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Limit Sets of Weil–Petersson Geodesics

Jeffrey Brock[1], **Christopher Leininger**[2], **Babak Modami**[3,*], **and Kasra Rafi**[4]

[1]Department of Mathematics, Brown University, Providence, RI, [2]Department of Mathematics, University of Illinois, 1409 W Green ST, Urbana, IL, [3]Department of Mathematics, Yale University, 10 Hillhouse Ave, New Haven, CT, and [4]Department of Mathematics, University of Toronto, Toronto, ON

**Correspondence to be sent to: e-mail: babak.modami@yale.edu*

In this paper we prove that the limit set of any Weil–Petersson geodesic ray with uniquely ergodic ending lamination is a single point in the Thurston compactification of Teichmüller space. On the other hand, we construct examples of Weil–Petersson geodesics with minimal non-uniquely ergodic ending laminations and limit set a circle in the Thurston compactification.

1 Introduction

Given a surface S, let Teich(S) denote the Teichmüller space of hyperbolic metrics on S, and Mod(S) the mapping class group of S. Thurston compactified Teich(S) by adjoining the space of projective measured laminations $\mathcal{PML}(S)$, and used this in his classification of elements of Mod(S) [16, 33]. On the other hand, Teich(S) has two important, Mod(S)-invariant, unique-geodesic metrics and hence has natural visual compactifications. These metrics have their own drawbacks—the Teichmüller metric

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is not negatively curved [25] and Weil–Petersson metric is incomplete [34]—and hence the standard results about visual compactification do not readily apply to any of these metrics. For example, the action of Mod(S) extends continuously to neither the Teichmüller visual boundary [18] nor the Weil–Petersson visual boundary [9].

In [26], Masur showed that the Thurston boundary and the Teichmüller visual boundary are not so different, proving that almost every Teichmüller ray converges to a point on the Thurston boundary (though positive dimensional families of rays based at a single point can converge to the same point). Lenzhen [20] constructed the first examples of Teichmüller geodesic rays that do not converge to a unique point in the Thurston boundary, and recent constructions have illustrated increasingly exotic behavior [2, 12, 21, 23].

In this paper we begin an investigation into the behavior of how the Weil–Petersson visual compactification relates to Thurston's compactification. Specifically, we study the behavior of Weil–Petersson geodesic rays in the Thurston compactification. Our results are stated in terms of the *ending laminations* of Weil–Petersson geodesic rays introduced by Brock, Masur, and Minsky in [5], see 2.7. The first theorem is a version of Masur's convergence for Teichmüller geodesics. We say that a lamination is *uniquely ergodic* if it admits a unique transverse measure, up to scaling. Moreover, we say that the lamination is *minimal* if every leaf of the lamination is dense in the lamination, and *filling* if the lamination intersects every simple closed geodesic on the surface non-trivially.

Theorem 1.1. Suppose that the ending lamination of a Weil–Petersson geodesic ray is minimal, filling, and uniquely ergodic. Then the ray converges in the Thurston compactification to the unique projective class of transverse measures on the ending lamination.

On the other hand, we prove that there are geodesic rays for which the ending lamination is minimal but non-uniquely ergodic, and whose limit sets are positive dimensional and in fact are non-simply connected.

Theorem 1.2. There exist Weil–Petersson geodesic rays with minimal, filling, non-uniquely ergodic ending laminations whose limit sets in the Thurston compactification are topological circles.

See Theorem 5.7 for a more precise statement. Without the minimality assumption, the construction of Weil–Petersson geodesic rays that do not limit to a single point requires some different ideas. This construction is given in [3].

1.1 Outline of the paper

Section 2 is devoted to background about Teichmüller theory, curve complexes, and laminations. In Section 2.5 we state our technical results about sequences of curves on surfaces that limit to non-uniquely ergodic laminations. These results are minor variations of those in [2], and their proofs are sketched in the appendix of the paper. In Section 3 we construct explicit examples of non-uniquely ergodic laminations on punctured spheres, appealing to the results from Section 2.5. In Section 4 we study the limiting picture of axes for pseudo-Anosov mapping classes of a punctured sphere arising in the construction of non-uniquely ergodic laminations in Section 3. In Section 5, we use this analysis to determine limit sets of our WP geodesic rays with non-uniquely ergodic ending laminations, and thus prove Theorem 1.2. In Section 6 we prove Theorem 1.1 about limit sets of geodesic rays with uniquely ergodic ending laminations.

2 Background

Notation 2.1. Our notation for comparing quantities in this paper is as follows: Let $K \geq 1$ and $C \geq 0$. Given two functions $f, g : X \to \mathbb{R}^{\geq 0}$, we write $f \asymp_{K,C} g$ if $\frac{1}{K} g(x) - C \leq f(x) \leq K g(x) + C$ for all $x \in X$, $f \overset{+}{\asymp}_C g$ if $g(x) - C \leq f(x) \leq g(x) + C$ and $f \overset{*}{\asymp}_K g$ if $\frac{1}{K} g(x) \leq f(x) \leq K g(x)$. The notation $f \overset{+}{\prec}_C g$ means that $f(x) \leq g(x) + C$ for all $x \in X$ and $f \overset{*}{\prec}_K g$ means that $f(x) \leq K g(x)$ for all $x \in X$.

When the numbers K, C are understood from the context we drop them from the notation.

2.1 Surfaces and subsurfaces:

In this paper surfaces are connected, orientable, and of finite type with boundaries or punctures. We denote a surface with genus g and b boundary curves or punctures by $S_{g,b}$ and define the complexity of the surface by $\xi(S_{g,b}) = 3g + b - 3$. The main surfaces we consider always have only punctures; however, we consider subsurfaces of the main surfaces with both punctures and boundary curves.

2.2 Curves and laminations

Notation 2.2. Throughout this paper, by a *curve* we mean the isotopy class of an essential, simple, closed curve. When convenient, we do not distinguish between a curve and a representative of the isotopy class. A *multicurve* is a collection of pairwise disjoint curves (i.e., curves with pairwise disjoint representatives).

By a *subsurface* of S, we mean the isotopy class of an embedded essential subsurface: one whose boundary consists of essential curves and whose punctures agree with those of S.

We say that two curves α and β overlap and denote it by $\alpha \pitchfork \beta$ if the curves α and β cannot be represented disjointly on the surface S. Two multicurves σ and τ overlap if there are curves $\alpha \in \sigma$ and $\beta \in \tau$ which overlap. A curve α and a subsurface $Y \subseteq S$ overlap, denoted by $\alpha \pitchfork Y$, if α cannot be realized disjointly from Y (up to homotopy). A multicurve and a subsurface overlap if a component of the multicurve overlaps with the subsurface. We say that two subsurfaces Y and Z overlap, and denote it by $Y \pitchfork Z$, if $\partial Y \pitchfork Z$ and $\partial Z \pitchfork Y$.

We refer the reader to [27, 28] for background about the curve complex and subsurface projection maps. Denote the curve complex of a surface S by $\mathcal{C}(S)$ and the set of vertices of the complex by $\mathcal{C}_0(S)$. The set $\mathcal{C}_0(S)$ is in fact the set of essential simple closed curves on S.

A *pants decomposition* on the surface S is a multicurve with a maximal number of components. A *(partial) marking* μ on the surface consists of a pants decomposition, called the base of μ, and a choice of transversal curves for (some) all curves in the base. For background about pants and marking graphs and hierarchical structures and (hierarchy) resolution paths in pants and marking graphs we refer the reader to [28]. We denote the pants graph of the surface S by $P(S)$. Here we only recall that hierarchy paths are certain quasi-geodesics in $P(S)$ with quantifiers that only depend on the topological type of S.

Let $Y \subseteq S$ be an essential subsurface. The *Y–subsurface projection coefficient* of two multicurves, markings or laminations μ, μ' is defined by

$$d_Y(\mu,\mu') := \mathrm{diam}_{\mathcal{C}(Y)}, \Big(\pi_Y(\mu) \cup \pi_Y(\mu')\Big). \tag{2.1}$$

Here π_Y is the subsurface projection (coarse) map and $\mathrm{diam}_{\mathcal{C}(Y)}(\cdot)$ denotes the diameter of the given subset of $\mathcal{C}(Y)$. When Y is an annular subsurface with core curve γ we also denote $d_Y(\mu, \mu')$ by $d_\gamma(\mu, \mu')$.

Our results in this paper are formulated in terms of subsurface coefficients which can be thought of as an analog of continued fraction expansions which provide a kind of symbolic coding for us.

We assume that the reader is familiar with basic facts about laminations and measured laminations on hyperbolic surfaces (see, e.g., [31] for an introduction). We denote the space of measured laminations equipped with the weak* topology by $\mathcal{ML}(S)$ and the space of projective classes of measured laminations equipped with the quotient topology by $\mathcal{PML}(S)$.

Recall that a measurable lamination is *uniquely ergodic* if it supports exactly one transverse measure up to scale. Otherwise, the lamination is *non-uniquely ergodic*.

An important property of curve complexes is that they are Gromov hyperbolic [27]. By the result of Klarreich [19] the Gromov boundary of the curve complex is homeomorphic to the quotient space of the space of (projective) measured laminations with minimal filling supports by the measure forgetting map equipped with the quotient topology, denoted by $\mathcal{EL}(S)$.

The Masur–Minsky distance formula [28] provides a coarse estimate for the distance between two pants decompositions in the pants graph $P(S)$. More precisely, there exists a constant $M > 0$ depending on the topological type of S with the property that for any threshold $A \geq M$ there are constants $K \geq 1$ and $C \geq 0$ so that for any P, $Q \in P(S)$ we have

$$d(P, Q) \asymp_{K,C} \sum_{\substack{Y \subseteq S \\ \text{non-annular}}} \{d_Y(P, Q)\}_A \tag{2.2}$$

where the cut-off function is defined by $\{x\}_A = \begin{cases} x & \text{if } x \geq A \\ 0 & \text{if } x \leq A \end{cases}$.

The following theorem is a straightforward consequence of [28, Theorem 3.1].

Theorem 2.3. (Bounded geodesic image) Given $K \geq 1$ and $C \geq 0$. Suppose that $\{\gamma_i\}_i$ is a sequence of curves which forms a 1 –Lipschitz (K, C)–quasi-geodesic in $\mathcal{C}(S)$. Furthermore, suppose that for a subsurface $Y \subsetneq S$, $\gamma_i \pitchfork Y$ for all i. Then there is a constant $G > 0$ depending on K, C so that

$$\operatorname{diam}_{\mathcal{C}(Y)}\left(\{\pi_Y(\gamma_i)\}_i\right) \leq G.$$

For the following inequality, see [1, 24].

Theorem 2.4. (Behrstock inequality) There exists a constant $B_0 > 0$ such that for any two subsurfaces $Y, Z \subsetneq S$ with $Y \pitchfork Z$ and a fixed marking or lamination μ we have that

$$\min\left\{d_Y(\partial Z, \mu), d_Z(\partial Y, \mu)\right\} \leq B_0.$$

We also need the following *no back-tracking* property of hierarchy paths, which follows from inequality (6.3) in [28, §6.3].

Theorem 2.5. There is a constant $C > 0$ depending only on the topological type of S so that given a hierarchy path $\varrho : [m, n] \to P(S)$ ($[m, n] \subset \mathbb{Z} \cup \{\pm\infty\}$), for parameters $i_1 \leq i_2 \leq i_3 \leq i_4$ in $[m, n]$, and a non-annular subsurface $Y \subseteq S$ we have

$$d_Y(\varrho(i_1), \varrho(i_4)) \geq d_Y(\varrho(i_2), \varrho(i_3)) - C.$$

2.3 Twist parameter

We define the twist parameter of a curve δ about γ at a point X in Teichmüller space by

$$\mathrm{tw}_\gamma(\delta, X) := d_\gamma(\mu, \delta) \tag{2.3}$$

where μ is a Bers marking at X (for definition of Bers marking see §2.6).

Note that for a filling set of bounded length curves Γ at X we have $\mathrm{tw}_\gamma(\delta, X) \stackrel{+}{\asymp} \mathrm{diam}_{\mathcal{C}(\gamma)}(\Gamma \cup \delta)$.

2.4 The Thurston compactification

Recall that a point in the Teichmüller space Teich(S) is a *marked complete hyperbolic surface* $[f : S \to X]$. The mapping class group of S, denoted by Mod(S), acts by remarking on Teich(S) and the quotient is the moduli space of hyperbolic surfaces $\mathcal{M}(S)$.

Given a curve $\alpha \in \mathcal{C}_0(S)$, the hyperbolic length of α at $[f : S \to X]$ is defined to be the hyperbolic length of the geodesic homotopic to $f(\alpha)$ in X. Abusing notation and denoting the point in Teich(S) by X, we write the hyperbolic length simply as $\ell_\alpha(X)$. For an $\epsilon > 0$, the ϵ*–thick part* of Teichmüller space consists of points $X \in$ Teich(S) with $\ell_\alpha(X) \geq 2\epsilon$ for all curves α. The projection of this set to the moduli space is the ϵ–thick part of moduli space.

The hyperbolic length function extends to a continuous function

$$\ell_\cdot(\cdot) \colon \mathrm{Teich}(S) \times \mathcal{ML}(S) \to \mathbb{R}.$$

Let ν be a measurable lamination and $\bar{\nu}$ a measured lamination with support ν. Moreover, denote the projective class of $\bar{\nu}$ by $[\bar{\nu}]$. The Thurston compactification,

$\widehat{\mathrm{Teich}(S)} = \mathrm{Teich}(S) \cup \mathcal{PML}(S)$ is constructed so that a sequence $\{X_n\}_n \subseteq \mathrm{Teich}(S)$ converges to $[\bar{\nu}] \in \mathcal{PML}(S)$ if and only if

$$\lim_{n\to\infty} \frac{\ell_\alpha(X_n)}{\ell_\beta(X_n)} = \frac{\mathrm{i}(\alpha,\bar{\nu})}{\mathrm{i}(\beta,\bar{\nu})}$$

for all simple closed curves α, β with $\mathrm{i}(\bar{\nu},\beta) \neq 0$. Here and throughout this paper the bi-homogenous function $\mathrm{i}(\cdot,\cdot)$ denotes the geometric intersection number of two curves and its extension to the space of measured laminations $\mathcal{ML}(S)$. See [7, 16] for more details on the intersection function and Thurston compactification.

2.5 Sequences of curves

In [21] and [2] the authors studied infinite sequences of curves on a surface that limit to non-uniquely ergodic laminations. The novelty in this work is that local estimates on subsurface projections and intersection numbers are promoted to global estimates on these quantities. We require minor modifications of some of the key results from [2] so as to be applicable to the sequences of curves on punctured spheres described in §3. We state the results here and sketch their proofs in the appendix for completeness.

Given a curve γ let D_γ be the positive (left) Dehn twist about γ.

Definition 2.6. Fix positive integers $m \leq \xi(S)$ and $b' \geq b > 0$, and a sequence $\mathcal{E} = \{e_k\}_{k=0}^\infty \subseteq \mathbb{N}$. We say that a sequence of curves $\{\gamma_k\}_{k=0}^\infty$ satisfies $\mathcal{P} = \mathcal{P}(\mathcal{E})$ if the following hold:

(i) any m consecutive curves are pairwise disjoint,

(ii) any consecutive $2m$ curves fill S, and

(iii) for all $k \geq m$, $\gamma_{k+m} = D_{\gamma_k}^{e_k}(\gamma'_{k+m})$, where γ'_{k+m} is a curve such that

$$\mathrm{i}\left(\gamma'_{k+m}, \gamma_j\right) \begin{cases} \leq b' & \text{for } j = k-m,\ldots,k+m-1 \\ = b & \text{for } j = k, k-1 \\ = 0 & \text{for } j = k+1,\ldots,k+m-1. \end{cases}$$

Remark 2.7. The only real difference between this definition and the one given in our previous paper [2, §3] is that this one requires fewer of the intersection numbers to be non-zero.

For the remainder of this subsection, we will assume that $\Gamma = \{\gamma_k\}_k$ satisfies $\mathcal{P}$ for some m, b, b' and $\mathcal{E} = \{e_k\}_k$ (see §3 for explicit examples). Furthermore, we assume there exists an $a \geq 1$ such that $e_{k+1} \geq ae_k$ for all $k \geq 0$.

The first result describes the behavior of $\{\gamma_k\}_k$ in the curve complex of S and its subsurfaces. Let $\mathbb{M}$ be the monoid generated by m and $m+1$, that is

$$\mathbb{M} = \left\{im + j(m+1) \mid i, j \in \mathbb{Z}^{\geq 0}\right\}.$$

Theorem 2.8. There exist constants E, K, $C > 0$ such that if $e_0 \geq E$, then $\{\gamma_k\}_k$ is a 1–Lipschitz (K, C)–quasi-geodesic. In particular, there exists a $\nu \in \mathcal{EL}(S)$ such that any accumulation point of $\{\gamma_k\}_k$ in $\mathcal{PML}(S)$ is supported on ν.

Furthermore, there exists a constant $R > 0$, and for a marking μ another constant $R(\mu) > 0$ depending on μ, so that for $i < k < j$, with $k-i, j-k \in \mathbb{M}$, we have $\gamma_i \pitchfork \gamma_k$, $\gamma_j \pitchfork \gamma_k$, and

$$d_{\gamma_k}(\gamma_i, \gamma_j), d_{\gamma_k}(\gamma_i, \nu) \stackrel{+}{\asymp}_R e_i \quad \text{and} \quad d_{\gamma_k}(\mu, \gamma_j), d_{\gamma_k}(\mu, \nu) \stackrel{+}{\asymp}_{R(\mu)} e_i. \tag{2.4}$$

Also, for any $i < j$ and a subsurface W which is not an annulus with core curve γ_k for some k we have

$$d_W(\gamma_i, \gamma_j), d_W(\gamma_i, \nu) \leq R \quad \text{and} \quad d_W(\mu, \gamma_j), d_W(\mu, \nu) \leq R(\mu). \tag{2.5}$$

The next result provides estimates on intersection numbers for curves in our sequence. To describe the estimates, for all $i < k$, define the integers

$$A(i,k) := \prod_{\substack{i+m \leq j < k, \\ j \equiv k \mod m}} be_j \tag{2.6}$$

where the product is taken to be 1 whenever the index set is empty.

Theorem 2.9. If $a > 1$ is sufficiently large and $e_{k+1} \geq ae_k$, then there exists $\kappa_0 \geq 1$ such that $\mathrm{i}(\gamma_i, \gamma_k) \leq \kappa_0 A(i, k)$ for all $i < k$, and

$$\mathrm{i}(\gamma_i, \gamma_k) \stackrel{*}{\asymp}_{\kappa_0} A(i,k). \tag{2.7}$$

if $k - i \geq 2m$ and $i \equiv k \mod m$, or if $i \leq 2m - 1$ and $k - i \geq m^2 + m - 1$.

For any curve δ, there exists $\kappa(\delta) \geq 1$ such that for all k sufficiently large

$$\mathrm{i}(\delta, \gamma_k) \stackrel{*}{\asymp}_{\kappa(\delta)} A(0, k). \tag{2.8}$$

For reference, we also record the following simple fact (see [2, Lemma 5.6]).

Lemma 2.10. Suppose that $\mathcal{E} = \{e_k\}_k$ satisfies $e_k \geq ae_{k-1}$ for all k and some $a > 1$. Then whenever $k < l$, we have

$$\frac{A(i,k)}{A(i,l)} \leq a^{1-\lfloor\frac{l-i}{m}\rfloor}.$$

The final result tells us that $\{\gamma_k\}_k$ splits into m subsequences, each projectively converging to a distinct ergodic measure on ν.

Theorem 2.11. If $\{\gamma_k\}_k$ is as in Theorems 2.8 and 2.9, then the sequence determines a $\nu \in \mathcal{EL}(S)$ which is non-uniquely ergodic and supports m ergodic measures, $\bar{\nu}^0, \ldots, \bar{\nu}^{m-1}$, given by

$$\lim_{i\to\infty} \frac{\gamma_{h+im}}{A(0, h+im)} = \bar{\nu}^h,$$

for each $h = 0, \ldots, m-1$.

2.6 Weil–Petersson metric

The Weil–Petersson (WP) metric is an incomplete, mapping class group invariant, Riemannian metric with negative curvature on the Teichmüller space. The WP completion of Teichmüller space $\overline{\mathrm{Teich}(S)}$ is a stratified CAT(0) space. Each stratum is canonically isometric to the product of Teichmüller spaces of lower complexity, each equipped with the WP metric. More precisely, for any possibly empty multicurve σ on S the stratum $\mathcal{S}(\sigma)$ consists of finite type Riemann surfaces appropriately marked by $S\backslash\sigma$, and this is isometric to the product of Teichmüller spaces of the connected components of $S\backslash\sigma$. An important property of completion strata is the following *non-refraction* property.

Theorem 2.12. (Non-refraction; [15, 37]) The interior of the geodesic segment connecting a point $X \in \mathcal{S}(\sigma)$ to a point $Y \in \mathcal{S}(\sigma')$ lies in the stratum $\mathcal{S}(\sigma \cap \sigma')$.

Let $L_S > 0$ be the *Bers constant* of S (see [10, §5]). Then each point $X \in \mathrm{Teich}(S)$ has a pants decomposition P (*Bers pants decomposition*) with the property that the length of every curve in P with respect to X is at most L_S. Any curve in a Bers pants decomposition is called a *Bers curve*. Moreover, a marking whose base is a Bers pants decomposition and has transversal curves with shortest possible length is called a *Bers marking*.

Brock [8] showed that the coarse map

$$Q : \mathrm{Teich}(S) \to P(S) \tag{2.9}$$

which assigns to a point in the Teichmüller space a Bers pants decomposition at that point is a quasi-isometry.

Using the non-refraction property of completion strata Wolpert [35] gives a picture for the limits of sequences of bounded length WP geodesic segments in Teichmüller space after remarkings. The following strengthening of the picture was proved in [29, §4]. For reference, given a multicurve σ let $\mathrm{tw}(\sigma) < \mathrm{Mod}(S)$ denote the subgroup generated by positive Dehn twists about the curves in σ.

Theorem 2.13. (Geodesic Limit) Given $T > 0$, let $\zeta_n : [0, T] \to \mathrm{Teich}(S)$ be a sequence of geodesic segments parametrized by arc length. After possibly passing to a subsequence, there exist a partition $0 = t_0 < \ldots < t_{k+1} = T$ of the interval $[0, T]$, multicurves σ_l, $l = 0, \ldots, k+1$, a multicurve $\hat{\tau}$ with $\hat{\tau} = \sigma_l \cap \sigma_{l+1}$ for all $l = 0, \ldots, k$ and a piecewise geodesic segment

$$\hat{\zeta} : [0, T] \to \overline{\mathrm{Teich}(S)}$$

such that

1. $\hat{\zeta}(t_l) \in \mathcal{S}(\sigma_l)$ for each $l = 0, \ldots, k+1$,
2. $\hat{\zeta}((t_l, t_{l+1})) \subset \mathcal{S}(\hat{\tau})$ for each $l = 0, \ldots, k$,
3. there exist elements $\psi_n \in \mathrm{Mod}(S)$ which are either trivial or unbounded as $n \to \infty$ and elements $\mathcal{T}_{l,n} \in \mathrm{tw}(\sigma_l - \hat{\tau})$ such that for any $\gamma \in \sigma_l - \hat{\tau}$ the power of the positive Dehn twist D_γ about γ is unbounded as $n \to \infty$, and we have:

$$\lim_{n\to\infty} \psi_n(t) = \hat{\zeta}(t)$$

for any $t \in [0, t_1]$. Moreover, for each $l = 1, \ldots, k$ let

$$\varphi_{l,n} = \mathcal{T}_{l,n} \circ \ldots \circ \mathcal{T}_{1,n} \circ \psi_n, \tag{2.10}$$

then

$$\lim_{n\to\infty} \varphi_{l,n}(\zeta_n(t)) = \hat{\zeta}(t)$$

for any $t \in [t_l, t_{l+1}]$.

In [29] controls on length-functions along WP geodesics in terms of subsurface coefficients are developed. The following are corollaries 4.10 and 4.11 in [29]. Here we denote a Bers marking at a point $X \in \mathrm{Teich}(S)$ by $\mu(X)$.

Theorem 2.14. Given T, ϵ_0 and $\epsilon < \epsilon_0$ positive, there is an $N \in \mathbb{N}$ with the following property. Let $\zeta : [0, T'] \to \mathrm{Teich}(S)$ be a WP geodesic segment parametrized by arc length, of length $T' \leq T$, such that

$$\sup_{t \in [0,T']} \ell_\gamma(\zeta(t)) \geq \epsilon_0.$$

Then if $d_\gamma(\mu(\zeta(0)), \mu(\zeta(T'))) > N$ we have

$$\inf_{t \in [0,T']} \ell_\gamma(\zeta(t)) \leq \epsilon.$$

Theorem 2.15. Given T, ϵ_0, s positive with $T > 2s$ and $N \in \mathbb{N}$, there is an $\epsilon \in (0, \epsilon_0)$ with the following property. Let $\zeta : [0, T'] \to \mathrm{Teich}(S)$ be a WP geodesic segment parametrized by arc length of length $T' \in [2s, T]$. Let $J \subseteq [s, T' - s]$ be a subinterval. Suppose that for some $\gamma \in \mathcal{C}_0(S)$ we have

$$\sup_{t \in [0,T']} \ell_\gamma(\zeta(t)) \geq \epsilon_0.$$

Then, if $\inf_{t \in J} \ell_\gamma(\zeta(t)) \leq \epsilon$, we have

$$d_\gamma\left(\mu(\zeta(0)), \mu(\zeta(T'))\right) > N.$$

In this paper we will frequently use the following result of Wolpert for estimating distance of a point and a completion stratum. It is part of [36, Corollary 4.10].

Corollary 2.16. Let $X \in \mathrm{Teich}(S)$ and let σ be a multicurve, then

$$d_{\mathrm{WP}}(X, \mathcal{S}(\sigma)) \leq \sqrt{2\pi \sum_{\alpha \in \sigma} \ell_\alpha(X)} \text{ and}$$
$$d_{\mathrm{WP}}(X, \mathcal{S}(\sigma)) = \sqrt{2\pi \sum_{\alpha \in \sigma} \ell_\alpha(X)} + O\Big(\sum_{\alpha \in \sigma} \ell_\alpha(X))^{5/2}\Big)$$

where the constant of the O notation depends only on an upper bound for the length of curves in σ at X.

2.7 End invariants

Brock, Masur, and Minsky [5] introduced the notion of *ending lamination* for Weil–Petersson geodesic rays as follows: Note that here and throughout the paper all geodesics would be parametrized by the arc length. Let $r: [a, b) \to \mathrm{Teich}(S)$ be a complete WP geodesic ray (a ray whose domain cannot be extended to the left end point b). First, the weak* limit of an infinite sequence of weighted distinct Bers curves at times $t_i \to b$ is an *ending measure* of the ray r, and any curve α with $\lim_{t\to b} \ell_\alpha(r(t_i)) = 0$ is a *pinching curve* of r. Now the union of supports of ending measures and pinching curves of r is the *ending lamination* of r which we denote by $\nu(r)$.

Let $g: I \to \mathrm{Teich}(S)$ be a WP geodesic, where $I \subseteq \mathbb{R}$ is an interval. Denote the left and right end points of I by a, b, respectively, and let c be a point in the interior of I. If g is extendable to b in $\overline{\mathrm{Teich}(S)}$, including the situation that $b \in I$, then the forward end invariant of g, denoted by ν^+, is a (partial) Bers marking at $g(b)$. If not, the forward end invariant of g (also called the forward ending lamination) is the ending lamination of the geodesic ray $g(t)|_{[c,b)}$ defined above. The backward end invariant (ending lamination) ν^- of g is defined similarly considering the ray $g(-t)|_{(a,c]}$. Finally, the pair (ν^-, ν^+) is called the end invariant of g.

For example, the end invariant of a geodesic segment $g: [a, b] \to \mathrm{Teich}(S)$ is the pair of markings $(\mu(g(a)), \mu(g(b)))$.

For more detail about end invariants of WP geodesics and their application to study the geometry and dynamics of Weil–Petersson metric see [4–6, 17, 29, 30].

2.8 Bounded combinatorics

Given $R > 0$, a pair of (partial) markings or laminations (μ, ν) has *R–bounded combinatorics* if for any proper subsurface $Y \subsetneq S$ the bound

$$d_Y(\mu, \nu) \le R \tag{2.11}$$

holds. If the bound holds only for non-annular subsurfaces of S we say that the pair has *non-annular R–bounded combinatorics*.

The following theorem relates the non-annular bounded combinatorics of end invariants to the behavior of WP geodesics.

Theorem 2.17. For any $R > 0$ there is an $\epsilon > 0$ so that any WP geodesic ray $r: [0, \infty) \to \mathrm{Teich}(S)$ whose end invariant has non-annular R–bounded combinatorics visits the ϵ–thick part of $\mathrm{Teich}(S)$ infinitely often.

Proof. The fact that an individual ray r visits an ϵ–thick part of Teich(S) infinitely often is [4, Theorem 4.1]. To show that ϵ can be chosen uniformly for all geodesic rays r whose end invariants have non-annular R–bounded combinatorics consider a decreasing sequence $\epsilon_n \to 0$ and a sequence of WP geodesic rays $r_n : [0, \infty) \to \text{Teich}(S)$ with non-annular R–bounded combinatorics end invariants $(\mu(r_n(0)), \nu_n^+)$ and assume that ϵ_n is the largest number that r_n visits the ϵ_n–thick part of Teichmüller space infinitely often. In particular, for each $n \geq 1$ there is a time t_n so that $r_n([t_n, \infty))$ does not intersect the $2\epsilon_n$–thick part of Teich(S).

Since the end invariant of r_n has non-annular R–bounded combinatorics, a hierarchy path ϱ_n between the end invariant is stable in the pants graph $P(S)$ [6, Theorem 4.3] [29, Theorem 5.13], in particular ϱ_n and $Q(r_n)$, the image of r_n under Brock's quasi-isometry (2.9), D–fellow travel in $P(S)$ where the constant D depends only on R. Theorem 2.5 then guarantees that any two points along ϱ_n also satisfy the non-annular bounded combinatorics condition (2.11) with a larger constant.

Now for any two times $t_1, t_2 \in [0, \infty)$, let i_1, i_2 be so that $d(\varrho_n(i_1), Q(r_n(t_1)))$ and $d(\varrho_n(i_2), Q(r_n(t_2)))$ are at most D. Then from the distance formula (2.2) we see that all subsurface coefficients of the pair $(\varrho_n(i_1), Q(r_n(t_1)))$ and the pair $(\varrho_n(i_2), Q(r_n(t_2)))$ are bounded by $\max\{A, KD+KC\}$ for a choice of threshold the A in the formula. This together with the fact that the pair $(\varrho_n(i_1), \varrho_n(i_2))$ has non-annular bounded combinatorics imply that the pair $(Q(r_n(t_1)), Q(r_n(t_2)))$ also satisfies the non-annular bounded combinatorics condition for some $R' > R$ which depends only on R.

For $R' > 0$ as above let the constants $T_0 > 0$ and $\epsilon_0 > 0$ be as in Lemma 4.2 of [4]. Let $I_n \subset [t_n, \infty)$ be an interval of length T_0 which contains a time s_n so that $r_n(s_n)$ is in the ϵ_n–thick part of Teichmüller space and $s_n \to \infty$ as $n \to \infty$. By the previous paragraph the end points of $r_n|_{I_n}$ have non-annular R'–bounded combinatorics, so by Lemma 4.2 of [4] after possibly passing to a subsequence the geodesic segments $r_n|_{I_n}$ intersect the ϵ_0–thick part of Teichmüller space. But for n sufficiently large $2\epsilon_n < \epsilon_0$ and by the choice of I_n the segment $r_n|_{I_n}$ does not intersect the $2\epsilon_n$–thick part, which contradicts that $r_n|_{I_n}$ intersects the ϵ_0–thick part. The existence of a uniform ϵ for all r with non-annular R–bounded combinatorics follows from this contradiction. ∎

2.9 Isolated annular subsurfaces

In this section we recall the relevant aspects of the notion of an isolated annular subsurface along a hierarchy path from [29, §6] and its consequences for our purposes in this paper.

Let (μ, ν) be a pair of (partial) markings or laminations with non-annular R–bounded combinatorics. A hierarchy path $\varrho : [m, n] \to P(S)$, $[m, n] \subseteq \mathbb{Z} \cup \{\pm\infty\}$, with end points (μ, ν) is stable in the pants graph of S [6]. In particular, $Q(g)$ the image of a WP geodesic g with end invariant (μ, ν) under Brock's quasi-isometry (2.9) D–fellow travels ϱ for a $D > 0$ depending only on R. For a parameter $i \in [m, n]$, we say that the time t corresponds to i, if $Q(g(t))$ is within distance D of $\varrho(i)$, and vice versa.

Let $i \in [m, n]$, and let Q be a pants decomposition within distance D of the point $\varrho(i)$, moreover let γ be a curve in Q. By [29, Definition 6.3] the annular subsurface with core curve γ is isolated at i along ϱ and hence by [29, Lemma 6.4] we have:

Lemma 2.18. (Annular coefficient comparison) There are positive constants $\bar{w}, b$, and B depending on R and a constant L depending only on the topological type of S, so that for the curve γ, a time t corresponding to i, any $s \geq \bar{w}$ and $s' \stackrel{+}{\asymp}_b s$, we have:

$$d_\gamma(\mu(g(t-s')), \mu(g(t+s'))) \stackrel{+}{\asymp}_B d_\gamma(\varrho(i-s), \varrho(i+s)), \tag{2.12}$$

where $\mu(\cdot)$ is a choice of Bers marking at the point. Moreover,

$$\min\{\ell_\gamma(g(t-s')), \ell_\gamma(g(t+s'))\} \geq L. \tag{2.13}$$

3 Sequences of curves on punctured spheres

In this section we construct a sequence of curves that satisfies the condition $\mathcal{P}$ of Definition 2.6. This construction is a generalization of the one in [21] to spheres with more punctures. Fix a sequence of positive integers $\mathcal{E} = \{e_k\}_{k=0}^{\infty}$.

Let $p \geq 5$ be an odd integer and $S = S_{0,p}$ be a sphere with p punctures. We visualize S as the double of a regular p-gon (with vertices removed), admitting an order *p rotational symmetry*, as in Figure 1. Let $\rho \colon S \to S$ be the counterclockwise rotation by angle $4\pi/p$. Set $m = \frac{p-1}{2}$.

Next, let γ_0 be a curve obtained by doubling an arc connecting two sides of the polygon adjacent to a common side. Then $\{\rho^j(\gamma_0)\}_{j=0}^{p-1}$ is a set of p curves that pairwise intersection 0 or 2 times; see Figure 1. We let $\alpha = \rho^m(\gamma_0)$, and recall that D_α denotes the positive Dehn twist about the curve α. For $k \geq 1$, set

$$\begin{aligned} \phi_k &= D_\alpha^{e_{k+m-1}}\rho, \text{ and} \\ \Phi_k &= \phi_1\phi_2\cdots\phi_k, \end{aligned} \tag{3.1}$$

(in particular, $\Phi_0 = id$).

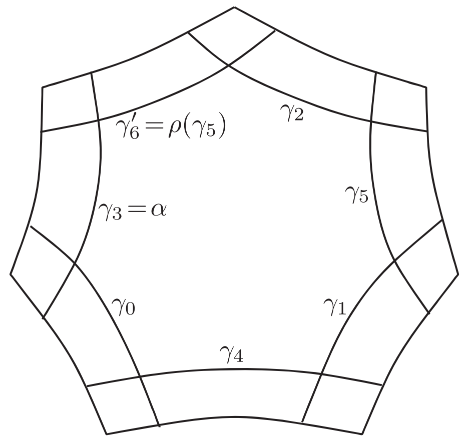

FIG. 1. $S_{0,7}$ as a double of a 7–gon. The curves $\gamma_0, \gamma_1, \gamma_2, \gamma_3 = \alpha, \gamma_4, \gamma_5$ and $\gamma'_6 = \rho(\gamma_5)$ are shown.

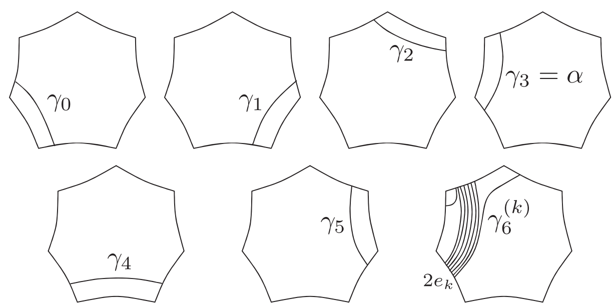

FIG. 2. For $S_{0,7}$ and any $k \geq 3$, applying Φ_{k-3} to any seven consecutive curves in the sequence, $\gamma_{k-3}, \ldots, \gamma_{k+3}$, gives $\gamma_0, \ldots, \gamma_5, \gamma_6^{(k)} = \Phi_{k-3}(\gamma_{k+3})$ as shown here.

Define a sequence of curves $\Gamma = \Gamma(\mathcal{E}) = \{\gamma_k\}_{k=0}^{\infty}$, starting with γ_0, by the formula

$$\gamma_k = \Phi_k(\gamma_0). \tag{3.2}$$

Since a twist about α has no effect on a curve disjoint from it, for $0 \leq j \leq 2m - 1$,

$$\gamma_k = \Phi_{k-j}(\gamma_j) = \Phi_{k-m}(\alpha), \tag{3.3}$$

for all $k \geq m$. See Figure 2 for a picture illustrating $2m + 1$ consecutive curves.

Proposition 3.1. The sequence $\Gamma(\mathcal{E}) = \{\gamma_k\}_{k=0}^{\infty}$ satisfies condition $\mathcal{P}(\mathcal{E})$ in Definition 2.6 for $m = \frac{p-1}{2}$ and $b' = b = 2$.

Proof. The proof boils down to showing that, after applying an appropriate homeomorphism, any $2m+1=p$ consecutive curves differ from $\gamma_0, \ldots, \gamma_{2m}$ only in the amount of relative twisting of γ_0 and γ_{2m} around γ_m; see Figure 2 and compare with the construction from [21]. We now explain this in more detail.

First, observe that for $j = 1, \ldots, p-2 = 2m-1$, $\mathrm{i}(\alpha, \rho^j(\gamma_0)) = 0$. Thus

$$\gamma_j = \rho^j(\gamma_0),$$

for $j = 0, \ldots, 2m-1$. By construction, any two of these curves intersect 0 or 2 times, while the first m are pairwise disjoint. Furthermore, the entire set of $2m$ curves fills S; see Figure 1.

Next, for any $k \geq m$, applying Φ_{k-m}^{-1} to the $2m-1$ consecutive curves $\{\gamma_{k-m}, \ldots, \gamma_{k+m-1}\}$, (3.3) implies

$$\Phi_{k-m}^{-1}(\gamma_{k-m+j}) = \gamma_j \tag{3.4}$$

for each $j = 0, \ldots, 2m-1$. Since k was arbitrary, it follows that any m consecutive curves are pairwise disjoint and any $2m$ consecutive curves fill S. Thus, conditions (i) and (ii) of $\mathcal{P}$ are satisfied.

For part (iii), let $\gamma'_{k+m} = \Phi_{k-m}(\rho(\gamma_{2m-1})) = \Phi_{k-m}(\rho^{2m}(\gamma_0))$. Then, for $j = 0, \ldots, 2m-1$, we may apply Φ_{k-m}^{-1}, and we have

$$\mathrm{i}(\gamma'_{k+m}, \gamma_{k-m+j}) = \mathrm{i}(\rho^{2m}(\gamma_0), \gamma_j) = \mathrm{i}(\rho^{2m}(\gamma_0), \rho^j(\gamma_0)) = \begin{cases} 2 & \text{for } j = m, \text{ and } m-1 \\ 0 & \text{otherwise,} \end{cases}$$

which implies the intersection number requirement for (iii), with $b' = b = 2$.

Finally, applying Φ_{k-m} to γ_{k+m} we get

$$\begin{aligned} \Phi_{k-m}^{-1}(\gamma_{k+m}) &= \phi_{k-m+1} \cdots \phi_{k+m}(\gamma_0) = \phi_{k-m+1}\left(\rho^{2m-1}(\gamma_0)\right) \\ &= D_\alpha^{e_{(k-m+1)+m-1}} \rho(\rho^{2m-1}(\gamma_0)) = D_{\gamma_m}^{e_k}\left(\rho^{p-1}(\gamma_0)\right), \end{aligned}$$

where we have used the fact that $\alpha = \rho^m(\gamma_0) = \gamma_m$. Therefore,

$$\begin{aligned} \gamma_{k+m} &= \Phi_{k-m}\left(D_{\gamma_m}^{e_k}(\rho^{p-1}(\gamma_0))\right) \\ &= \Phi_{k-m} D_{\gamma_m}^{e_k} \Phi_{k-m}^{-1} \Phi_{k-m} \rho^{p-1}(\gamma_0) \\ &= D_{\Phi_{k-m}(\gamma_m)}^{e_k}\left(\gamma'_{k+m}\right) = D_{\gamma_k}^{e_k}\left(\gamma'_{k+m}\right). \end{aligned}$$

Therefore, part (iii) from $\mathcal{P}$ is also satisfied. ∎

Corollary 3.2. If $\mathcal{E} = \{e_k\}_{k=0}^{\infty}$ satisfies $e_{k+1} \geq a e_k$ for all k and for an $a > 1$ sufficiently large, then the conclusions of Theorem 2.8, Theorem 2.9, and Theorem 2.11 hold for the sequence $\Gamma(\mathcal{E}) = \{\gamma_k\}_{k=0}^{\infty}$. In particular, the sequence determines a minimal, filling non-uniquely ergodic lamination ν.

4 Limits of closed geodesics

Let $S = S_{0,p}$ be the p–punctured sphere where $p \geq 5$ is an odd integer. Let α, ρ then be as in §3, e be a positive integer, and $f_e = D_\alpha^e \rho$. To relate this to the previous section, note that for any fixed e, the sequence of mapping classes $\{\phi_k\}_{k=0}^{\infty}$ obtained from the constant sequence $\mathcal{E} = \{e\}_{k=0}^{\infty}$ is constant; $\phi_k = f_e$ for all k. Consequently, the sequence of curves $\Gamma(\mathcal{E})$ is obtained by iteration: $\Gamma(\mathcal{E}) = \{f_e^k(\alpha)\}_k$ (after a shift of indices).

We assume in the following that $e > E$ from Theorem 2.8. Then by Proposition 3.1 and Theorem 2.8, $k \mapsto f_e^k(\alpha)$ is a f_e–invariant quasi-geodesic in the curve complex, and hence f_e is pseudo-Anosov. By [19, Theorem 4.1] the sequence of curves $\{f_e^k(\alpha)\}_{k=0}^{\infty}$ determines a projective measured lamination $[\bar{\nu}_e^+]$ and $\{f_e^k(\alpha)\}_{k=0}^{-\infty}$ determines a projective measured lamination $[\bar{\nu}_e^-]$.

A key ingredient in our construction of a Weil–Petersson geodesic ray in §5 will be a very precise understanding of the limiting picture of the axes g_e of the pseudo-Anosov mapping classes f_e, as e tends to infinity. The main results of this section are Proposition 4.5 in which we describe a biinfinite piecewise geodesic in $\overline{\mathrm{Teich}(S)}$ which approximate the geodesics g_e in the Hausdorff topology and gives us the necessary limiting picture for g_e as $e \to \infty$.

Our analysis of the axes g_e of f_e begins with an analysis of the action of ρ on Teich(S) and certain strata in the Weil–Petersson completion. Observe that the quotient of S by $\langle\rho\rangle$ is a sphere with one puncture and two cone points. A fixed point of ρ in Teich(S) is a ρ–invariant conformal structure on S or, equivalently, a conformal structure obtained by pulling back a conformal structure on the quotient $S/\langle\rho\rangle$. Since the sphere with three marked points is rigid, there is a unique such conformal structure, and hence exactly one fixed point $Z \in$ Teich(S) for the action of ρ.

Proposition 4.1. For the stratum $\mathcal{S}(v)$ defined by a multicurve v, there exists a point $X_0 \in \overline{\mathcal{S}(v)}$ so that

$$d_{\mathrm{WP}}\left(X_0, \rho(X_0)\right) = \inf_{Y \in \mathcal{S}(v)} d_{\mathrm{WP}}(Y, \rho(Y)).$$

Remark 4.2. Note that unless $\mathcal{S}(v)$ is a point (i.e., v is a pants decomposition), $\overline{\mathcal{S}(v)}$ is not compact.

Now we define the function

$$F : \overline{\mathrm{Teich}(S)} \to \mathbb{R}^{\geq 0},$$

by $F(X) = d_{WP}(X, \rho(X))$. The proposition is then equivalent to showing that the restriction of F to the closure $\overline{\mathcal{S}(v)}$ attains a minimum value. We begin with a lemma.

Lemma 4.3. The function $F : \overline{\mathrm{Teich}(S)} \to \mathbb{R}^{\geq 0}$ is convex, 2–Lipschitz, and for any $R > 0$, $F^{-1}([0, R])$ is a bounded set.

Proof. Since the completion of the Weil–Petersson metric is CAT(0), the distance function on $\overline{\mathrm{Teich}(S)}$ is convex, and hence so is F. The triangle inequality proves that F is 2–Lipschitz, since

$$\begin{aligned} |F(Y) - F(X)| &= \left| d_{\mathrm{WP}}(Y, \rho(Y)) - d_{\mathrm{WP}}(X, \rho(X)) \right| \\ &\leq \left| d_{\mathrm{WP}}(X, Y) + d_{\mathrm{WP}}(X, \rho(Y)) - d_{\mathrm{WP}}(X, \rho(X)) \right| \\ &\leq d_{\mathrm{WP}}(X, Y) + d_{\mathrm{WP}}(\rho(X), \rho(Y)) \\ &= 2 d_{\mathrm{WP}}(X, Y). \end{aligned}$$

Let $Z \in \mathrm{Teich}(S)$ be the fixed point of the action of ρ on $\mathrm{Teich}(S)$ and suppose that $R_0 > 0$ is sufficiently small so that $B_{R_0}(Z)$, the closed ball of radius R_0 in $\overline{\mathrm{Teich}(S)}$ about Z, is contained in $\mathrm{Teich}(S)$, and thus is compact. Let $R_1 > 0$ be the minimum value of F on $\partial B_{R_0}(Z)$.

For any $Y \in \overline{\mathrm{Teich}(S)} \setminus B_{R_0}(Z)$, let Y_0 be the unique point of intersection of the geodesic from Z to Y with the sphere $\partial B_{R_0}(Z)$. Then it follows that $d_{WP}(Y, Z) = R_0 + d_{WP}(Y, Y_0)$, and so convexity of F implies

$$F(Y_0) \leq \frac{R_0}{d_{\mathrm{WP}}(Y, Z)} F(Y) + \frac{d_{\mathrm{WP}}(Y, Y_0)}{d_{\mathrm{WP}}(Y, Z)} F(Z)$$

But then since $F(Z) = 0$ and $F(Y_0) \geq R_1$ we have

$$F(Y) \geq \frac{d_{WP}(Z, Y)}{R_0} F(Y_0) \geq \frac{d_{WP}(Z, Y)}{R_0} R_1.$$

Rearranging the above inequality, we have

$$d_{WP}(Z,Y) \leq \frac{R_0}{R_1}F(Y)$$

and hence if $R > R_0$ and $F(Y) \leq R$, then we have $d_{\mathrm{WP}}(Z,Y) \leq \frac{R_0 R}{R_1}$. That is, $F^{-1}([0,R]) \subset B_{R_0R/R_1}(Z)$, as required. ■

Proof of Proposition 4.1. Any stratum in $\overline{\mathcal{S}(v)}$ has the form $\mathcal{S}(v')$ for a multicurve v' containing v. Observe that the infimum of the function F on any stratum $\mathcal{S}(v')$ in $\overline{\mathcal{S}(v)}$ is no less than the infimum of F on $\mathcal{S}(v)$. Let $\mathcal{S}(v')$ in $\overline{\mathcal{S}(v)}$ be a stratum in the closure having minimal dimension, so that the infimum of F on $\mathcal{S}(v')$ is equal to the infimum on $\mathcal{S}(v)$. It suffices to show that the infimum of F on $\mathcal{S}(v')$ is realized on $\mathcal{S}(v')$.

Let $\{X_n\}_{n=1}^{\infty} \subset \mathcal{S}(v')$ be an *infimizing sequence* for F on $\mathcal{S}(v')$; that is

$$\lim_{n\to\infty} F(X_n) = \inf_{X\in\mathcal{S}(v')} F(X). \tag{4.1}$$

Let $R < \infty$ be such that $F(X_n) \leq R$ for all $n \geq 1$. Lemma 4.3 then implies that there exists $D > 0$ such that $d_{WP}(Z, X_n) \leq D$ for all $n \geq 1$.

By the triangle inequality, the lengths of the geodesic segments $[X_1, X_n]$ are bounded by $2D$. Let $S_1, \ldots, S_m$ be the connected component of $S \backslash v'$. Then $\mathcal{S}(v')$ is isometric to $\mathrm{pr}\,od_{j=1}^{m}\,\mathrm{Teich}(S_j)$ with the product of WP metrics on each factor. Let $\zeta_n : [0, T_n] \to \mathrm{pr}\,od_{j=1}^{m}\,\mathrm{Teich}(S_j)$ be the parametrization of $[X_1, X_n]$ by arc length. Let

$$\mathrm{pr}_j : \prod_{j=1}^{m} \mathrm{Teich}(S_j) \to \mathrm{Teich}(S_j)$$

be the projection to the j-the factor and let $\zeta_n^j : [0, T_n^j] \to \mathrm{Teich}(S_j)$ be parametrization by arc length of $\mathrm{pr}_j \circ \zeta_n$. Note that $T_n^j \leq 2D$ for $j = 1, \ldots, m$. So for a fixed j, trimming the intervals and reparametrization we get a sequence of geodesic segments $\zeta_n^j : [0, T^j] \to \mathrm{Teich}(S_j)$ of equal length. We may then apply Theorem 2.13 (Geodesic Limit Theorem) to the sequence of geodesic segments $\zeta_n^j : [0, T^j] \to \mathrm{Teich}(S_j)$. Let the multicurves σ_i^j, $i = 0, \ldots, k_j + 1$, the multicurve $\hat{\tau}^j$, the partition $t_0^j < \ldots < t_{k_j+1}^j$, and the piecewise geodesic $\hat{\zeta}^j$ be as in the theorem. Also let the elements of mapping class group ψ_n^j and $\varphi_{l,n}^j$, $l = 1, \ldots, k_j$ be as in the theorem. Note that by the theorem when $k_j \geq 1$ we have that $\hat{\zeta}^j(t_1^j) \in \mathcal{S}(\sigma_1^j)$ and $\lim_{n\to\infty} \varphi_{1,n}^j(\zeta_n^j(t_1^j)) = \hat{\zeta}^j(t_1^j)$.

Since the geodesics ζ_n^j have a common starting point $\mathrm{pr}_j(X_1)$, it follows that ψ_n^j is the identity map for all n. Hence, if $k_j = 0$, then after possibly passing to a subsequence the points $\mathrm{pr}_j(X_n)$ converge.

First suppose that $k_j = 0$ for all $j = 1, \ldots, m$, then after possibly passing to a subsequence all sequences $\mathrm{pr}_j(X_n)$ converge as $n \to \infty$. As a result the points X_n converge and we are done.

Now we suppose that $k_j \geq 1$ for some j, let $\beta \in \sigma_1^j$, and we derive a contradiction. Note that $\mathrm{pr}_j(X_1)$ and $\mathrm{pr}_j(X_n)$ are in $\mathrm{Teich}(S_j)$. Claim 4.9 in the proof of Theorem 4.1 in [29] tells us that for Bers markings $\mu(\mathrm{pr}_j(X_1))$ and $\mu(\mathrm{pr}_j(X_n))$ and curves $\beta_n = (\varphi_{1,n}^j)^{-1}(\beta)$,

$$d_{\beta_n}\Big(\mu(\mathrm{pr}_j(X_1)), \mu(\mathrm{pr}_j(X_n))\Big) \to \infty.$$

as $n \to \infty$.

Now recall that $\varphi_{1,n}^j = \mathcal{T}_{1,n}^j \circ \psi_n^j$, also that $\mathcal{T}_{1,n}^j$ is the composition of a power of the Dehn twist about the curve β_n and powers of Dehn twists about curves disjoint from β_n. Moreover, as we saw above ψ_n is identity. Thus $\beta_n \equiv \beta$ for all n. Therefore the above limit becomes

$$d_{\beta}\Big(\mu(\mathrm{pr}_j(X_1)), \mu(\mathrm{pr}_j(X_n))\Big) \to \infty$$

as $n \to \infty$. We may then choose a sequence $\{n_k\}_{k=1}^{\infty}$ so that

$$d_{\beta}\Big(\mu(\mathrm{pr}_j(X_{n_k})), \mu(\mathrm{pr}_j(X_{n_{k+1}}))\Big) \to \infty \tag{4.2}$$

as $k \to \infty$. ■

Claim 4.4. There exists a sequence of points $\{Y_{n_k}\}_k$ on the geodesic segments $[X_{n_k}, X_{n_{k+1}}]$ with the property that the distance between Y_{n_k} and $\mathcal{S}(v' \cup \beta)$ goes to 0.

Proof. It suffices to show that there is a sequence of points Y_{n_k} on $[X_{n_k}, X_{n_{k+1}}]$ so that the distance between $\mathrm{pr}_j(Y_{n_k})$ and $\mathcal{S}(\beta) \subset \overline{\mathrm{Teich}(S_j)}$ goes to 0. If the distance between $\mathrm{pr}_j(X_{n_k})$ and $\mathcal{S}(\beta)$ goes to 0, the sequence $Y_{n_k} = X_{n_k}$ is the desired sequence. Otherwise, there is a lower bound for the distance between $\mathrm{pr}_j(X_{n_k})$ and $\mathcal{S}(\beta)$. Moreover, by Corollary 2.16 we have

$$d_{\mathrm{WP}}(\mathrm{pr}_j(X_{n_k}), \mathcal{S}(\beta)) \leq \sqrt{2\pi \ell_\beta(\mathrm{pr}_j(X_{n_k}))}.$$

Thus we obtain a lower bound for $\ell_\beta(\mathrm{pr}_j(X_{n_k}))$. Appealing to Theorem 2.14, the annular coefficient limit (4.2) provides a point $Y^j_{n_k}$ on $[\mathrm{pr}_j(X_{n_k}), \mathrm{pr}_j(X_{n_{k+1}})]$ so that $\ell_\beta(Y^j_{n_k}) \to 0$, and hence again by Corollary 2.16 the distance between $Y^j_{n_k}$ and $\mathcal{S}(\beta)$ goes to 0. Now the points Y_{n_k} on $[X_{n_k}, X_{n_{k+1}}]$ with $\mathrm{pr}_j(Y_{n_k}) = Y^j_{n_k}$ are the desired points. ∎

It follows from the above claim and the convexity of the function F that

$$F(Y_{n_k}) \leq \max\left\{F(X_{n_k}), F(X_{n_{k+1}})\right\}. \tag{4.3}$$

Therefore, $\{Y_{n_k}\}$ is also an infimizing sequence for the function F on $\mathcal{S}(v)$. Let $Y_k{}'$ be the closest point to Y_{n_k} in $\mathcal{S}(v' \cup \beta)$. Since the distance of the points Y_{n_k} and $\mathcal{S}(v' \cup \beta)$ goes to 0 we have that

$$d_{\mathrm{WP}}\left(Y_{n_k}, Y'_k\right) \to 0$$

and therefore $F(Y_{n_k})$ and $F(Y_n{}')$ have the same limit since F is 2–Lipschitz. Therefore $\{Y_k{}'\}_k$ is a infimizing sequence for the function F in the stratum $\mathcal{S}(v' \cup \beta)$, but this stratum has dimension less than that of $\mathcal{S}(v')$. This contradiction finishes the proof of the proposition.

Now we can describe the biinfinite piecewise geodesics $g^\omega_e \subset \overline{\mathrm{Teich}(S)}$ which approximate the geodesics g_e, the axes of the pseudo-Anosov mapping classes f_e as follows. First, appealing to Proposition 4.1, let $X_0 \in \overline{\mathcal{S}(\rho^{-1}(\alpha))}$ be a point where the function $F(X) = d_{WP}(X,\ \rho(X))$ is minimized in the closure of the stratum $\mathcal{S}(\rho^{-1}(\alpha))$. As already observed, on $\overline{\mathcal{S}(\rho^{-1}(\alpha))}$, we have $f_e = D^e_\alpha\rho = \rho$ since D^e_α acts trivially on $\rho(\overline{\mathcal{S}(\rho^{-1}(\alpha))}) = \overline{\mathcal{S}(\alpha)}$. Consequently, $f_e(X_0) = \rho(X_0)$, and we may concatenate the geodesic segment $\omega = [X_0,\ \rho(X_0)]$ with its f_e–translates to produce an f_e–invariant, biinfinite piecewise geodesic in $\overline{\mathrm{Teich}(S)}$:

$$g^\omega_e = \cdots \cup f_e^{-2}(\omega) \cup f_e^{-1}(\omega) \cup \omega \cup f_e(\omega) \cup f_e^2(\omega) \cup \cdots \tag{4.4}$$

Proposition 4.5. The path g^ω_e is a biinfinite piecewise geodesic that fellow travels g_e, and the Hausdorff distance between g^ω_e and g_e tends to 0 as $e \to \infty$.

For the proof of the proposition we need the following theorem which is a *characterization of the short curves* along the geodesic g_e. In the following let $E > 0$ be the constant from Theorem 2.8.

Theorem 4.6. There exists $\epsilon > 0$ so that for all $e > E$ and every point of g_e, at most one curve on S has length less than ϵ, and such a curve is in the set $\{f_e^k(\alpha)\}_{k\in\mathbb{Z}}$ (with α as in §3). Moreover, let t_e be the translation length of f_e, then after reparametrization of g_e we have that the minimal length of the curve $f_e^k(\alpha),\ k \in \mathbb{Z}$ along g_e is realized at kt_e and tends to zero as $e \to \infty$.

Proof. Let $\nu_e^{\pm}$, as before, be the laminations determined by the sequences of curves $\{f_e^k(\alpha)\}_{k=0}^{\pm\infty}$. There is a uniform bound for all subsurface coefficients of the pairs (ν_e^-, ν_e^+) except those of $\{f_e^k(\alpha)\}_{k\in\mathbb{Z}}$. This follows from the fact that in Theorem 2.8 the upper bound R depends only on the parameters from Definition 2.6 and the initial marking μ which is the same for all f_e.

Similarly we have

$$d_{f_e^k(\alpha)}\left(\nu_e^-, \nu_e^+\right) \overset{+}{\asymp} e \tag{4.5}$$

for all $k \in \mathbb{Z}$, where the additive error is independent of e.

Let $\varrho_e : [-\infty, +\infty] \to P(S)$ be a hierarchy path between the pair (ν_e^-, ν_e^+) (see [28]). Since the pair has non-annular R–bounded combinatorics ϱ_e is stable in $P(S)$ [6, Theorem 4.3] [29, Theorem 5.13]. Therefore, ϱ_e and $Q(g_e)$, the image of g_e under Brock's quasi-isometry (2.9) D–fellow travel, where the constant $D \geq 0$ depends only on R. ■

Lemma 4.7. There is an $\epsilon_2 > 0$, so that for all $e > E$, g_e visits the ϵ_2–thick part of Teich(S) infinitely often in both forward and backward times.

Proof. Let $\mu(g_e(0))$ be a Bers marking at $g_e(0)$, and let i be so that $\varrho_e(i)$ is within distance D of $Q(g_e(0))$. Then all non-annular subsurface coefficients of the pair $(\varrho_e(i), Q(g_e(0)))$ are bounded by $\max\{KD + KC, A\}$ by the distance formula (2.2) for a choice of threshold A. Moreover, by Theorem 2.5 all non-annular subsurface coefficients of the pair $(\varrho_e(i), \nu_e^+)$ are bounded by an enlargement of R. Combining the bounds with the triangle inequality in the curve complex of each subsurface then implies that $(\mu(g_e(0)), \nu_e^+)$, the end invariant of $g_e|_{[0,\infty)}$, has non-annular bounded combinatorics, independent of e. Theorem 2.17, then guarantees that for an $\epsilon_2 > 0$, independent of e, the geodesic ray $g_e|_{[0,\infty)}$ visits the ϵ_2–thick part of Teich(S) infinitely often. The proof of that the geodesic ray $g_e|_{[0,-\infty)}$ visits the ϵ_2–thick part of Teich(S) infinitely often is similar. ■

Now we prove the following:

Lemma 4.8. There exists $\epsilon_1 > 0$, depending only on R, so that for all $e > E$ the length of each curve $\gamma \notin \{f_e^k(\alpha)\}_k$ is bounded below by ϵ_1 along g_e.

Proof. Suppose that for a $t \in \mathbb{R}$ the length of γ at $g_e(t)$ is less than the Bers constant. Then, $\gamma \in Q(g_e(t))$ and thus γ is isolated at some i along ϱ_e; for the discussion about isolated annular subsurfaces see §2.9. By Lemma 2.18, there are constants $\bar{w}, b$ depending only on R and a constant L such that for any $s > \bar{w}$ and $s' \stackrel{+}{\asymp}_b s$,

$$\min\left\{\ell_\gamma(g_e(t-s')), \ell_\gamma(g_e(t+s'))\right\} \geq L.$$

Fix s, s' as above and fix $u < s'$. Let $J = [t - s' + u, t + s' - u]$. Then, Theorem 2.15 applies to the geodesic segment $g_e|_{[t+s',t-s']}$ and implies that for any integer $N \geq 1$, there is an $\epsilon \in (0, L/2)$ so that

$$\text{if } \inf_{r\in J} \ell_\gamma(g_e(r)) < \epsilon, \text{then } d_\gamma(\mu(g_e(t-s')), \mu(g_e(t+s'))) > N, \tag{4.6}$$

where $\mu(\cdot)$ denotes a Bers marking at the given point.

According to Lemma 2.17 there is a constant $B > 0$ depending only on R such that

$$d_\gamma\left(\mu\left(g_e(t-s')\right), \mu\left(g_e(t+s')\right)\right) \stackrel{+}{\asymp}_B d_\gamma\left(\varrho_e(i-s), \varrho_e(i+s)\right). \tag{4.7}$$

Further, suppose that γ is not in the set $\{f_e^k(\alpha)\}_{k\in\mathbb{Z}}$. Then the upper bound for $d_\gamma(\nu_e^-, \nu_e^+)$ and Theorem 2.5 for the parameters $-\infty, i-s, i+s, \infty$ of ϱ_e give us an upper bound for the subsurface coefficient

$$d_\gamma(\varrho_e(i-s), \varrho_e(i+s))$$

depending only on R. So by (4.7) we get an upper bound for

$$d_\gamma(\mu(g_e(t-s')), \mu(g_e(t+s'))$$

depending only on R. On the other hand, since $t \in J$ by (4.6) if $\ell_\gamma(g(t))$ gets arbitrary small, then $d_\gamma(\mu(g_e(t-s')), \mu(g_e(t+s')))$ would become arbitrary large, which contradicts the upper bound we just obtained. Therefore, there is a lower bound $\epsilon_1 > 0$ for the length of γ at time t which depends only on R. Since t was arbitrary the proof of the lemma is complete. ∎

The length of each one of the curves in the set $\{f_e^k(\alpha)\}_{k\in\mathbb{Z}}$ is strictly convex along g_e ([36]), and so has a unique minimum. The unique minimum for $f_e^k(\alpha)$ occurs at the f_e^k–image of the point where α is minimized. Thus, we can parameterize g_e by arc length so that for t_e the WP translation length of f_e, the length of the curve $f_e^k(\alpha)$ is minimized at $g_e(kt_e)$.

By Lemma 4.7 there is an $\epsilon_2 > 0$ so that for all $e > E$, g_e visits the ϵ_2–thick part infinitely often in both forward and backward times. Let $t_e{}' \in (0, t_e)$ be a time for which $g_e(t_e{}')$ is in the ϵ_2–thick part. But then $g_e(kt_e + t_e{}')$ is in the thick part for all k. In particular, by convexity of the length of α, it follows that outside the interval $(-t_e + t_e{}', t_e{}')$, $2\epsilon_2$ is a uniform lower bound for the length of α. Likewise, the length of $f_e^k(\alpha)$ is uniformly bounded below by $2\epsilon_2$ outside the interval $((k-1)t_e + t_e{}', kt_e + t_e{}')$. Consequently, for $k \neq k'$, the curves $f_e^k(\alpha)$ and $f_e^{k'}(\alpha)$ cannot simultaneously have length less than $2\epsilon_2$.

As we saw in Lemma 4.8 the only curves which can get shorter than ϵ_1 along g_e are $\{f_e^k(\alpha)\}_k$. Moreover, since we saw above that two of these curves cannot get shorter than ϵ_2 at the same time, the first statement of the theorem holds for $\epsilon = \min\{\epsilon_1, 2\epsilon_2\}$.

Let the laminations $\nu_e^{\pm}$ be as before, and let ϱ_e be a hierarchy path between ν_e^- and ν_e^+. Recall that ϱ_e is stable and that ϱ_e and $Q(g)$, D–fellow travel for a D that depends only on R.

Note that by (4.5) each curve $f_e^k(\alpha)$ $(k \in \mathbb{Z})$ for e sufficiently large has a big enough subsurface coefficient that $f_e^k(\alpha)$ is in $\varrho_e(i)$ for an i in the domain of ϱ_e by [28, Lemma 6.2 (Large link)]. Thus $f_e^k(\alpha)$ is isolated at i along ϱ_e (see §2.9). Let t be a time so that $Q(g_e(t))$ is within distance D of $\varrho_e(i)$. Then for constants $\bar{w}, b, B$ from Lemma 2.18 and any $s \geq \bar{w}$ and $s' \stackrel{+}{\asymp}_b s$ we have that

$$d_\gamma(\mu(g_e(t-s')), \mu(g_e(t+s'))) \stackrel{+}{\asymp}_B d_\gamma(\varrho_e(i-s), \varrho_e(i+s)).$$

Thus the bound $d_{f_e^k(\alpha)}(\nu_e^-, \nu_e^+) \stackrel{+}{\asymp} e$ and Theorem 2.5 for the parameters $-\infty, i-s, i+s, \infty$ of ϱ_e imply that

$$d_\gamma(\mu(g_e(t-s')), \mu(g_e(t+s'))) \stackrel{+}{\asymp} e.$$

Moreover, for the constant $L > 0$ form Lemma 2.18 we have that

$$\min\{\ell_\gamma(g_e(t-s')), \ell_\gamma(g_e(t+s'))\} \geq L.$$

Now fix s, s', then Theorem 2.14 applies to geodesic segment $g_e|_{[t-s',t+s']}$ and implies that

$$\inf_{r\in[t-s',t+s']} \ell_{f_e^k(\alpha)}(g_e(t)) \to 0$$

as $e \to \infty$. But the minimal length of $f_e^k(\alpha)$ is realized at kt_e so

$$\lim_{e\to\infty} \ell_{f_e^k(\alpha)}(g_e(kt_e)) = 0.$$

This completes the proof of the second statement of the theorem.

We continue to use $t_e > 0$ to denote the WP translation length of f_e and assume the geodesic g_e is parameterized as in the proof of the theorem above. Then, in particular the minimal length of $f_e^k(\alpha)$ along g_e is realized at time kt_e and $\ell_{f_e^k(\alpha)}(g_e(kt_e)) \to 0$ as $e \to \infty$. Likewise, $\{(k-1)t_e + t'_e\}_{k\in\mathbb{Z}}$ denotes times when g_e intersects the fixed thick part of Teich(S). Also, note that the minimum of the length of $f_e^k(\alpha)$ is realized at kt_e ($k \in \mathbb{Z}$) and $\lim_{e\to\infty} \ell_{f_e^k(\alpha)}(kt_e) = 0$.

To prove Proposition 4.5 we also need the following lemma about the limit of translation length of f_e.

Lemma 4.9. The translation distance t_e of f_e limits to $|\omega|$ the length of ω; that is,

$$\lim_{e\to\infty} t_e = d_{\mathrm{WP}}(X_0, \rho(X_0)) = |\omega|,$$

where $X_0 \in \overline{\mathcal{S}(\rho^{-1}(\alpha))}$, as before, is the point where $d_{\mathrm{WP}}(X_0, \rho(X_0)) = \inf_{X\in\mathcal{S}(\rho^{-1}(\alpha))} d_{\mathrm{WP}}(X, \rho(X))$.

Proof. Let $Y_e \in \mathcal{S}(\rho^{-1}(\alpha))$ be the closest point to $g_e(-t_e)$ (and hence closest to the entire geodesic g_e). Then $f_e(Y_e) = D_\alpha^e \rho(Y_e) = \rho(Y_e)$, and hence

$$d_{\mathrm{WP}}(f_e(Y_e), Y_e) = d_{\mathrm{WP}}(\rho(Y_e), Y_e) \geq d_{\mathrm{WP}}(X_0, \rho(X_0)).$$

Moreover, $\rho^{-1}(\alpha) = f_e^{-1}(\alpha)$ so the minimal length of $\rho^{-1}(\alpha)$ along g_e is realized at time $-t_e$ and $\ell_{\rho^{-1}(\alpha)}(g_e(-t_e)) \to 0$ as $e \to \infty$. By Corollary 2.16 the distance between $g_e(-t_e)$ and $\mathcal{S}(\rho^{-1}(\alpha))$ is bounded above by $\sqrt{2\pi \ell_{\rho^{-1}(\alpha)}(g_e(-t_e))}$, so we obtain

$$d_{\mathrm{WP}}(Y_e, g_e(-t_e)) = d_{\mathrm{WP}}(f_e(Y_e), g_e(0)) \to 0$$

as $e \to \infty$. It follows then from the triangle inequality that

$$\begin{aligned}\liminf_{e\to\infty} t_e &= \liminf_{e\to\infty} d_{\mathrm{WP}}(g_e(-t_e), g_e(0)) \\ &\geq \liminf_{e\to\infty}\Big(d_{\mathrm{WP}}(Y_e, f_e(Y_e)) - d_{\mathrm{WP}}(g_e(-t_e), Y_e) - d_{\mathrm{WP}}(f_e(Y_e), g_e(0))\Big) \\ &\geq d_{\mathrm{WP}}(X_0, \rho(X_0)).\end{aligned}$$

On the other hand, since g_e is the geodesic axis of f_e, t_e is less than the distance that f_e translates along g_e^ω, which is precisely $|\omega| = d_{WP}(X_0, \rho(X_0))$. That is, $t_e \leq d_{WP}(X_0, \rho(X_0))$, and hence

$$\limsup_{e\to\infty} t_e \leq d_{\mathrm{WP}}(X_0, \rho(X_0)).$$

Combining this with the above, we have $\lim_{e\to\infty} t_e = d_{\mathrm{WP}}(X_0, \rho(X_0))$, completing the proof of the lemma. ■

We are now ready for the proof of Proposition 4.5.

Proof of Proposition 4.5. We recall that g_e intersects a fixed thick part of Teichmüller space, independent of e, at the times $(k-1)t_e + t_e'$, for all $k \in \mathbb{Z}$. Denote the closest point on g_e^ω to the point $g_e((k-1)t_e + t_e')$ by $X_{e,k}$. The distance between $X_{e,k}$ and $g_e((k-1)t_e + t_e')$ must tend to zero as $e \to \infty$. Otherwise, the strict negative curvature in the thick part of Teich(S) would imply a definite contraction factor $\delta < 1$ for the closest point projection to g_e restricted to g_e^ω for all e sufficiently large. Since $X_{e,k+1} = f_e(X_{e,k})$, $d_{WP}(X_{e,k}, X_{e,k+1}) = |\omega|$. Now by the contraction of the projection on g_e and Lemma 4.9 we would have that

$$|\omega| = \lim_{e\to\infty} t_e \leq \delta|\omega|$$

an obvious contradiction. The sequence of points $\{X_{e,k}\}_{k\in\mathbb{Z}}$ is f_e–invariant and its distance to g_e tends to 0 as $e \to \infty$. Appealing to the CAT(0) property of $\overline{\mathrm{Teich}(S)}$, the furthest point of g_e^ω to g_e must also have distance tending to 0, and hence the Hausdorff distance between g_e and g_e^ω tends to 0, as desired. ■

Corollary 4.10. The point $X_0 \in \overline{\mathcal{S}(\rho^{-1}(\alpha))}$ where the minimum of the function $F(X) = d_{WP}(X, \rho(X))$ (restricted to $\overline{\mathcal{S}(\rho^{-1}(\alpha))}$) is realized lies in $\mathcal{S}(\rho^{-1}(\alpha))$. Moreover,

$$\lim_{e\to\infty} g_e([-t_e, 0]) = \omega.$$

Proof. First recall that $\ell_{f_e^{-1}(\alpha)}(g_e(-t_e)) = \ell_{\rho^{-1}(\alpha)}(g_e(-t_e))$ and that $\ell_\alpha(g_e(0))$ goes to 0 as $e \to \infty$. The distance between the point $g_e(-t_e)$ and the stratum $\mathcal{S}(\rho^{-1}(\alpha))$ by Corollary 2.16 is bounded above by $\sqrt{2\pi \ell_{\rho^{-1}(\alpha)}(g_e(-t_e))}$, and hence tends to zero. Thus the point $g_e(-t_e)$ converges to the closure of $\mathcal{S}(\rho^{-1}(\alpha))$. From Theorem 4.6, the only curve which is very short (has length less than ϵ) at $g_e(-t_e)$ is $\rho^{-1}(\alpha)$, so the point $g_e(-t_e)$ converges to $\mathcal{S}(\rho^{-1}(\alpha))$. Similarly we can see that $g_e(0)$ converges to $\mathcal{S}(\alpha)$.

Moreover, since g_e is a geodesic and $g_e(0) = \rho(g_e(-t_e))$ the point $g_e(-t_e)$ converges to X_0 at which the minimum of the function F is realized. Also, $g_e(0)$ converges to $\rho(X_0)$.

By the non-refraction of the property of WP geodesics (Theorem 2.12) then the interior of ω lies in Teich(S). The limiting behavior of the geodesic follows from the CAT(0) property of the metric on $\overline{\text{Teich}(S)}$. ■

The geodesic axis g_e descends to a closed geodesic $\hat{g}_e$ in $\mathcal{M}(S_{0,p})$ and ω descends to a geodesic segment $\hat{\omega}$ in $\overline{\mathcal{M}(S_{0,p})}$. The previous corollary immediately implies the following.

Corollary 4.11. As $e \to \infty$, we have convergence $\hat{g}_e \to \hat{\omega} \subset \overline{\mathcal{M}(S_{0,p})}$.

For any $e > 0$, we let δ_e denote the geodesic segment from the midpoint of ω to the midpoint of $f_e(\omega)$. We also let ω^- and ω^+ denote the first and second half-segments of ω, respectively (so $\omega = \omega^- \cup \omega^+$ and $\omega^- \cap \omega^+$ is the midpoint of ω). Our construction in the next section will use the following.

Lemma 4.12. Given $\epsilon > 0$, there exists $N > 0$ so that for all $e \geq N$, the triangle with sides δ_e, ω^+, and $f_e(\omega^-)$ has angles less than ϵ at the endpoints of δ_e, and the Hausdorff distance between δ_e and $\omega^+ \cup f_e(\omega^-)$ is at most ϵ.

Proof. By Proposition 4.5 and Corollary 4.10, the segment $g_e([-\frac{t_e}{2}, \frac{t_e}{2}])$ can be made as close as we like to $\omega^+ \cup f_e(\omega^-)$. Since $g_e([-\frac{t_e}{2}, \frac{t_e}{2}])$ and δ_e are both geodesics in a CAT(0) space, and since their endpoints become closer and closer as e tends to infinity, it follows that the distance between $g_e([-\frac{t_e}{2}, \frac{t_e}{2}])$ and δ_e tends to zero as $e \to \infty$. Therefore, the distance between δ_e and $\omega^+ \cup f_e(\omega^-)$ tends to zero as $e \to \infty$. This proves the second statement of the lemma.

Short initial segments of δ_e and ω^+ are both geodesics in a Riemannian manifold; they have a common initial point, and the initial segment of δ_e converges to that of ω^+ as $e \to \infty$. It follows that the angle between δ_e and ω^+ tends to zero as $e \to \infty$. A similar argument (composing with f_e^{-1}) shows that the angle at endpoint of δ_e and $f_e(\omega^-)$ tends to zero as $e \to \infty$. This proves the first statement of the lemma. ∎

5 The non-uniquely ergodic case

Given a sequence of integers $\mathcal{E} \subset \mathbb{N}$ in this section first we construct a WP geodesic ray r that is strongly asymptotic to the piecewise geodesic $g^{\omega}_{\mathcal{E}}$ in $\overline{\mathrm{Teich}(S)}$ similar to the construction in §4, but now for a non-constant sequence $\mathcal{E}$; see (5.3). The proof of strong asymptoticity involves producing regions with definite total negative curvature on ruled surfaces and an application of the Gauss–Bonnet Theorem (c.f. [6, 30]). The asymptoticity to $g^{\omega}_{\mathcal{E}}$ helps us to develop good control on lengths of curves along r in §5.2 and determine the limit set of r in the Thurston compactification of Teichmüller space in §5.3. In §5.4 we prove a technical result required for determining the limit sets of rays in §5.3.

5.1 Infinite geodesic ray

Consider a sequence $\mathcal{E} = \{e_k\}_k \subset \mathbb{N}$ with $e_0 > E$ and $e_{k+1} \geq ae_k$ for some $a > 1$ and all k, to which we will impose further constraints later. We write ϕ_k, $\Phi_k = \phi_1 \cdots \phi_{k-1}\phi_k$, and $\Gamma(\mathcal{E}) = \{\gamma_k\}_{k=0}^{\infty}$ as in (3.1) and (3.2) in §3. Recall from (3.3) that $\gamma_k = \Phi_{k-m}(\alpha)$ for all $k \geq m$. Moreover, recall that the sequence $\{\gamma_k\}_k$ converges to a minimal non-uniquely ergodic lamination ν in $\mathcal{EL}(S)$ by Corollary 3.2.

Let ω denote the Weil–Petersson geodesic segment connecting the point $X_0 \in \mathcal{S}(\rho^{-1}(\alpha))$ to $\rho(X_0) \in \mathcal{S}(\alpha)$ as in §4. Note that $X_0 \in \mathcal{S}(\rho^{-1}(\alpha))$ and so

$$\Phi_k(X_0) \in \mathcal{S}(\Phi_k(\rho^{-1}(\alpha))) = \mathcal{S}(\Phi_{k-1}(\alpha)) = \mathcal{S}(\gamma_{k+m-1}). \tag{5.1}$$

Write δ_k for the geodesic segment connecting midpoints of ω and $\phi_k(\omega)$ (compare with §4 and Lemma 4.12). The endpoint of δ_k on ω will be called its initial endpoint, and the one on $\phi_k(\omega)$ its terminal endpoint. The image of δ_k under any mapping class will have its endpoints labeled as initial and terminal according to those of δ_k.

With this notation, we claim that the terminal endpoint of $\Phi_k(\delta_{k+1})$ is the same as the initial endpoint of $\Phi_{k+1}(\delta_{k+2})$. Indeed, applying Φ_k^{-1} to this pair of arcs, we have δ_{k+1} and $\phi_{k+1}(\delta_{k+2})$. The terminal endpoint of δ_{k+1} is the midpoint of $\phi_{k+1}(\omega)$. This is the ϕ_{k+1}–image of the midpoint of ω, which is also the ϕ_{k+1}–image of the initial endpoint of δ_{k+1}, as claimed.

Concatenating segments of this type defines a half-infinite path:

$$R_{\mathcal{E}} = \delta_1 \cup \Phi_1(\delta_2) \cup \Phi_2(\delta_3) \cup \Phi_3(\delta_4) \cup \cdots . \tag{5.2}$$

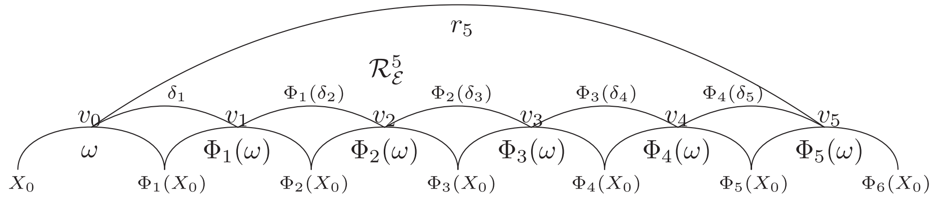

FIG. 3. The concatenation of geodesic segments δ_1, $\Phi_1(\delta_2)$, ..., $\Phi_4(\delta_5)$ defining $R^5_{\mathcal{E}} \subset R_{\mathcal{E}}$ and $\omega \cup \Phi_1(\omega) \cup \ldots \cup \Phi_5(\omega) \subset g^{\omega}_{\mathcal{E}}$, together with the geodesic segment r_5 connecting the endpoints of $R^5_{\mathcal{E}}$.

This path fellow-travels the concatenation of ω and its translates:

$$g^{\omega}_{\mathcal{E}} = \omega \cup \Phi_1(\omega) \cup \Phi_2(\omega) \cup \Phi_3(\omega) \cup \cdots . \tag{5.3}$$

By (5.1), projecting $R_{\mathcal{E}}$ and $g^{\omega}_{\mathcal{E}}$ to the curve complex (via the systole map) gives paths fellow traveling $\{\gamma_k\}_{k=0}^{\infty}$. By Proposition 3.1 and Theorem 2.8, it follows that these are quasi-geodesics in the curve complex. Since the projection to the curve complex is coarsely Lipschitz, so $R_{\mathcal{E}}$ and $g^{\omega}_{\mathcal{E}}$ are also quasi-geodesics.

We will also be interested in a truncation of $R_{\mathcal{E}}$ after k steps:

$$R^k_{\mathcal{E}} = \delta_1 \cup \Phi_1(\delta_2) \cup \cdots \cup \Phi_{k-1}(\delta_k),$$

and let r_k denote the geodesic segment connecting the initial and terminal point of the broken geodesic segment $R^k_{\mathcal{E}}$; see Figure 3.

The angle between consecutive segments $\Phi_{k-1}(\delta_k)$ and $\Phi_k(\delta_{k+1})$ will be denoted θ_k. Applying Φ_{k-1}, this is the same as the angle between δ_k and $\phi_k(\delta_{k+1})$. Observe that the angle θ_k is at least π minus the sum of the angle between δ_k and $\phi_k(\omega)$ and $\phi_k(\delta_{k+1})$ and $\phi_k(\omega)$ (with appropriate directions chosen). Since $\phi_k = D_{\alpha}^{e_{k+m-1}}\rho = f_{e_{k+m-1}}$, by taking e_{k+m-1} and e_{k+m} sufficiently large, appealing to Lemma 4.12 we can ensure that θ_k is as close to π as we like. In particular, we additionally assume that our sequence $\{e_k\}_k$ grows fast enough that

$$\sum_{k=1}^{\infty} \pi - \theta_k < 1. \tag{5.4}$$

We can also (clearly) assume that the integers e_k are all large enough so that $\theta_k \geq \frac{\pi}{2}$ for all k.

Remark 5.1 While we have imposed growth conditions here to control angles, it is worth mentioning that these are in addition to those conditions already imposed to prove non-unique ergodicity.

Proposition 5.2. The geodesic segments r_k limit to a geodesic ray r as $k \to \infty$, and all three of r, $R_{\mathcal{E}}$, and $g^{\omega}_{\mathcal{E}}$ are strongly asymptotic (the distance between any pair of them tends to zero).

Proof. According to the last part of Lemma 4.12, $R_{\mathcal{E}}$ and $g^{\omega}_{\mathcal{E}}$ are strongly asymptotic. Therefore, it suffices to prove that r_k has a limit r, and that this is asymptotic to $R_{\mathcal{E}}$. Before proceeding, we note that $\Phi_k(X_0)$ lies in the stratum $\mathcal{S}(\Phi_k(\rho^{-1}(\alpha))) = \mathcal{S}(\Phi_{k-1}(\alpha)) = \mathcal{S}(\gamma_{k+m})$.

Let $\{v_i\}_{i=0}^{\infty}$ denote the concatenation points of $R_{\mathcal{E}}$. Denote by P_k a ruled polygon bounded by r_k and $R^k_{\mathcal{E}}$. This polygon has vertices $v_0, \ldots, v_k$. Let θ^k_i denote the interior angles of P_k at v_i, for $i = 0, \ldots, k$, and observe that for $0 < i < k$, we have $\theta_i \leq \theta^k_i$. In addition, there are constants $c_0 < 0$ and $d_0 > 0$ so that the d_0–neighborhood of v_i in P_k has Gaussian curvature $K \leq c_0$. Consequently, for any $d < d_0$, if r_k is disjoint from the d–neighborhood $N_d(v_i)$ of v_i, then since $\theta^k_i \geq \frac{\pi}{2}$, $N_d(v_i) \cap P_k$ contains a quarter-sector of a disk of radius d centered at v_i in a surface of curvature at most c_0. Therefore, the integral of the curvature K over $N_d(v_i) \cap P_k$ satisfies

$$\int_{N_d(v_i)\cap P_k} K\, dA \leq \frac{c_0 \pi d^2}{4}.$$

By the Gauss–Bonnet Theorem (see, e.g., [11, Theorem V.2.5]), we have

$$\int_{P_k} KdA + \sum_{i=0}^{k} \left(\pi - \theta^k_i\right) = 2\pi$$

which implies

$$\theta^k_0 + \theta^k_k - \sum_{i=1}^{k-1} \left(\pi - \theta^k_i\right) = \int_{P_k} KdA.$$

For any $d > 0$, let $i_1, i_2, \ldots, i_j$ denote those indices i for which $R^k_{\mathcal{E}}$ is more than d away from v_i. Then by our assumption on the angles θ_i in (5.4) we have

$$\begin{aligned} \theta^k_0 + \theta^k_k - 1 \leq \theta^k_0 + \theta^k_k - \sum_{i=1}^{k-1} \left(\pi - \theta^k_i\right) &= \int_{P_k} KdA \\ &\leq \sum_{\ell=1}^{j} \int_{N_{v_\ell}\cap P_k} KdA \leq \frac{j c_0 \pi d^2}{4}. \end{aligned}$$

Since $c_0 < 0$, this implies

$$j \leq \frac{4\left(\theta_0^k + \theta_k^k - 1\right)}{c_0 \pi d^2} \leq \frac{4|1 - \left(\theta_0^k + \theta_k^k\right)|}{|c_0| \pi d^2} \leq \frac{4(1 + 2\pi)}{|c_0| \pi d^2}.$$

This bounds the number of vertices along $R_{\mathcal{E}}^k$ that can be further than d away from r_k by some number $J(d)$, which is independent of k. Therefore, for any $N > 0$ and k, $k' \geq N + 2J(d) + 1$, there is a vertex v_i of $R_{\mathcal{E}}$ with $N \leq i \leq \min\{k, k'\}$ so that r_k and $r_{k'}$ contain points x_k and $x_{k'}$, respectively, which are within distance d of v_i. Therefore, x_k and $x_{k'}$ are within distance $2d$ of each other. Since $R_{\mathcal{E}}$ is a quasi-geodesic, the distance from v_i to v_0 tends to infinity with i. Consequently, as N tends to infinity, the distance from x_k and $x_{k'}$ to v_0 also tends to infinity. By convexity of the distance function between two geodesic segments in a CAT(0) space, it follows that for any $D > 0$, the initial segments of $\{r_k\}_k$ of length D form a Cauchy sequence in the topology of uniform convergence. By completeness of $\overline{\text{Teich}(S)}$, these segments of length D converge. Letting D tend to infinity, it follows that r_k converges (locally uniformly) to a geodesic ray r.

For any $d > 0$, suppose that v_i is a vertex of $R_{\mathcal{E}}$ further than $2d$ away from any point of r. For k sufficiently large, it follows that r_k is further than d from v_i. Since there are at most $J(d)$ of the latter indices i, it follows that r must come closer than $2d$ from all but $J(d)$ vertices. In particular, there exists $N(d)$ so that for all $i \geq N(d)$, r comes within $2d$ of v_i. By convexity of the distance between geodesics in the WP metric, the distance of any point on $\mathcal{R}_{\mathcal{E}}$ lying between consecutive vertices v_i and v_{i+1} (for $i \geq N(d)$) and r is no more than $2d$. Therefore, the tail of $R_{\mathcal{E}}$ starting at $v_{N(d)}$ is within Hausdorff distance $2d$ from some tail of r. Since d was arbitrary, it follows that $R_{\mathcal{E}}$ and r are strongly asymptotic, as required. ■

In the rest of this section let $r : [0, \infty) \to \text{Teich}(S)$ be the geodesic ray from Proposition 5.2.

5.2 Curves along r

The following lemma is a straightforward consequence of the setup of curves $\{\gamma_k\}_k$ in §3 and the choice of the segment ω in the previous section which we record as a convenient reference.

Lemma 5.3. For any $k \geq m-1$ the initial and terminal endpoints of $\Phi_{k-m+1}(\omega)$ are in the strata $\mathcal{S}(\gamma_k)$ and $\mathcal{S}(\gamma_{k+1})$, respectively. Furthermore, for any compact subsegment $I \subset int(\omega)$, the $2m$ consecutive curves

$$\{\gamma_{k-m+1}, \ldots, \gamma_{k+m}\}$$

have bounded length on $\Phi_{k-m+1}(I)$, with the bound depending on the choice of interval I, but independent of k.

Proof. Recall that $\alpha = \rho^m(\gamma_0) = \rho(\gamma_{m-1})$, and hence $X_0 \in \mathcal{S}(\alpha) = \mathcal{S}(\gamma_{m-1})$. Consequently $\Phi_{k-m+1}(X_0) \in \mathcal{S}(\gamma_k)$, since $\Phi_{k-m+1}(\gamma_{m-1}) = \gamma_k$; see (3.4). Thus the initial endpoint of $\Phi_{k-m+1}(\omega)$ is in $\mathcal{S}(\gamma_k)$. Since the terminal endpoint of $\Phi_{k-m+1}(\omega)$ is the initial endpoint of $\Phi_{k-m+2}(\omega)$, this common endpoint lies in $\mathcal{S}(\gamma_{k+1})$, proving the first statement.

The compact subsegment $I \subset int(\omega)$ is entirely contained in Teichmüller space, and hence the curves $\gamma_0, \ldots, \gamma_{2m-1}$ have bounded length in I. Since the Φ_{k-m+1}–image of these curves are precisely those listed in the lemma, the second statement also follows. ■

Theorem 5.4. There exists a sequence $\{t_k\}_{k=1}^{\infty}$ which is eventually increasing, such that $\lim_{k\to\infty} \ell_{\gamma_k}(r(t_k)) = 0$. Furthermore, for any $\epsilon > 0$ sufficiently small, the set of curves with length less than ϵ along r is contained in $\{\gamma_k\}_{k=0}^{\infty}$ and contains a tail of this sequence, $\{\gamma_k\}_{k\geq N}$, for some $N = N(\epsilon) \in \mathbb{Z}$.

Proof. Since r is strongly asymptotic to $g_{\mathcal{E}}^{\omega}$ by Proposition 5.2 we may choose t_k so that

$$d_{\mathrm{WP}}(r(t_k), \mathcal{S}(\gamma_k)) \to 0$$

as $k \to \infty$. Then by the formula

$$d_{\mathrm{WP}}(r(t_k), \mathcal{S}(\gamma_k)) = \sqrt{2\pi \ell_{\gamma_k}(r(t_k))} + O\Big(\ell_{\gamma_k}(r(t_k))^{5/2}\Big)$$

from Corollary 2.16, where the constant of the O notation depends only on an upper bound for the length of γ_k at the point $r(t_k)$, we see that

$$\lim_{k\to\infty} \ell_{\gamma_k}(r(t_k)) = 0.$$

Since $g_{\mathcal{E}}$ passes through the strata $\{\mathcal{S}(\gamma_k)\}_k$ in order (i.e., $g^{\omega}_{\mathcal{E}}$ intersects $\mathcal{S}(\gamma_k)$ before $\mathcal{S}(\gamma_{k+1})$), the times when r comes close to $\{\mathcal{S}(\gamma_k)\}_k$ also occur in order. This proves the first statement.

For the second statement, we note that the first statement implies that for any $\epsilon > 0$, there exists $N(\epsilon) > 0$ so that for all $k \geq N(\epsilon)$, γ_k has length less than ϵ at some point along r (in fact, at the point $r(t_k)$). Moreover, by Lemma 5.3 $\Phi_{k-m+1}(\omega)$ goes from $\mathcal{S}(\gamma_k)$ to $\mathcal{S}(\gamma_{k+1})$, and no other curves become very short along $\Phi_{k-m+1}(\omega)$. Again appealing to the fact that r is asymptotic to $g^{\omega}_{\mathcal{E}}$, it follows that for k sufficiently large, the only curves of length less than ϵ on $r([t_k, t_{k+1}])$ are γ_k and γ_{k+1}. Therefore, for ϵ sufficiently small, the only curves that can have length less than ϵ along r are from $\{\gamma_k\}_k$. ■

Because r is asymptotic to $g^{\omega}_{\mathcal{E}}$, there is a version of Lemma 5.3 for r. We will consider sequences $\{s_k\}_k \subset [0, \infty)$ satisfying one of the following:

(C1) There exists $\epsilon > 0$ such that $t_k + \epsilon < s_k < t_{k+1} - \epsilon$, or

(C2) $\lim_{k\to\infty} |t_{k+1} - s_k| = 0$.

Corollary 5.5. Suppose that $\{s_k\}_k \subset [0, \infty)$ is a sequence.

- If $\{s_k\}_k$ satisfies (C1), then the $2m$ consecutive curves $\gamma_{k-m+1}, \ldots, \gamma_{k+m}$ have bounded length at $r(s_k)$, independent of k, but depending on ϵ.
- If $\{s_k\}_k$ satisfies (C2), then $\lim_{k\to\infty} \ell_{\gamma_{k+1}}(r(s_k)) = 0$, and the $2m - 1$ consecutive curves $\gamma_{k-m+2}, \ldots, \gamma_{k+m}$ have bounded length at $r(s_k)$, independent of k.

Proof. Suppose that we are in case (C1). Then there exists a compact interval $I \subset int(\omega)$ so that the Hausdorff distance between $\Phi_{k-m+1}(I)$ and $r([t_k + \epsilon, t_{k+1} - \epsilon])$ tends to zero as $k \to \infty$. By Lemma 5.3, $\gamma_{k-m+1}, \ldots, \gamma_{k+m}$ have bounded length along $\Phi_{k-m+1}(I)$. Since $\Phi_{k-m+1}(I)$ remains bounded away from the completion strata of Teich(S), $\gamma_{k-m+1}, \ldots, \gamma_{k+m}$ also have bounded length along $r([t_k + \epsilon, t_{k+1} - \epsilon])$, as required.

For case (C2), the assumptions imply that $d_{WP}(r(s_k), \mathcal{S}(\gamma_{k+1})) \to 0$ as $k \to \infty$, and hence $\ell_{\gamma_{k+1}}(r(s_k)) \to 0$ as $k \to \infty$. The bound on the lengths of $\gamma_{k-m+2}, \ldots, \gamma_{k+m}$ follows from case (C1) and convexity of the length-functions ([36]). Indeed, from case (C1), we know that the curves $\gamma_{k-m+2}, \ldots, \gamma_{k+m}$ have uniformly bounded lengths at $r(\frac{t_k+t_{k+1}}{2})$ and $r(\frac{t_{k+1}+t_{k+2}}{2})$, and hence all the curves have uniformly bounded length along $r([\frac{t_k+t_{k+1}}{2}, \frac{t_{k+1}+t_{k+2}}{2}])$ by convexity of length-functions. ■

As another application of Theorem 5.4, we can identify the ending lamination of r.

Corollary 5.6. The lamination ν is the ending lamination of the ray r.

Proof. By Theorem 5.4, $\lim_{k\to\infty} \ell_{\gamma_k}(r(t_k)) = 0$. Since, by Theorem 2.11, the subsequence

$$\{\gamma_k \mid k \equiv 0 \bmod m\}$$

converges to $\bar{\nu}_0$ in $\mathcal{ML}(S)$ (after appropriately scaling), it follows that the ending lamination of r contains ν. Moreover, $\nu \in \mathcal{EL}(S)$, and hence ν is the ending lamination of r. ∎

5.3 The Limit set

By Corollary 5.6 the ending lamination of r is the minimal non-uniquely ergodic lamination ν. Let $\bar{\nu}^h$, $h = 0, \ldots, m-1$, be the ergodic measures supported on ν as in Theorem 2.11. Theorem 1.2 follows immediately from the following theorem.

Theorem 5.7. The limit set of r in $\mathcal{PML}(S)$ is the concatenation of the edges

$$\big[[\bar{\nu}^0], [\bar{\nu}^1]\big], \ldots, \big[[\bar{\nu}^{m-1}], [\bar{\nu}^0]\big]$$

in the 1 $-$skeleton of the simplex of projective measures supported on ν.

We will reduce this to a more technical statement, and then in the next subsection, we will prove that technical statement. As we will be exclusively interested in lengths of curves along r, for any curve δ and $s \in [0, \infty)$, we write

$$\ell_\delta(s) = \ell_\delta(r(s)).$$

Our main technical result is the following theorem.

Theorem 5.8. Suppose that $\{s_k\}_k \subset [0, \infty)$ is a sequence.

- If $\{s_k\}_k$ satisfies (C1), then there exists $x_k > 0$ such that for any simple closed curve δ, we have

$$\lim_{k\to\infty} \frac{x_k \mathrm{i}(\delta, \gamma_k)}{\ell_\delta(s_k)} = 1.$$

- If $\{s_k\}_k$ satisfies (C2), then there exist $x_k, y_k \geq 0$ with $x_k + y_k > 0$ such that for any simple closed curve δ, we have

$$\lim_{k\to\infty} \frac{x_k \mathrm{i}(\delta, \gamma_k) + y_k \mathrm{i}(\delta, \gamma_{k+1})}{\ell_\delta(s_k)} = 1.$$

Proof of Theorem 5.7 assuming Theorem 5.8. We will pass to subsequences in the following, and to avoid double subscripts, for a subsequence of a sequence $\{c_k\}_{k=1}^{\infty}$, we simply write $\{c_k\}_{k\in J}$, where J is the index set defining the subsequence. Likewise $\lim_{k\in J} c_k$ will denote the limit of the subsequence as the indices from J tend to infinity.

Now suppose that $[\bar{\mu}] \in \mathcal{PML}(S)$ is a limit point of the ray r. That is, for some sequence of times $\{s_j\}_j \subset [0,\infty)$ and any two curves δ, δ', we have

$$\lim_{j\to\infty} \frac{\ell_\delta(s_j)}{\ell_{\delta'}(s_j)} = \frac{\mathrm{i}(\delta, \bar{\mu})}{\mathrm{i}(\delta', \bar{\mu})}$$

(see §2.4). Since s_j must tend to infinity as $j \to \infty$, by passing to a subsequence, we may assume that the sequence is increasing, and there exists an increasing sequence $\{k_j\}_j$ such that either $|s_j - t_{k_j}| \to 0$ or else there exists $\epsilon > 0$ so that $t_{k_j} + \epsilon < s_j < t_{k_j+1} - \epsilon$. Consequently, after reindexing, we assume (as we may) that our sequence is a subsequence $\{s_k\}_{k\in J}$ of some sequence $\{s_k\}_{k=1}^{\infty}$ satisfying either (C1) or (C2).

Suppose first that $\{s_k\}_{k=1}^{\infty}$ satisfies (C1), and pass to a further subsequence (with index set still denoted J for simplicity) so that all $k \in J$ are congruent to some $h \in \{0, \ldots, m-1\}$ mod m. Then by our assumption and Theorem 5.8 we have

$$\frac{\mathrm{i}(\delta, \bar{\mu})}{\mathrm{i}(\delta', \bar{\mu})} = \lim_{k\in J} \frac{\ell_\delta(s_k)}{\ell_{\delta'}(s_k)} = \lim_{k\in J} \frac{x_k \mathrm{i}(\delta, \gamma_k)}{x_k \mathrm{i}(\delta', \gamma_k)} = \frac{\mathrm{i}(\delta, \bar{\nu}^h)}{\mathrm{i}(\delta', \bar{\nu}^h)}$$

where the last equality follows from the fact that $[\gamma_k] \to [\bar{\nu}^h]$ in $\mathcal{PML}(S)$, for $k \in J$ by Theorem 2.11. But this implies that $[\bar{\mu}] = [\bar{\nu}^h]$ since δ, δ' were arbitrary.

We further observe that if $h \in \{0, \ldots, m-1\}$, then $\{s_k := \frac{t_k + t_{k+1}}{2}\}_k$ satisfies (C1), and the computations just given show that for any subsequence $\{s_k\}_{k\in J}$ such that $k \equiv h$ mod m for all $k \in J$, we have $\lim_{k\in J} r(s_k) = [\bar{\nu}^h]$ in the Thurston topology. Consequently, all the vertices of the simplex are in fact accumulation points.

Next, suppose that $\{s_k\}_{k=1}^{\infty}$ satisfies (C2), and again pass to yet another subsequence so that all $k \in J$ are congruent to some $h \in \{0, \ldots, m-1\}$ mod m. In this case, we must pass to yet another subsequence so that $[x_k\gamma_k + y_k\gamma_{k+1}]$ converges to some $[\bar{\mu}'] \in \mathcal{PML}(S)$, for $k \in J$. Note that, since by Theorem 2.11 $[\gamma_k] \to [\bar{\nu}^h]$ and $[\gamma_{k+1}] \to [\bar{\nu}^{h+1}]$

(where we replace $h+1$ by 0, if $h+1=m$), we have $[\bar{\mu}'] \in [[\bar{\nu}^h], [\bar{\nu}^{h+1}]]$. Then, by similar reasoning we have

$$\begin{aligned}\frac{\mathrm{i}(\delta,\bar{\mu})}{\mathrm{i}(\delta',\bar{\mu})} &= \lim_{k\in J}\frac{\ell_\delta(s_k)}{\ell_{\delta'}(s_k)} = \lim_{k\in J}\frac{x_k\mathrm{i}(\delta,\gamma_k)+y_k\mathrm{i}(\delta,\gamma_{k+1})}{x_k\mathrm{i}(\delta',\gamma_k)+y_k\mathrm{i}(\delta',\gamma_{k+1})}\\ &= \lim_{k\in J}\frac{\mathrm{i}(\delta,x_k\gamma_k+y_k\gamma_{k+1})}{\mathrm{i}(\delta',x_k\gamma_k+y_k\gamma_{k+1})} = \frac{\mathrm{i}(\delta,\bar{\mu}')}{\mathrm{i}(\delta',\bar{\mu}')}.\end{aligned}$$

Here the second to the last equality follows from bilinearity of intersection number, while the last equality follows since $[x_k\gamma_k + y_k\gamma_{k+1}] \to [\bar{\mu}']$ in $\mathcal{PML}(S)$, for $k \in J$. Thus again we see that $[\bar{\mu}] = [\bar{\mu}']$.

So, the limit set of r is contained in the required loop in the 1–skeleton of the simplex of projective classes of measures on ν. If we fix $h \in \{0, \ldots, m-1\}$ and consider the arcs

$$\{r([\tfrac{t_k+t_{k+1}}{2}, \tfrac{t_{k+1}+t_{k+2}}{2}]) \mid k \equiv h \bmod m\},$$

it follows that the initial endpoints converge to $[\bar{\nu}^h]$ while the terminal endpoints converge to $[\bar{\nu}^{h+1}]$ (again replacing $h+1$ with 0 if $h+1=m$). Moreover, the accumulation set of this sequence of arcs is a connected subset of $[[\bar{\nu}^h], [\bar{\nu}^{h+1}]]$. Any such set is necessarily the entire 1–simplex. Therefore, the ray r accumulates on the entire loop, as required. ■

5.4 Proof of Theorem 5.8

Here we prove the required technical theorem used in the proof of Theorem 5.7. Throughout what follows, we assume that $\{s_k\}_k$ satisfies (C1) or (C2). Many of the estimates can be carried out for both cases simultaneously.

From Corollary 5.5, the $2m-1$ curves $\gamma_{k-m+2}, \ldots, \gamma_{k+m}$ have bounded lengths in $r(s_k)$, and since

$$\gamma_{k-m+2}, \ldots, \gamma_k, \gamma_{k+2}, \ldots, \gamma_{k+m}$$

fill $S - \gamma_{k+1}$, there is a pants decomposition P_k containing the m–component multicurve

$$\sigma_k := \gamma_k \cup \ldots \cup \gamma_{k+m-1}$$

such that $\ell_\beta(s_k)$ is bounded for all $\beta \in P_k$, independent of k (though the bounds depend on ϵ in case (C1)). Write $P_k^c = P_k \setminus \sigma_k$.

For an arbitrary curve δ and a curve $\beta \in P_k$, the *contribution to the length of* δ *from* β *in* $r(s_k)$ is defined by the equation:

$$\ell_\delta(s_k, \beta) := \mathrm{i}(\delta, \beta)\Big(w_{s_k}(\beta) + \mathrm{tw}_\beta(\delta, s_k)\ell_\beta(s_k)\Big) \tag{5.5}$$

where $\mathrm{tw}_\beta(\delta, s_k)$ is the twist of δ about β at $r(s_k)$ as is defined in (2.3), and $w_{s_k}(\beta)$ is the width of the largest embedded tubular neighborhood of β in $r(t_k)$ (i.e., the minimal distance between boundary components of the neighborhood). By [10, §4.1], we have

$$w_{s_k}(\beta) = 2\log\left(\tfrac{1}{\ell_\beta(s_k)}\right). \tag{5.6}$$

The following estimate for the hyperbolic length of a curve δ from [14, Lemmas 7.2, 7.3] will be our primary tool.

Theorem 5.9. Suppose that the sequence $\{s_k\}_k$ satisfies (C1) or (C2). Then, for any curve δ we have

$$\Big|\ell_\delta(s_k) - \sum_{\beta \in P_k} \ell_\delta(s_k, \beta)\Big| = O\Big(\sum_{\beta \in P_k} \mathrm{i}(\delta, \beta)\Big). \tag{5.7}$$

Here the constant of the O notation depends only on the upper bound for the length of the curves in P_k.

The proof of Theorem 5.8 now follows from estimating various terms in the sum in the above theorem, and finding that one (in case (C1)) or two (in case (C2)) dominate not only the other terms, but also the error term on the right.

Recall that for any simple closed curve δ, Theorem 2.9 implies that for all j sufficiently large we have

$$\mathrm{i}(\delta, \gamma_j) \overset{*}{\asymp} A(0, j) \tag{5.8}$$

where the multiplicative error depends on δ, but not on j. Combining (5.8) and Lemma 2.10, we see that for all $0 \le h \le m-1$, we have

$$\lim_{k\to\infty} \frac{\mathrm{i}(\delta, \gamma_{k+h})}{A(0, k+m)} = 0. \tag{5.9}$$

Observe that the curves γ_{k+h} here are precisely the components of σ_k. It turns out that the intersection numbers with the other curves in P_k (not just those in σ_k)

are also controlled by $A(0, k + m)$. This is essentially the Weil–Petersson analog of [2, Theorem 9.15].

Lemma 5.10. For any $\beta_k \in P_k$ we have

$$\lim_{k\to\infty} \frac{\mathrm{i}(\delta, \beta_k)}{A(0, k+m)} = 0.$$

Proof. By (5.9) it suffices to prove the lemma when $\beta_k \in P_k^c$, for all k.

Let μ be any fixed marking on S and let Y_k be the component of the complement $S \setminus \sigma_k$ that contains β_k.

Claim 5.11. There exists $I > 0$, depending only on μ and δ so that $\mathrm{i}(\pi_Y(\delta), \beta_k) \le I$.

Since δ and μ are a fixed curve and marking, we can assume that their projections to all subsurfaces are uniformly close. Let $Z_k \subseteq Y_k$ be any subsurface with $\beta_k \pitchfork Z_k$, and observe that since Z_k is disjoint from the m consecutive curves in σ_k, Theorem 2.8 implies that it cannot be an annulus with core curve in the sequence $\{\gamma_i\}_i$. By Corollary 5.5, at the point $r(s_k)$ the $2m - 1$ curves $\gamma_{k-m+2}, \ldots, \gamma_{k+m}$ have length bounded independent of k, and hence $\mathrm{i}(\beta_k, \gamma_l)$ is uniformly bounded for each $l = k - m + 2, \ldots, k + m$. Since these curves fill $S \setminus \gamma_{k+1}$ and $\gamma_{k+1} \in \sigma_k$, $\pi_{Z_k}(\gamma_l) \neq \emptyset$ for some $k - m + 2 \le l \le k + m$, and hence $d_{Z_k}(\gamma_l, \beta_k)$ is uniformly bounded. Thus, by the triangle inequality and (2.5) we have

$$d_{Z_k}(\beta_k, \delta) \overset{+}{\asymp} d_{Z_k}(\gamma_l, \mu) \le R(\mu).$$

Since this holds for all subsurfaces $Z_k \subseteq Y_k$, [13, Corollary D] tells us that $\mathrm{i}(\pi_{Y_k}(\delta), \beta_k)$ is uniformly bounded, as required.

Every arc of $\pi_{Y_k}(\delta)$ comes from a pair of intersection points with curves in σ_k. Consequently, taking $\kappa(\delta)$ as the second paragraph of Theorem 2.9 and noting that $A(0, j)$ is increasing in j we have

$$\mathrm{i}(\delta, \beta_k) \le I \sum_{d=k}^{k+m-1} \mathrm{i}(\delta, \gamma_d) \overset{*}{\asymp}_{\kappa(\delta)} I \sum_{d=k}^{k+m-1} A(0, d) \le mIA(0, k+m-1).$$

Thus, setting $K = mI\kappa(\delta)$ the proof of lemma is complete. ■

Next, we estimate the various terms of $\ell_\delta(s_k, \beta)$ for $\beta \in P_k$.

Lemma 5.12. Suppose that $\{s_k\}_k$ is a sequence satisfying either (C1) or (C2). Then for all k sufficiently large and $\beta \in P_k$, we have

$$\mathrm{tw}_\beta(\delta, s_k) \stackrel{+}{\asymp} \mathrm{tw}_\beta(\gamma_0, s_k) \stackrel{+}{\asymp} \begin{cases} e_k & \text{if } \beta = \gamma_k \\ 0 & \beta \neq \gamma_k \text{ or } \gamma_{k+1}. \end{cases}$$

If $\{s_k\}_k$ satisfies (C1), then for all k sufficiently large, $\mathrm{tw}_{\gamma_{k+1}}(\delta, s_k) \stackrel{+}{\asymp} 0$.

Proof. By Theorem 2.8 and Proposition 3.1, $\{\gamma_k\}_k$ is a quasi-geodesic ray in $\mathcal{C}(S)$ (the curve complex of S). Thus, for any fixed curve δ and j sufficiently large, Theorem 2.3 implies that $d_{\gamma_j}(\gamma_0, \delta) \stackrel{+}{\asymp} 0$. To see this, note that for j sufficiently large the curve complex distance between γ_j and every curve on a geodesic connecting γ_0 to δ is at least 3 and hence γ_j intersects all curves on the connecting geodesic. Thus Theorem 2.3 implies a uniform bound on $d_{\gamma_j}(\gamma_0, \delta)$. Since each $\beta \in P_k$ is within distance 1 of γ_k, similarly we have $d_\beta(\gamma_0, \delta) \stackrel{+}{\asymp} 0$ for all $\beta \in P_k$, once k is sufficiently large.

Suppose that $\{s_k\}_k$ satisfies (C1). The filling set of curves $\gamma_{k-m+1}, \ldots, \gamma_{k+m}$ have bounded length in $r(s_k)$. So, for all k sufficiently large and $\beta \in P_k$

$$\mathrm{tw}_\beta(\delta, s_k) \stackrel{+}{\asymp} \mathrm{tw}_\beta(\gamma_0, s_k) \stackrel{+}{\asymp} \mathrm{diam}_{\mathcal{C}(\beta)}(\gamma_0 \cup \gamma_{k-m+1} \cup \cdots \cup \gamma_{k+m}).$$

If $\beta \in P_k^c$, then $\beta \notin \{\gamma_j\}_j$, and so the term on the right is uniformly close to 0 by Theorem 2.8. If $\beta = \gamma_j \neq \gamma_k$, then $j > k$, and the only curves in the set $\gamma_{k-m+1}, \ldots,$ γ_{k+m} which actually intersect γ_j nontrivially must have index *less* than j. In this case, Theorem 2.8 implies that the term on the right is also uniformly close to 0. When $\beta = \gamma_k$, again appealing to Theorem 2.8, the right-hand side is estimated (up to a bounded additive error) by

$$d_{\gamma_k}(\gamma_0, \gamma_{k+m}) \stackrel{+}{\asymp} e_k.$$

This proves the lemma when $\{s_k\}_k$ satisfies (C1). The proof when $\{s_k\}_k$ satisfies (C2) is nearly identical since the curves $\gamma_{k-m+2}, \ldots, \gamma_k, \gamma_{k+2}, \ldots, \gamma_{k+m}$ have bounded length and fill $S \setminus \gamma_{k+1}$, so the only curve whose twisting we can no longer estimate is γ_{k+1}. Since the conclusion of the lemma is silent regarding the twisting about this curve in case (C2), we are done. ∎

Proof of Theorem 5.8. In either case that $\{s_k\}_k$ satisfies (C1) or (C2), define $x_k = w_{s_k}(\gamma_k) + \mathrm{tw}_{\gamma_k}(\gamma_0, s_k)\ell_{\gamma_k}(s_k)$. Observe that $\ell_{\gamma_k}(s_k) \stackrel{*}{\asymp} 1$, so by (5.6) $w_{s_k}(\gamma_k) \stackrel{+}{\asymp} 1$ (these estimates depend on ϵ in case (C1), but not k). Moreover, from Lemma 5.12, for any curve δ and k sufficiently large we have

$$\mathrm{tw}_{\gamma_k}(\gamma_0, s_k) \stackrel{+}{\asymp} \mathrm{tw}_{\gamma_k}(\delta, s_k) \stackrel{+}{\asymp} e_k \to \infty$$

as $k \to \infty$. Consequently, we have $x_k \stackrel{*}{\asymp} e_k$ and

$$\frac{\ell_\delta(s_k, \gamma_k)}{x_k \mathrm{i}(\delta, \gamma_k)} = \frac{w_{s_k}(\gamma_k) + \mathrm{tw}_{\gamma_k}(\delta, s_k)\ell_{\gamma_k}(s_k)}{w_{s_k}(\gamma_k) + \mathrm{tw}_{\gamma_k}(\gamma_0, s_k)\ell_{\gamma_k}(s_k)} \to 1 \tag{5.10}$$

as $k \to \infty$. Combining this with (5.8) and using the setup of integers $A(0, k)$, for large k, we have

$$\ell_\delta(s_k, \gamma_k) \stackrel{*}{\asymp} x_k \mathrm{i}(\delta, \gamma_k) \stackrel{*}{\asymp} e_k \mathrm{i}(\delta, \gamma_k) \stackrel{*}{\asymp} A(0, k+m) \tag{5.11}$$

Now suppose that we are in case (C1) and $\beta_k \in P_k$, but $\beta_k \neq \gamma_k$. As for γ_k above, we have $\ell_{\beta_k}(s_k) \stackrel{*}{\asymp} 1 \stackrel{+}{\asymp} w_{s_k}(\beta_k)$ (with errors depending on ϵ, but not k). Combining this with (5.8) and Lemma 5.12, we have

$$\ell_\delta(s_k, \beta_k) \stackrel{*}{\asymp} \mathrm{i}(\delta, \beta_k).$$

Therefore, by (5.11) and Lemma 5.10, we have

$$\frac{\ell_\delta(s_k, \beta_k)}{x_k i(\delta, \gamma_k)} \stackrel{*}{\asymp} \frac{i(\delta, \beta_k)}{A(0, k+m)} \to 0 \tag{5.12}$$

as $k \to \infty$.

Combining Theorem 5.9 with Lemma 5.10, (5.10), and (5.12), for any curve δ we have

$$\lim_{k\to\infty} \frac{\ell_\delta(s_k)}{x_k \mathrm{i}(\delta,\gamma_k)} = \lim_{k\to\infty} \frac{\ell_\delta(s_k,\gamma_k)}{x_k \mathrm{i}(\delta,\gamma_k)} + \frac{1}{x_k \mathrm{i}(\delta,\gamma_k)} \Bigg(\sum_{\substack{\beta_k \in P_k \text{ and} \\ \beta_k \neq \gamma_k}} \ell_\delta(s_k, \beta_k) + O\Big(\sum_{\beta_k \in P_k} \mathrm{i}(\delta, \beta_k) \Big) \Bigg) = 1,$$

as required.

When $\{s_k\}_k$ satisfies (C2), x_k is defined as above, and we define

$$y_k = w_{s_k}(\gamma_{k+1}) + \mathrm{tw}_{\gamma_{k+1}}(\gamma_0, s_k)\ell_{\gamma_{k+1}}(s_k).$$

According to Corollary 5.5, $\ell_{\gamma_{k+1}}(s_k) \to 0$ as $k \to \infty$, and so by (5.6) we have

$$w_{s_k}(\gamma_{k+1}) \to \infty$$

as $k \to \infty$. Moreover, Lemma 5.12 ensures that for any curve δ and k sufficiently large, we have

$$\mathrm{tw}_{\gamma_{k+1}}(\gamma_0, s_k) \stackrel{+}{\asymp} \mathrm{tw}_{\gamma_{k+1}}(\delta, s_k).$$

Therefore,

$$\lim_{k\to\infty} \frac{\ell_\delta(s_k, \gamma_{k+1})}{y_k \mathrm{i}(\delta, \gamma_{k+1})} = 1,$$

and combining this with (5.10) we have

$$\lim_{k\to\infty} \frac{\ell_\delta(s_k, \gamma_k) + \ell_\delta(s_k, \gamma_{k+1})}{x_k \mathrm{i}(\delta, \gamma_k) + y_k \mathrm{i}(\delta, \gamma_{k+1})} = 1.$$

Because the estimate (5.12) still holds for any curve $\beta_k \in P_k$ where $\beta_k \neq \gamma_k$ or γ_{k+1}, we can again apply Theorem 5.9 and Lemma 5.10 to deduce that for any curve δ we have

$$\lim_{k\to\infty} \frac{\ell_\delta(s_k)}{x_k \mathrm{i}(\delta, \gamma_k) + y_k \mathrm{i}(\delta, \gamma_{k+1})} = \lim_{k\to\infty} \frac{\ell_\delta(s_k, \gamma_k) + \ell_\delta(s_k, \gamma_{k+1})}{x_k \mathrm{i}(\delta, \gamma_k) + y_k \mathrm{i}(\delta, \gamma_{k+1})} = 1,$$

completing the proof in case (C2), and hence in general. ■

6 The uniquely ergodic case

Let $r : [0, \infty) \to \mathrm{Teich}(S)$ be a WP geodesic ray, and denote the ending lamination of r by ν; see §2.7. The following immediately implies Theorem 1.1 from the introduction.

Theorem 6.1. Suppose that ν is uniquely ergodic, then the limit set of r in $\mathcal{PML}(S)$ (Thurston boundary) is the point $[\bar{\nu}]$.

Proof. The proof of the theorem closely follows Masur's proof of the analogous fact for Teichmüller geodesics [26, Theorem 1]. Assuming $[\bar{\xi}]$ is any accumulation point of r, let $\{t_i\}_i$ be a sequence of times so that $r(t_i) \to [\bar{\xi}]$ as $i \to \infty$ in the Thurston compactification of Teichmüller space $\mathrm{Teich}(S) \cup \mathcal{PML}(S)$; see §2.4. According to the *Fundamental Lemma* of [16, exposé 8], there exists a sequence $\{\bar{\mu}_i\}_i \subset \mathcal{ML}(S)$, such that

$$\mathrm{i}(\bar{\mu}_i, \delta) \leq \ell_\delta(r(t_i)),$$

for all simple closed curves δ, as well as a sequence of positive real numbers $\{b_i\}_i$, so that $b_i\bar{\mu}_i \to \bar{\xi} \in \mathcal{ML}(S)$ and $b_i \to 0$, as $i \to \infty$.

Let $\bar{\nu}$ be a transverse measure on ν. Since ν is uniquely ergodic, and hence minimal, there exists a sequence of Bers curves $\{\gamma_i\}_i$ and positive real numbers $\{c_i\}_i$, so that $c_i\gamma_i \to \bar{\nu} \in \mathcal{ML}(S)$ and $c_i \to 0$, as $i \to \infty$. Since the γ_i are Bers curves, by the inequality above there exists $C > 0$ so that

$$\mathrm{i}(\bar{\mu}_i, \gamma_i) \le \ell_{\gamma_i}(r(t_i)) \le C.$$

Consequently, by continuity of the intersection form, we have

$$\mathrm{i}(\bar{\xi}, \bar{\nu}) = \lim_{i\to\infty} \mathrm{i}(b_i\bar{\mu}_i, c_i\gamma_i) = \lim_{i\to\infty} b_ic_i\mathrm{i}(\bar{\mu}_i, \gamma_i) \le C \lim_{i\to\infty} b_ic_i = 0,$$

and hence $\mathrm{i}(\bar{\xi}, \bar{\nu}) = 0$. Because ν is uniquely ergodic, [26, Lemma 2] implies $\bar{\xi}$ is a multiple of $\bar{\nu}$, and therefore $[\bar{\xi}] = [\bar{\nu}]$. Since $[\bar{\xi}]$ was an arbitrary accumulation point of r in $\mathcal{PML}(S)$, the proof is complete. ■

7 Appendix

In this appendix we provide the proofs of the results of §2.5 about sequences of curves. As we mentioned there, many of the proofs closely follow the ones in [2], while others have been streamlined since the writing of that paper. Here we mainly outline the proofs that are similar, incorporating the required changes, and otherwise provide the streamlined proofs.

7.1 Subsurface coefficient estimates

In the next Lemma, B_0 is the constant from Theorem 2.4.

Lemma 7.1. (Local to Global) Fix any $B \ge B_0 + 1$, and let $\{\delta_k\}_{k=0}^{\omega}$ ($\omega \in \mathbb{Z}^{\ge 0} \cup \{\infty\}$) be a (finite or infinite) sequence of curves in $\mathcal{C}(S)$, with the property that $\delta_{k-1} \pitchfork \delta_k, \delta_{k+1} \pitchfork \delta_k$ and that $d_{\delta_k}(\delta_{k-1}, \delta_{k+1}) \ge 3B$ for all $k \ge 1$. Then for all $0 \le i < k < j$, we have that $\delta_i \pitchfork \delta_k, \delta_j \pitchfork \delta_k$ and that

$$|d_{\delta_k}(\delta_i, \delta_j) - d_{\delta_k}(\delta_{k-1,}, \delta_{k+1})| \le 2B_0. \tag{7.1}$$

Proof. To simplify the notation, write $d_k(i,j) = d_{\delta_k}(\delta_i, \delta_j)$. The proof is by induction on $n = j - i$. The base case is $n = 2$, in which case $i = k - 1$, $j = k + 1$, and the conclusions of the lemma hold trivially.

We suppose that $\gamma_i \pitchfork \gamma_k$ and that (7.1) holds for all i, j with $i < k < j$ and $j - i \leq n$, and prove them for $n + 1$. To that end, suppose that $0 \leq i < k < j$ are such that $j - i = n + 1$. We claim that $d_k(i, k - 1) \leq B_0$. To see this, note that if $i = k - 1$, then the claim holds obviously. Otherwise, $i < k - 1 < k$ and $k - i \leq n$, so by hypothesis of the induction $\gamma_i \pitchfork \gamma_k$ and

$$d_{k-1}(i,k) \geq e_k \geq 3B - 2B_0 > B_0,$$

note that $\delta_{k-1} \pitchfork \delta_k$ by assumptions of the lemma. Then Theorem 1.6 implies that $d_k(i, k - 1) \leq B_0$. By a similar reasoning, we have that $d_k(k + 1, j) \leq B_0$, and so by the triangle inequality

$$|d_k(i,j) - d_k(k-1,k+1)| \leq d_k(i,k-1) + d_k(j,k+1) \leq 2B_0,$$

which is (7.1). Moreover, since $d_k(k, k + 1) \geq 3B_0$ for the above inequality we have that $d_k(i, j) \geq B_0 + 1 > 1$ which implies that $\gamma_i \pitchfork \gamma_j$, finishing the proof of lemma by induction. ■

Set $B = \max\{3, B_0 + 1, G_0\}$, where G_0 is the constant from Theorem 2.3 for a geodesic in $\mathcal{C}(S)$. Set $E_0 = 3B + 4$, and for the remainder of this subsection assume that the sequence $\Gamma(\mathcal{E}) = \{\gamma_k\}_{k=0}^{\infty}$ satisfies $\mathcal{P}(\mathcal{E})$ from Definition 2.6, where $\mathcal{E} = \{e_k\}_{k=0}^{\infty}$, $e_k \geq ae_{k-1}$ for some $a \geq 1$ and all k, and $e_0 \geq E_0$ (and hence $e_k \geq E_0$ for all k). Also throughout this subsection let $\mathbb{M}$ be the monoid generated by $\{m, m + 1\}$. A simple arithmetic computation shows that any integer which is greater than or equal to $m^2 - 1$ is in $\mathbb{M}$.

Lemma 7.2. For all $i < k < j$ such that $k - i, j - k \in \mathbb{M}$, for example, if $k - i, j - k \geq m^2 - 1$, we have that $\gamma_i \pitchfork \gamma_k, \gamma_j \pitchfork \gamma_k$ and that

$$|d_{\gamma_k}(\gamma_i, \gamma_j) - e_k| \leq 2B_0 + 4.$$

Proof. As in the previous proof, we write $d_k(i,j) = d_{\gamma_k}(\gamma_i, \gamma_j)$, and also write $\mathrm{i}(i, j) = \mathrm{i}(\gamma_i, \gamma_j)$ and $\pi_i(j) = \pi_{\gamma_i}(\gamma_j)$.

We make a few observations from Definition 2.6. First, $\mathrm{i}(j, j + 1) = 0$ for all j, and hence if $\pi_k(j), \pi_k(j + 1) \neq \emptyset$ for some k, then $d_k(j, j + 1) = 1$. Second, for all k, $\mathrm{i}(k, k + m), \mathrm{i}(k, k + m + 1) \neq 0$. Consequently, $\pi_k(j) \neq \emptyset$ for all j, k with $|j - k| \in \{m, m$

+ 1}. Finally, observe that $|d_k(k-m, k+m) - e_k| \leq 2$. Thus, if $i < k < j$ and $k-i, j-k \in \{m, m+1\}$, by the triangle inequality

$$|d_k(i,j) - e_k| \leq 4. \tag{7.2}$$

Now, for any sequence of integers $\{k_j\}_j$ such that $k_{j+1} - k_j \in \{m, m+1\}$ for all j, the sequence $\{\gamma_{k_j}\}_j$ has the property that $\gamma_{k_{j-1}}, \gamma_{k_{j+1}} \pitchfork \gamma_{k_j}$ and that $d_{k_j}(k_{j-1}, k_{j+1}) \geq 3B$ by (7.2) and since $e_k \geq E_0$. Hence the sequence $\{\gamma_{k_j}\}_j$ satisfies the assumptions of Lemma 7.1. Then by the lemma, for any ℓ with $i < \ell < j$, $\gamma_\ell \pitchfork \gamma_i, \gamma_\ell \pitchfork \gamma_j$ and

$$|d_{k_\ell}(k_i, k_j) - e_{k_\ell}| \leq 2B_0 + 4.$$

Therefore, if $i < k < j$ and $k-i, j-k \in \mathbb{M}$, then

$$|d_k(i,j) - e_k| \leq 2B_0 + 4.$$

This completes the proof of the inequality in the statement of the lemma. ■

Lemma 7.3. The map $k \mapsto \gamma_k$ is a 1–Lipschitz, (K, C)–quasi-geodesic, where $K = C = 2m^2 + 2m - 1$.

Proof. First, suppose that $i < j$ with $j - i \geq 2m^2 + 2m - 1$. Then by Lemma 7.2 for each $k \in \{i + m^2, i + m^2 + 1, \ldots, j - m^2\}$, we have that $\gamma_k \pitchfork \gamma_i, \gamma_k \pitchfork \gamma_j$ and that

$$d_{\gamma_k}(\gamma_i, \gamma_j) \geq e_k - 2B_0 - 4 \geq B \geq 3. \tag{7.3}$$

Thus the curves γ_i, γ_j fill the annulus with core curve γ_k. This implies that any curve that intersects γ_k must intersect one of γ_i or γ_j. Moreover, the $2m$ curves γ_k for $k = i + m^2, \ldots, i + m^2 + 2m - 1$ fill S, so γ_i, γ_j also fill S.

Next, suppose that $j > i + 2m^2 + 2m - 1$ and write $j = i + q(2m^2 + 2m - 1) + r$, where q, r are non-negative integers with $0 \leq r < 2m^2 + 2m - 1$. Set the curve $\delta_k = \gamma_{i+k(2m^2+2m-1)}$, for $k = 1, \ldots, q-1$. Then the curves

$$\gamma_i, \delta_1, \ldots, \delta_{q-1}, \gamma_j$$

form a sequence in $\mathcal{C}(S)$. As we saw above any two distinct curves in the sequence fill S, and by (7.3) for all $0 < k < q$ we have

$$d_{\delta_k}(\gamma_i, \gamma_j) \geq B > G_0.$$

So, by Theorem 2.3, a geodesic from γ_i to γ_j must have a vertex disjoint from δ_k, for all $k = 0, \ldots, q$. Since any two curves δ_k, $\delta_{k'}$ in the sequence fill S, no curve can be disjoint from more than one of them, and hence the geodesic must contain at least $q+1$ vertices, so

$$d(\gamma_i, \gamma_j) \geq q = \frac{j-i-r}{2m^2+2m-1} \geq \frac{1}{K}(j-i-C)$$

where $K = 2m^2 + 2m - 1$ and $C = 2m^2 + 2m - 1$. Since this inequality trivially holds if $j - i < C$ and $i < j$, the required lower bound follows. Moreover, since γ_k, γ_{k+1} are disjoint, the map is 1–Lipschitz. Finally the upper bound for the (K, C)-quasi-geodesic is immediate from the fact that the map is 1-Lipschitz. ■

For each $k \geq 0$, let $\mu_k := \{\gamma_k, \ldots, \gamma_{k+2m-1}\}$.

Lemma 7.4. There exists $M > 0$ such that for any subsurface $W \subsetneq S$ which is neither S nor an annulus with core curve some γ_k, we have

$$d_W(\mu_i, \mu_j) \leq M$$

for all i, j.

Proof. First, let $\mu_k{}' = \{\gamma_k, \ldots, \gamma_{k+2m-2}, \gamma'_{k+2m-1}\}$ where γ'_{k+2m-1} is as in Definition 2.6. From the definition, any curve in μ_k and curve in μ'_{k+1} have uniformly bounded intersection number (bounded by $m^2 b_2$). Consequently, there exists $M_0 > 0$ such that for any subsurface W and any k, we have

$$d_W\left(\mu_k, \mu'_{k+1}\right) \leq M_0.$$

Next, observe that μ'_{k+1} and μ_{k+1} differ by Dehn twisting γ'_{k+2m} about γ_{k+m} (which has zero intersection number with all curves in μ'_{k+1} except γ'_{k+2m}). Therefore, there exists another constant $M_1 > 0$ so that as long as W is not the annulus with core γ_{k+m}, we have

$$d_W(\mu'_{k+1}, \mu_{k+1}) \leq M_1.$$

Indeed, for any such W, $\pi_W(\mu'_{k+1}) \cap \pi_W(\mu_{k+1}) \neq \emptyset$, and so we may take M_1 to be at most the sum of the diameters of $\pi_W({\mu'}_{k+1})$ and $\pi_W({\mu'}_{k+1})$ which is at most 4.

From these two inequalities, we see that for any subsurface W which is not the annulus with core γ_{k+m}, the triangle inequality implies

$$d_W(\mu_k, \mu_{k+1}) \leq M_0 + M_1.$$

From this it follows that for any $D > 0$ and $|j - i| \leq D$,

$$d_W(\gamma_i, \gamma_j) \leq D(M_0 + M_1) \tag{7.4}$$

whenever W is not an annulus with core curve γ_k, for some k.

Finally, suppose that W is any subsurface which is not S and not an annulus with core curve γ_k, for some k. By Lemma 7.3, $k \mapsto \gamma_k$ is a quasi-geodesic in $\mathcal{C}(S)$. So there is a uniform bound for the number its vertices that are within distance 1 of ∂W. Consequently, there exists $D_0 > 0$ (independent of W) and i_0 so that if $k \notin [i_0, i_0 + D_0]$, then $\pi_W(\gamma_k) \neq \emptyset$. By Theorem 2.3, there exists $G = G(K, C)$ so that the projections of both sequences $\{\gamma_k\}_{k=0}^{i_0}$ and $\{\gamma_k\}_{k=i_0+D_0}^{\infty}$ to W have diameter at most G. Combining this with (7.4) and setting $M = 2G + D_0(M_0 + M_1)$, we have

$$\operatorname{diam}_{\mathcal{C}(W)}\left(\{\gamma_k\}_{k=0}^{\infty}\right) \leq 2G + D_0(M_0 + M_1) = M.$$

Since M is independent of the subsurface W, this completes the proof. ■

Proof of Theorem 2.8. The fact that $\{\gamma_k\}_k$ is a 1–Lipschitz (K, C)–quasi-geodesic is Lemma 7.3. Klarreich's work [19, Theorem 4.1]) describing the Gromov boundary of the curve complex then implies that there exists a $\nu \in \mathcal{EL}(S)$, so that any accumulation point of $\{\gamma_k\}_k$ in $\mathcal{PML}(S)$ is supported on ν.

Any accumulation point of $\{\gamma_k\}_k$ in the Hausdorff topology of closed subset of S contains ν, and hence for a subsurface $W \subseteq S$, $\pi_W(\nu) \subseteq \pi_W(\gamma_k)$ for all k sufficiently large. Consequently the equations on the left of (2.4) and (2.5) follow from Lemmas 7.2 and 7.4, respectively, setting $R = \max\{M, 2B_0 + 4\}$. For any marking μ, the pairwise intersection between curves in μ and in μ_0 are bounded by some finite number, and hence $d_W(\mu, \mu_0)$ is uniformly bounded by some constant $D > 0$, independent of W. Setting $R(\mu) = R + D$, the equations on the right-hand side of (2.4) and (2.5) then follow from those on the left-hand side, together with the triangle inequality. ■

7.2 Intersection number estimates

We now assume that $\mathcal{E} = \{e_k\}_k$ grows exponentially, with $e_k \geq ae_{k-1}$ for some $a > 1$ and all $k \geq 1$. The aim is to estimate intersection numbers in terms of the numbers $A(i, k)$ defined in (2.6). We begin with the upper bound.

Lemma 7.5. If $\Gamma(\mathcal{E}) = \{\gamma_k\}_k$ satisfies $\mathcal{P}(\mathcal{E})$ with $e_k \geq ae_{k-1}$ for some $a > 1$, then there exists $\kappa > 0$ such that

$$\mathrm{i}(\gamma_i, \gamma_k) \leq \kappa A(i, k).$$

Moreover, we may take κ to be decreasing as a function of a.

Sketch of proof. The proof is a rather complicated induction, but is essentially identical to the proof of Proposition 5.5 from [2]. We sketch the proof for completeness.

We first recall from [16, exposé 4], that for any simple closed curves β, δ, δ', and integer e, we have

$$\left|\mathrm{i}(D_\beta^e(\delta'), \delta) - |e|\mathrm{i}(\delta', \beta)\mathrm{i}(\delta, \beta)\right| \leq \mathrm{i}(\delta, \delta'). \tag{7.5}$$

Since $\gamma_{k+m} = D_{\gamma_k}^{e_k}(\gamma'_{k+m})$ and $\mathrm{i}(\gamma'_{k+m}, \gamma_k) = b$, we can apply this to estimate $\mathrm{i}(\gamma_i, \gamma_{k+m})$ to obtain

$$\left|\mathrm{i}(\gamma_i, \gamma_{k+m}) - be_k\mathrm{i}(\gamma_i, \gamma_k)\right| \leq \mathrm{i}(\gamma_i, \gamma'_{k+m}).$$

The right-hand side can be bounded as follows. Since the curves $\gamma_{k-m}, \ldots, \gamma_{k+m-1}$ fill S, these cut any simple closed curve δ into N arcs, where

$$N = \sum_{j=k-m}^{k+m-1} \mathrm{i}(\delta, \gamma_j).$$

Now apply this cutting procedure to both γ'_{k+m} and γ_i. Any pair of resulting arcs (one from γ'_{k+m} and one from γ_j) are either disjoint, intersect at most once if they lie in a complementary disk, or intersect at most twice if they lie in a once-punctured complementary disk. Thus

$$\mathrm{i}\left(\gamma_i, \gamma'_{k+m}\right) \leq 2 \sum_{j=k-m}^{k+m-1} \mathrm{i}(\gamma_i, \gamma_j) \sum_{j=k-m}^{k+m-1} \mathrm{i}\left(\gamma'_{k+m}, \gamma_j\right).$$

Moreover, the assumption on the values of intersection numbers of γ'_{k+m} and the curves $\gamma_j, j = k - m, \ldots, k + m - 1$ from Definition 2.6 implies that

$$\sum_{j=k-m}^{k+m-1} \mathrm{i}\left(\gamma_j, \gamma'_{k+m}\right) \leq (m+1)b'.$$

Consequently, setting $B = 2(m + 1)b'$, we have

$$\left|\mathrm{i}(\gamma_i, \gamma_{k+m}) - be_k \mathrm{i}(\gamma_i, \gamma_k)\right| \leq B \sum_{j=k-m}^{k+m-1} \mathrm{i}(\gamma_i, \gamma_j), \tag{7.6}$$

and hence

$$\mathrm{i}(\gamma_i, \gamma_{k+m}) \leq be_k \mathrm{i}(\gamma_i, \gamma_k) + B \sum_{j=k-m}^{k+m-1} \mathrm{i}(\gamma_i, \gamma_j).$$

The goal is to show that $\mathrm{i}(\gamma_i, \gamma_{k+m}) \leq \kappa A(i, k + m)$ for some $\kappa > 0$. The proof is by induction on $(k + m) - i$, and the constant κ is actually a limit of an increasing sequence of constants $K(1) < K(2) < K(3) < \ldots$. To see what these constants should be, we assume by induction that $\mathrm{i}(\gamma_i, \gamma_j) \leq K(j - i)A(i, j)$ for all $i < j$ with $j - i < k + m$, then dividing both sides of the inequality above by $A(i, k + m)$, we have

$$\begin{aligned}
\frac{\mathrm{i}(\gamma_i, \gamma_{k+m})}{A(i, k+m)} &\leq \frac{be_k \mathrm{i}(\gamma_i, \gamma_k)}{A(i, k+m)} + B \sum_{j=k-m}^{k+m-1} \frac{\mathrm{i}(\gamma_i, \gamma_j)}{A(i, k+m)} \\
&\leq K(k-i) \frac{be_k A(i, k)}{A(i, k+m)} + B \sum_{j=k-m}^{k+m-1} K(j-i) \frac{A(i, j)}{A(i, k+m)} \\
&\leq K(k+m-1-i)\left(1 + B \sum_{j=k-m}^{k+m-1} a^{-\lfloor \frac{k-i}{m} \rfloor}\right) \\
&= K((k+m-i)-1)(1 + 2mBa^{-\lfloor \frac{k-i}{m} \rfloor})
\end{aligned}$$

The right-hand side above suggests the recursive/inductive definition

$$K(k+m-i) := K((k+m-i)-1)(1 + 2mBa^{-\lfloor \frac{k-i}{m} \rfloor}).$$

(Note that the right-hand side depends only on the difference $k - i$ and not on i and k independently). For $a > 1$, one shows (using a comparison with a geometric series, after taking logarithms) that $K(j)$ so defined is bounded, and since it is increasing, it converges to a constant $\kappa > 0$. Since $K(j) < \kappa$ for all j, the inequality above proves the lemma. See [2, Proposition 5.5] for the details. ■

Since we are assuming fewer non-zero intersection numbers in the current work than in [2], the lower bounds we obtain are weaker, and the proof is slightly more complicated than the one in §5 of [2]. Fortunately, the weaker estimates suffice for our purposes.

Lemma 7.6. If $e_k \geq a e_{k-1}$ for $a > 1$ sufficiently large and all $k \geq 0$, then there exists $\kappa' > 0$ such that

$$\mathrm{i}(\gamma_i, \gamma_k) \geq \kappa' A(i,k)$$

whenever

1. $0 \leq i \leq 2m - 1$ and $k \geq i + m^2 + m - 1$, or
2. $k - i \geq 2m$ and $i \equiv k \bmod m$.

Proof. Fix some constant $a_0 > 1$ and assume to begin that $e_k \geq a_0 e_{k-1}$, and let $\kappa > 0$ be the constant from Lemma 7.5. We will eventually take a larger $a > a_0$, but we observe that upper bound on intersection numbers from Lemma 7.5 remains valid with this choice of κ.

Now, recalling that $A(i, k+m) = b e_k A(i, k)$, dividing (7.6) by $A(i, k+m)$, we have

$$\frac{\mathrm{i}(\gamma_i, \gamma_{k+m})}{A(i,k+m)} \geq \frac{\mathrm{i}(\gamma_i, \gamma_k)}{A(i,k)} - B \sum_{j=k-m}^{k+m-1} \frac{\mathrm{i}(\gamma_i, \gamma_j)}{A(i,k+m)}.$$

Combining this with Lemmas 2.10 and 7.5 we have

$$\begin{aligned} \frac{\mathrm{i}(\gamma_i, \gamma_{k+m})}{A(i,k+m)} &\geq \frac{\mathrm{i}(\gamma_i, \gamma_k)}{A(i,k)} - \kappa B \sum_{j=k-m}^{k+m-1} \frac{A(i,j)}{A(i,k+m)} \\ &\geq \frac{\mathrm{i}(\gamma_i, \gamma_k)}{A(i,k)} - 2m\kappa B a^{-\lfloor \frac{k-i}{m} \rfloor} \end{aligned}$$

Recursively substituting, we see that for any $n \geq 0$ such that $i < k - nm$,

$$\frac{\mathrm{i}(\gamma_i, \gamma_{k+m})}{A(i,k+m)} \geq \frac{\mathrm{i}(\gamma_i, \gamma_{k-nm})}{A(i,k-nm)} - 2m\kappa B \sum_{s=0}^{n} a^{-\lfloor \frac{k-i}{m} \rfloor + s} \tag{7.7}$$

In this situation, taking $a > a_0 > 1$ sufficiently large, we can make the term

$$2m\kappa B \sum_{s=0}^{n} a^{-\lfloor \frac{k-i}{m} \rfloor + s}$$

as small as we like, independent of n (since this is a partial sum of a geometric series with common ratio a). Specifically, we choose $a > a_0 > 1$ large enough so that for all $k - i \geq m$ and $n \geq 0$, the sum is bounded above by κ', where

$$\kappa' = \frac{1}{2} \min \left\{ \frac{1}{A(i,j)} \mid 0 \leq i \leq 2m-1 \text{ and } i < j \leq m^2 + 3m - 2 \right\}.$$

Now, suppose that $0 \leq i \leq 2m - 1$ and that $k \geq i + m^2 + m - 1$, and write $k = j + m$ (so $j \geq i + m^2 - 1$). Let $n \geq 0$ be such that

$$i + m^2 - 1 \leq j - nm \leq m^2 + 3m - 2,$$

which is possible since $i + m^2 - 1 \leq m^2 + 2m - 2$. By Lemma 7.2, $\gamma_i \pitchfork \gamma_{j-nm}$, so $\mathrm{i}(\gamma_i, \gamma_{j-nm}) \geq 1$, and therefore

$$\frac{\mathrm{i}(\gamma_i, \gamma_{j+m})}{A(i, j+m)} \geq \frac{\mathrm{i}(\gamma_i, \gamma_{j-nm})}{A(i, j-nm)} - 2m\kappa B \sum_{s=0}^{n} a^{-\lfloor \frac{j-i}{m} \rfloor + s} \geq 2\kappa' - \kappa' = \kappa'.$$

That is, $\mathrm{i}(\gamma_i, \gamma_k) \geq \kappa' A(i, k)$, proving part (i).

Next, let k, i be any two positive integers such that $k - i \geq 2m$ and $i \equiv k \bmod m$. Write $k = j + m$ and let $n \geq 0$ be such that $j - nm = i + m$. Then by (7.7) we have

$$\begin{aligned} \frac{\mathrm{i}(\gamma_i, \gamma_{j+m})}{A(i, j+m)} &\geq \frac{\mathrm{i}(\gamma_i, \gamma_{j-nm})}{A(i, j-nm)} - 2m\kappa B \sum_{s=0}^{n} a^{-\lfloor \frac{j-i}{m} \rfloor + s} \\ &= \frac{\mathrm{i}(\gamma_i, \gamma_{i+m})}{1} - \kappa' \geq 2\kappa' - \kappa' = \kappa'. \end{aligned}$$

The last inequality follows from the fact that $\kappa' \leq \frac{1}{2}$ and $\mathrm{i}(\gamma_i, \gamma_{i+m}) = b \geq 1$. This proves (ii), and completes the proof of the lemma. ■

Proof of Theorem 2.9. The first paragraph of the theorem follows immediately from Lemmas 7.5 and 7.6, setting $\kappa_0 = \max\{\kappa, \frac{1}{\kappa'}\}$. The second paragraph is the next lemma. ■

Lemma 7.7. For any curve δ, there exists $\kappa(\delta) > 0$ and $N(\delta) \in \mathbb{Z}$ so that for all $k \geq N(\delta)$, we have

$$\mathrm{i}(\delta, \gamma_k) \overset{*}{\asymp}_{\kappa(\delta)} A(0, k).$$

We only sketch the proof since this is exactly the same statement and proof as in Lemma 5.11 from [2].

Sketch of proof. The idea is that from the estimates in Lemma 7.5 and 7.6, we have some κ_0, $n_0 > 0$ so that for all $0 \leq i \leq 2m - 1$ and $k \geq n_0$,

$$\mathrm{i}(\gamma_i, \gamma_k) \overset{*}{\asymp}_{\kappa_0} A(0, k).$$

Since $\mu_0 = \gamma_0 \cup \ldots \cup \gamma_{2m-1}$ fills S, this means that the set of laminations $\{\frac{\gamma_k}{A(0,k)}\}_{k \geq n_0} \subset \mathcal{ML}(S)$ form a compact subset of $\mathcal{ML}(S)$. By Theorem 2.9, this sequence can only accumulate on points of $\mathcal{ML}(S)$ supported on ν. So, for any curve δ, there is a compact neighborhood of the accumulation points and a number $\kappa(\delta) > 0$ on which intersection number with δ lies in the interval $\left[\frac{1}{\kappa(\delta)}, \kappa(\delta)\right]$. But then for k sufficiently large, $\frac{\gamma_k}{A(0,k)}$ is in this neighborhood and hence

$$\frac{\mathrm{i}(\delta, \gamma_k)}{A(0, k)} = \left(\delta, \frac{\gamma_k}{A(0, k)}\right) \overset{*}{\asymp}_{\kappa(\delta)} 1$$

as required. ∎

7.3 Convergence in $\mathcal{ML}(S)$

Lemma 7.7, together with (7.5), are the key ingredients in the proof of the following, which is identical to Lemma 5.13 from [2].

Proposition 7.8. For each $h = 0, \ldots, m - 1$,

$$\lim_{i \to \infty} \frac{\gamma_{h+im}}{A(0, h + im)} = \bar{\nu}^h$$

in $\mathcal{ML}(S)$, where $\bar{\nu}^h$ is a measure supported on ν.

Sketch of proof. Applying (7.5) to estimate $\mathrm{i}(\gamma_{k+m}, \delta)$ using the fact that $\gamma_{k+m} = D_{\gamma_k}^{e_k}(\gamma'_{k+m})$, and applying Lemma 7.7 we may argue as in the proof of Lemma 7.5 we have

$$\begin{aligned} |\mathrm{i}(\gamma_{k+m}, \delta) - be_k \mathrm{i}(\gamma_k, \delta)| &\leq \mathrm{i}\left(\gamma'_{k+m}, \delta\right) \leq B \sum_{l=k-m}^{k+m-1} \mathrm{i}(\gamma_l, \delta) \\ &\leq B\kappa(\delta) \sum_{l=k-m}^{k+m-1} A(0, l). \end{aligned}$$

Dividing both side by $A(0, k+m)$ and applying Lemma 2.10 this implies

$$\left| \mathrm{i}\Big(\frac{\gamma_{k+m}}{A(0, k+m)}, \delta\Big) - \mathrm{i}\Big(\frac{\gamma_k}{A(0,k)}, \delta\Big) \right| \leq 2mB\kappa(\delta) a^{-\lfloor \frac{k}{m} \rfloor}.$$

From this and a geometric series argument, we deduce that for all $h = 0, \ldots, m-1$, the sequence $\left\{ \mathrm{i}\Big(\frac{\gamma_{h+im}}{A(0,h+im)}, \delta\Big)\right\}_{i=0}^{\infty}$ is a Cauchy sequence of real numbers, and hence converges. By Lemma 7.7, the limit is non-zero, and since this is true for every simple closed curve δ, the sequence $\left\{ \frac{\gamma_{h+im}}{A(0,h+im)} \right\}_{i=0}^{\infty}$ converges to some $\bar{\nu}^h \in \mathcal{ML}(S)$, supported on ν by Theorem 2.8. ■

The next lemma is the analog of Theorem 6.1 from [2]. The proof is essentially the same, but since the required intersection number estimates are weaker here, we sketch the proof nonetheless.

Lemma 7.9. For each $h, h' \in \{0, \ldots, m-1\}$ with $h \neq h'$, we have

$$\lim_{i \to \infty} \frac{\mathrm{i}(\gamma_{h+im}, \bar{\nu}^h)}{\mathrm{i}(\gamma_{h+im}, \bar{\nu}^{h'})} = \infty.$$

Consequently, $\bar{\nu}^h$ is not absolutely continuous with respect to $\bar{\nu}^{h'}$.

Sketch of Proof. As in the proof of [2, Theorem 6.1], it clearly suffices to prove that for i sufficiently large,

$$\mathrm{i}(\gamma_h, \gamma_{h+(i+1)m}) \mathrm{i}(\gamma_{h+im}, \bar{\nu}^h) \stackrel{*}{\asymp} 1$$

and

$$\lim_{i \to \infty} \mathrm{i}(\gamma_h, \gamma_{h+(i+1)m}) \mathrm{i}(\gamma_{h+im}, \bar{\nu}^{h'}) = 0.$$

Appealing to Proposition 7.8 to estimate $\bar{\nu}^h$ by $\frac{\gamma_{h+km}}{A(0,h+km)}$ for large $k \gg i$, and Theorem 2.9, we have

$$\mathrm{i}(\gamma_0, \gamma_{h+(i+1)m})\mathrm{i}(\gamma_{h+im}, \bar{\nu}^h) \stackrel{*}{\asymp} \frac{A(0,h+(i+1)m)A(h+im,h+km)}{A(0,h+km)} = 1$$

The last equality here follows from a simple calculation using the formula (2.6) for $A(i,j)$ (see the proof of [2, Theorem 6.1] for details). The multiplicative error here can be made arbitrarily close to κ_0^2 (taking k sufficiently large).

Similarly, we estimate $\bar{\nu}^{h'} = \frac{\gamma_{h'+km}}{A(0,h'+km)}$ and apply Theorem 2.9 to obtain

$$\mathrm{i}(\gamma_0, \gamma_{h+(i+1)m})\mathrm{i}(\gamma_{h+im}, \bar{\nu}^{h'}) \stackrel{*}{\prec} \frac{A(0,h+(i+1)m)A(h+im,h'+km)}{A(0,h'+km)} \stackrel{*}{\prec} a^{-i}$$

The first multiplicative error can be made arbitrarily close to κ_0^2 (again by taking k sufficiently large). The second bound follows from a calculation and Lemma 2.10, with multiplicative error depending only on whether $h > h'$ or $h' > h$ (see [2] for details). ■

Proof of Theorem 2.11. All that remains is to prove that $\bar{\nu}^0, \ldots, \bar{\nu}^{m-1}$ are ergodic measures. At this point, the proof is identical to the proof of the analogous statement Theorem 6.7 from [2], appealing to the facts proved so far. This proof involves a detailed analysis of Teichmüller geodesics, drawing specifically on results of Lenzhen–Masur [22] and the fourth author [32]. As this would take us too far afield of the current discussion, we refer the reader to that paper for the details. ■

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