal{T}(S_{1,1}, \ell)$ with the Thurston metric as usual (see [**10**] for details).

Let μ be any irrational measured lamination on $\mathcal{S}_{1,1}$. There is a unique way to complete μ to a *complete* lamination λ, such that the complement are two ideal triangles, by adding two bi-infinite leaves both tending to the cusp in one direction and wrapping around μ in the other. Hence, for any $U_0 \in \mathcal{T}(\mathcal{S}_{1,1})$, there exists a unique stretch path from U_0 with μ the stump of the maximally stretched lamination. We will denote by $\operatorname{stretch}(U_0, \lambda, t)$ this stretch path.

Proposition 6.1. *There is a constant ℓ_0 such that the following holds. For any $U_0 \in \mathcal{T}(\mathcal{S}_{1,1})$, any irrational measured lamination μ on $\mathcal{S}_{1,1}$ and the associated stretch path*

$$U_t = \operatorname{stretch}(U_0, \lambda, t), \quad U_t \in \mathcal{T}(\mathcal{S}_{1,1}),$$

where λ is the unique complete lamination containing μ as its stump, there is a bi-infinite Thurston geodesic $W_t \in \mathcal{T}(\mathcal{S}_{1,1}, \ell_0)$, $t \in \mathbb{R}$, where the stump of the maximally stretched lamination is still μ and, for any other curve α in $\mathcal{S}_{1,1}$,

$$\ell_{W_t}(\alpha) \stackrel{*}{\asymp} \ell_{U_t}(\alpha).$$

Proof. We will refer the reader to Figure 10 for this proof.

Choose $U_0 \in \mathcal{T}(\mathcal{S}_{1,1})$ and represent μ as a geodesic lamination on U_0. Let λ be the unique completion of μ and let A and B be the ideal triangles in the complement of λ in U_0. Each A and B has two sides a and b coming from the two leaves of λ tending toward the cusp, and a third side that wraps around the stump μ. There is an involution of U_0 fixing λ and switching A with B. Hence, the two anchor points of A at a and b are glued, respectively, to the two anchor points of B at a and b. See the upper right-hand side of Figure 10.

Now we double this picture. Let $U_0^+ = U_0$ and let U_0^- be an orientation-reversing copy of U_0^+. We also label $A = A^+$, $B = B^+$, $a = a^+$, $b = b^+$ and $\mu = \mu^+$ and we label the associated objects in U_0^- by A^-, B^-, a^-, b^- and μ^-. Cut $U_0^{\pm}$ open along $a^{\pm}$ and $b^{\pm}$. Via a reflection map, glue b^+ of A^+ to a^- of A^- such that the anchor point of A^+ in b^+ is glued to the anchor point of A^- in a^-. Similarly, via a reflection map, glue b^+ to b^-, a^+ to b^- and a^+ to a^- gluing the corresponding anchor points. Note that the third sides of $A^{\pm}$ and $B^{\pm}$ wrap about $\mu^{\pm}$ in $U^{\pm}$. This yields a genus two surface $\widetilde{U}_0$ with a geodesic lamination $\widetilde{\lambda}$ that contains $\mu^{\pm}$ as its stump and has two extra leaves each wrapping about μ^+ in one direction and wrapping about μ^- in the other direction.

In each $A^{\pm}$ or $B^{\pm}$, there is a geodesic arc connecting the midpoint of $a^{\pm}$ to the midpoint of $b^{\pm}$. Because the gluing maps were reflections, the angles of these arcs with $a^{\pm}$ and $b^{\pm}$ match and the four arcs glue together to form a separating geodesic γ in $\widetilde{U}_0$. Let ℓ_0 be the length of γ. By construction, ℓ_0 is independent of the irrational lamination μ and U_0.

Let $\widetilde{U}_t = \operatorname{stretch}(\widetilde{U}, \widetilde{\lambda}, t)$. Then $\widetilde{U}_t$ is also obtained from the doubling of U_t as above. Along the stretch paths, the length of geodesic arcs connecting the midpoint of $a^{\pm}$ to the midpoint

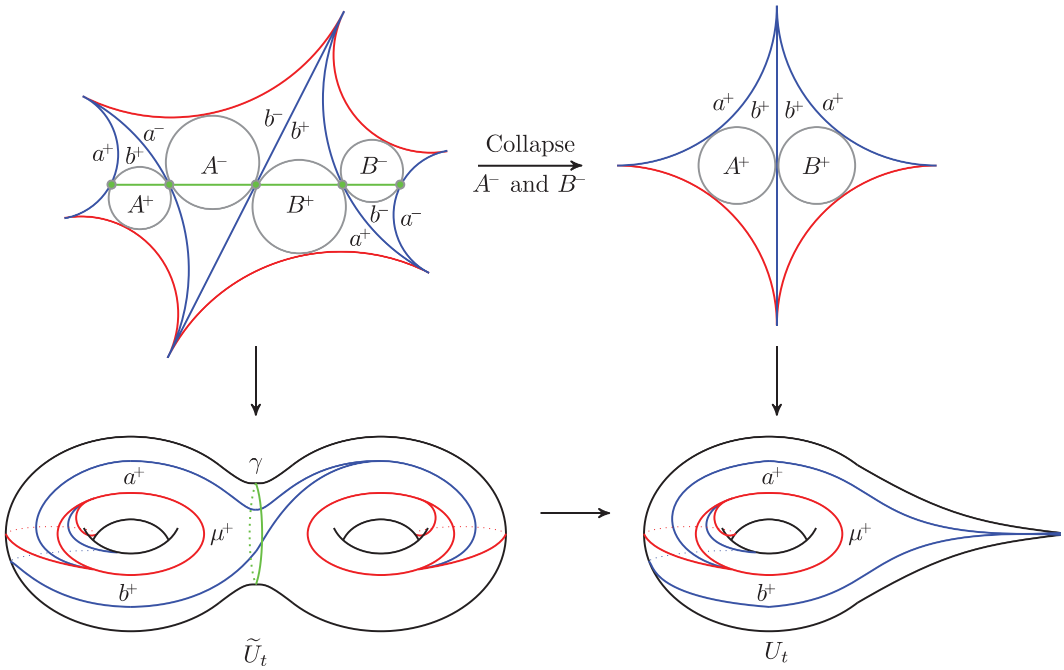

FIGURE 10. *We can double the stretch path on a punctured torus to obtain a stretch path in a surface of genus two. The length of the curve γ remains unchanged along the stretch path.*

of $b^{\pm}$ does not change. Hence, they glue the same way in $\widetilde{U}_t$ to form a separating geodesic of the same length. That is, the length of γ along $\widetilde{U}_t$ remains the constant ℓ_0.

The proposition now holds where W_t is the subsurface of $\widetilde{U}_t$ with boundary γ. The stretch map from U_s to U_t doubles to an e^{t-s}-Lipschitz homeomorphism between $\widetilde{U}_s$ and $\widetilde{U}_t$ that fixes γ pointwise. Hence, the length of curves grow by at most a factor of e^{t-s} both from $\widetilde{U}_s$ to $\widetilde{U}_t$ and from W_s to W_t.

To see the last assertion in the proposition, we note that there is a contraction map from W_t to U_t. Consider the restriction of A^- and B^- in W_t and foliate it with horocycles perpendicular to the boundary. Then collapse these regions sending each horocycle to a point. This is a distance-decreasing map: the derivative in the direction tangent to the horocycles is zero and in the direction perpendicular to the horocycles is 1. Since stretch paths preserve these horocycles, this collapsing map commutes with the stretch maps and the image of W_t under the collapsing map is exactly U_t. Hence, the length of a curve α in W_t is longer than the length of its image in U_t under the collapsing map, which is longer than the length of the geodesic representative of α. □

This proposition is the building block for all the examples we construct in this section. Essentially, we can glue W_t to a family of surfaces with desired behavior to obtain various examples.

Proof of Theorem 1.1. Let μ be a simple closed curve and let U_0 be a point in $\mathcal{T}(S_{1,1})$ where the length of μ is ϵ for some small $\epsilon > 0$. Let W_t be the family obtained by Proposition 6.1. Also, choose $V \in \mathcal{T}(S_{1,1}, \ell_0)$ to be a point in the thick part. Let β be a curve of bounded length in V and define $V^n = D_\beta^n V$, where D_β is the Dehn twist around the curve β. (The values of ϵ and n are to be determined below.)

Let s be the time when μ has length 1 in W_s. Define

$$X = W_0 \cup V, \quad Y = W_s \cup V \quad \text{and} \quad Z = W_{s/2} \cup V^n.$$

By $\cup$, we mean glue the two surfaces along the boundary and consider them as an element of $\mathcal{T}(S_{2,0})$. We mark the surfaces so that they have bounded relative twisting along the gluing curve γ. (Note that relative twisting is only well defined up to an additive error). We claim that if $\log 1/\epsilon \gg n$, then

$$d_{\mathrm{Th}}(X, Z) = d_{\mathrm{Th}}(Z, Y) = s/2.$$

First consider X and Z. Indeed, since the length of μ grows exponentially in W_t, $s/2$ is a lower bound for the distance $d_{\mathrm{Th}}(X, Z)$. We need to show that the length of any other curve grows by a smaller factor. This is true for any curve contained in W_t by Proposition 6.1. For any other curve α, let α_W be the restriction of α to W_0 and let $\bar{\alpha}_W$ be the restriction to the complement of W_0. Any representative of α in Z is no shorter than the geodesic representative. Therefore, there is a uniform constant C such that

$$\ell_Z(\alpha) \leqslant e^{s/2}\ell_X(\alpha_W) + C(\ell_X(\bar{\alpha}_W) + n\ell_X(\beta)\mathrm{i}(\alpha, \beta)).$$

But $\ell_X(\bar{\alpha}_W) \overset{*}{\succ} \mathrm{i}(\alpha, \beta)$. Hence, if $e^{s/2} \gg C + n$, then we obtain

$$e^{s/2}\ell_X(\bar{\alpha}_W) \geqslant (C + n)(\ell_X(\bar{\alpha}_W) + \ell_X(\beta)\mathrm{i}(\alpha, \beta)).$$

Therefore, for sufficiently large s, we have

$$\ell_Z(\alpha) \leqslant e^{s/2}(\ell_X(\alpha_W) + \ell_X(\bar{\alpha}_W)) = e^{s/2}\ell_X(\alpha).$$

That is, μ is the maximally stretched lamination from X to Z. The argument for the distance from Z to Y is similar.

Now define

$$\mathcal{G}_1(t) = W_t \cup V,$$

and let $\mathcal{G}_2(t)$ be the geodesic obtained by a concatenation of the geodesic connecting X to Z and Z to Y. Let α be a curve disjoint from W_t that has a bounded length in X. Then, for all t,

$$\ell_{\mathcal{G}_1(t)}(\alpha) = \ell_X(\alpha) \overset{*}{\asymp} 1.$$

But the length of α in $Z = \mathcal{G}_2(s/2)$ is of order n. That is,

$$d_{\mathrm{Th}}(\mathcal{G}_1(t), Z) \geqslant \log \frac{\ell_Z(\alpha)}{\ell_{\mathcal{G}_1(t)}(\alpha)} \overset{+}{\asymp} \log n,$$

which can be chosen to be much larger than D. Note that a lower bound for the distance in the other direction can also be found by replacing α with $D_\beta^{-n}(\alpha)$.

To obtain the second part of Theorem 1.1, we note that the distance from Y to X is only of order $\log\log(1/\epsilon)$ if ϵ is small enough. We now choose n and ϵ such that

$$\log \frac{1}{\epsilon} \gg \log n, \quad \log n \overset{+}{\asymp} D \quad D \gg \log\log \frac{1}{\epsilon}.$$

This way, if $s \geqslant 2D$, then $Z = \mathcal{G}_2(s/2)$ has distance at least D to any point on any geodesic connecting Y to X. This completes the proof of part 2. $\square$

Next, we construct an example showing that the set of short curves in a Thurston geodesic connecting two points is not the same as the set of short curves along the Teichmüller geodesic. This is in contrast with the following theorem.

THEOREM 6.2 [**14**]. *For every $K > 0$ and $\epsilon > 0$ there exists $\epsilon' > 0$ such that whenever $X, Y \in \mathcal{T}(\mathcal{S})$ are ϵ-thick and have K-bounded combinatorics, then any Thurston geodesic $\mathcal{G}$ from X to Y remains in the ϵ'-thick part.*

Proof of Theorem 1.4. Let ϕ be a pseudo-Anosov map in the mapping class group of $S_{1,1}$ and let $U_0 \in \mathcal{T}(S_{1.1})$ be on the Teichmüller axis of ϕ. For any $n \in \mathbb{Z}$, the maximally stretched lamination μ from U_0 to $\phi^n(U_0)$ is irrational, hence the stretch path from U_0 to $\phi^n(U_0)$ is the unique Thurston geodesic connecting them. Let $U_t = |(U_0, \mu, t)$ and let $U_s = \phi^n(U_0)$. The point $\phi^n(U_0)$ is also on the Teichmüller axis of ϕ and the Teichmüller geodesic segment connecting U_0 to $\phi^n(U_0)$ stays in a uniform thick part (independent of n) of Teichmüller space. From [**14**], we know that the Thurston geodesic connecting these two points also stays in this part and fellow-travels the Teichmüller geodesic.

Let W_t be the family of surfaces in $\mathcal{T}(S_{1.1}, \ell_0)$ obtained from Proposition 6.1 and let V be any point in the thick part of $\mathcal{T}(S_{1.1}, \ell_0)$. Now define

$$\mathcal{G}(t) = W_t \cup V, \quad t \in [0, s].$$

Then $\mathcal{G}(t)$ is a Thurston geodesic in $\mathcal{T}(S_{2,0})$. This is because the length of μ is growing exponentially and the length of every other curve is growing by a smaller factor. (The argument is an easier version of the arguments in the previous proof and is dropped.)

Since U_t is in the thick part, so is W_t and hence $\mathcal{G}(t)$. Let $X = \mathcal{G}(0)$ and $Y = \mathcal{G}(s)$. But, by taking n large enough, we can ensure that $d_W(X, Y)$ is as large as desired, where W is the subsurface of $S_{2,0}$ associated to W_t. It then follows from [**20**] that the boundary of W is short along the Teichmüller geodesic connecting X to Y, in fact, its minimum length is inversely proportional to $d_W(X, Y)$. That is, ∂W has bounded length along the Thurston geodesic $\mathcal{G}(t)$ but is arbitrarily short along the Teichmüller geodesic. This completes the proof of the theorem. □

Acknowledgements. Our Key Proposition 5.5 is modeled after [**2**, Proposition 6.4] where Bestvina–Feighn show the projection of a folding path to the free factor graph is a reparameterized quasi-geodesic which is in turn inspired by the arguments of Masur–Minsky [**15**]. We would like to thank Université de Rennes and Erwin Schrödinger International Institute for Mathematical Physics for their hospitality. We also thank the referee for carefully reading the paper and providing us all the useful comments.

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Anna Lenzhen
Laboratoire de Mathématiques
Université de Rennes 1
35042 Rennes
France

anna.lenzhen@univ-rennes1.fr

Kasra Rafi
Department of Mathematics
University of Toronto
Toronto
Canada M5S 2E4

rafi@math.toronto.edu

Jing Tao
Department of Mathematics
University of Oklahoma
Norman, OK 73069
USA

jing@math.ou.edu