Time reversal of degenerate diffusions

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1. When is the time reversal $\bar{x}_t = x_{T-t}$ of a diffusion x_t also a diffusion? The problem has been considered by several authors [HP,MN, F1,F2, CF, P]. It comes up in stochastic control theory and Nelson's stochastic mechanics. In the appendix we present a further application to particle systems.

Suppose we have a generator,

$$L_t = \nabla \cdot a(t,x)\nabla + b(t,x) \cdot \nabla \tag{1}$$

and a solution $P \in \mathcal{M}_1(\Omega)$, $\Omega = C([0,T], \mathbf{T}^d)$ of the martingale problem; For each $\phi \in C^\infty(\mathbf{T}^d)$, $\phi(x_t) - \int_0^t L_s\phi(x_s)ds$ is a martingale with respect to $\mathcal{F}_t = \sigma(x_s : 0 \leq s \leq t)$ under P. We ask whether the law $\bar{P}$ of $\bar{x}_t = x_{T-t}$ under P is also the solution of a martingale problem for some generator $\bar{L}_t$. Clearly the problem is not time reversal invariant, so the first question is what should $\bar{L}_t$ be.

To study the time reversed problem note that the marginal density $\rho(t,x)$ will be a solution of the forward equation

$$\partial_t \rho = L_t^* \rho. \tag{2}$$

Hence the time reversed density $\bar{\rho}(t,x) = \rho(T-t,x)$ will be a solution of the backward heat equation

$$\partial_t \bar{\rho} = -L_{T-t}^* \bar{\rho}.$$

The problem is well posed because we are only dealing with something that comes from (2) in the first place. But also $\bar{\rho}$ are the marginal densities of the time reversed process and hence should satisfy the time reversed forward equation

$$\partial_t \bar{\rho} = \bar{L}_t^* \bar{\rho}.$$

So consistency requires of $\bar{L}_t$ that

$$\bar{L}_t^* \bar{\rho} = -L_{T-t}^* \bar{\rho}. \tag{3}$$

The idea is that the second order term in (1) should be stable under time reversal and hence we guess that

$$\bar{L}_t = \nabla \cdot a(T-t,x)\nabla + \bar{b}(t,x) \cdot \nabla \tag{4}$$

for some $\bar{b}$. The choice consistent with both (3) and (4) is

$$\bar{b}(t,x) = 2a(T-t,x)\nabla \log \rho(T-t,x) - b(T-t,x). \tag{5}$$

So what we expect is that as long as the martingale problems for L_t and $\bar{L}_t$ are well posed, we can identify $\bar{P}$ as the unique solution of the martingale problem for $\bar{L}_t$.

This was done in [HP]. Amoung other things, it was shown there that if b and the matrix square root σ of a are uniformly Lipschitz, and the marginal density ρ satisfies a priori

$$\int_0^T \int_{\mathbf{T}^d} \left[|\rho|^2 + \nabla \rho \cdot a \nabla \rho \right] dx dt < \infty, \tag{6}$$

then $\bar{P}$ is the solution of the martingale problem for $\bar{L}_t$. (Note that they work on $\mathbf{R}^d$. Since the problem is local in nature, we work on the torus $\mathbf{T}^d$ in order to simplify matters and concentrate on the essential difficulties.) They also note that (6) holds if either 1. $\partial_t + L_t$ is hypoelliptic, or 2. the initial density is in L^2 and either 2i. a is uniformly elliptic, or 2ii. $a^{ij}_{x_i x_j}$ are uniformly bounded.

The purpose of the present article is to extend these results using methods from [QV], which provides a natural context for the time reversal problem in the degenerate case.

Suppose we are given $a(x,t)$ taking values in nonnegative definite symmetric $d \times d$ matrices and a d-vector valued $b(x,t)$ and density $\rho(x,t)$ satisfying

$$\int_0^T \int_{\mathbf{T}^d} \sum_{i,j,k,\ell=1}^{d} a^{jk}_{x_i} (a^{-1})^{k\ell} a^{\ell j}_{x_i} dx dt < \infty, \tag{7}$$

$$\sup_{x \in \mathbf{T}^d, t \in [0,T]} \rho(t,x) < \infty. \tag{8}$$

$$\int_0^T \int_{\mathbf{T}^d} b \cdot a^{-1} b \rho dx dt < \infty. \tag{9}$$

and $\rho(x,t)$ is a solution of the forward equation $\partial_t \rho = L_t^* \rho$ in the sense that for all $\phi \in C_0^\infty([0,T) \times \mathbf{T}^d)$,

$$-\int \rho(0,x)\phi(0,x) dx = \int_0^T \int_{\mathbf{T}^d} \rho \left[-\partial_t + L_t \right] \phi dx dt. \tag{10}$$

The main result of [QV] is that under these conditions there exists a solution P of the martingale problem for L_t with $\rho(x,t)$ as its marginals, and it is unique within the class of P with marginals satisfying (8) and (9).

Remark. Conditions (7) and (9) are stated somewhat informally. More precisely, (7) means that there is a $C < \infty$ such that for all smooth $\phi : [0,T] \times \mathbf{T}^d \to \mathbf{R}^d$,

$$\left| \int_0^T \int_{\mathbf{T}^d} a \phi_{x_i} dx dt \right|^2 \le C \int_0^T \int_{\mathbf{T}^d} \phi \cdot a \phi dx dt. \tag{7'}$$

and similarly (9) means that

$$\left(\int_0^T \int_{\mathbf{T}^d} b \cdot \phi \rho dx dt \right)^2 \le C \int_0^T \int_{\mathbf{T}^d} \phi \cdot a \phi \rho dx dt. \tag{9'}$$

Let $\bar{P}$ denote the law of the time reversed process. The main result of this article is

Theorem 1. *Under the above assumptions, $\bar{P}$ is the unique solution of the martingale problem with generator $\bar{L}_t$ given by (4) and (5).*

We need the following result from [QV].

Lemma [QV]. *Suppose that (7) holds and that $\rho(x,t)$ is a very weak solution of the forward equation $\partial_t \rho = L_t^* \rho$ in the sense of (8), (9) and (10). Then*

$$\int_0^T \int_{\mathbf{T}^d} \nabla\rho \cdot a \nabla\rho dx dt < \infty. \tag{11}$$

If in addition we suppose that ρ is nonnegative then

$$\int_0^T \int_{\mathbf{T}^d} \frac{\nabla\rho \cdot a\nabla\rho}{\rho} dx dt < \infty \tag{12}$$

and ρ is a weak solution of the forward equation in the sense that

$$\int_{\mathbf{T}^d} \rho\phi \Big|_0^T = \int_0^T \int_{\mathbf{T}^d} \left[\partial_t \phi + \left(\frac{a\nabla\rho}{\rho} + b \right) \cdot \nabla\phi \right] \rho dx dt \tag{13}$$

for all ϕ satisfying

$$\int_0^T \int_{\mathbf{T}^d} \nabla\phi \cdot a \nabla\phi \rho dx dt < \infty. \tag{14}$$

Notice that the apriori estimate one obtains (12) is exactly what is needed in order for $\bar{b}$ of (5) to satisfy the finite energy condition (9). Hence under our conditions for the well-posedness of the martingale problem for L, the martingale problem for $\bar{L}$ is also well posed. It only remains to check that we have enough estimates to show that $\bar{L}$ is really the generator of the time reversed process.

Lemma 2. *Suppose that (7) holds and that $\rho(x,t)$ is a nonnegative very weak solution of the forward equation $\partial_t \rho = L_t^* \rho$ in the sense of (8), (9) and (10), and that $v(x,t)$ is a nonnegative very weak solution of the time reversed forward equation $\partial_t v = -L_t^* v$ in the sense that (8) and (9) hold for v and (10) holds in the time reversed sense that for all $\phi \in C_0^\infty((0,t] \times \mathbf{T}^d)$,*

$$\int v(t,x)\phi(t,x) dx = \int_0^t \int_{\mathbf{T}^d} v \left[-\partial_t - L_t \right] \phi dx dt. \tag{15}$$

Then

$$u = \rho v$$

is a weak solution of

$$\partial_t u = \tilde{L}_t^* u$$

for

$$\tilde{L}_t = -\bar{L}_{T-t} = -\nabla \cdot a(t,x) \nabla - \left(2a \frac{\nabla\rho(t,x)}{\rho(t,x)} - b(t,x) \right) \cdot \nabla.$$

Proof. Let $\psi \in C^\infty(\mathbf{T}^d)$. By the previous lemma (11) holds for v and therefore, since ρ is bounded (14) holds for $\phi = \psi v$ so we can use it in (13). The same argument works for testing $\partial_t v = -L_t^* v$ with $\psi\rho$. So we obtain

$$\int_{\mathbf{T}^d} \rho v \psi \Big|_0^T = \int_0^T \int_{\mathbf{T}^d} \left[(\nabla\rho\psi + \rho\nabla\psi)\,(a\nabla v - bv) + \left(\frac{a\nabla\rho}{\rho} + b\right) \cdot (v\nabla\psi + \nabla v\psi)\,\rho \right] dxdt$$
$$= \int_0^T \int_{\mathbf{T}^d} [\nabla\psi a\nabla(\rho v) - 2a\nabla\rho\nabla\psi v + b\nabla\psi\rho v]\, dxdt \tag{16}$$

so that $u = \rho v$ is a weak solution of $\partial_t u = \tilde{L}_t^* u$.

Proof of Theorem 1. Clearly it suffices to show that for any $f, g \in C^\infty(\mathbf{T}^d)$, and any $0 \le s < t \le T$,

$$E\left[\left(f(\bar{X}_t) - f(\bar{X}_s) - \int_s^t \bar{L}_u f(\bar{X}_u) du\right) g(\bar{X}_s)\right] = 0.$$

Changing variables $T - u \mapsto u$ this becomes

$$E\left[\left(f(X_t) - f(X_s) - \int_s^t \tilde{L}_u f(X_u) du\right) g(X_t)\right] = 0.$$

For $0 < s \le t \le T$ let

$$v(s,x) = E_{s,x}[g(X_t)].$$

Then $v(s,x)$ is a very weak solution of $\partial_s v + L^* v = 0$ on $0 < s < t$ with $v(t) = g$. By the previous lemma we are done.

Remark. The mysterious condition (7) is closely related to more natural, and weaker, conditions such as

$$\int_0^T \int_{\mathbf{T}^d} \nabla \cdot a\; a^{-1} \nabla \cdot a \; dxdt < \infty \tag{17}$$

and the even weaker

$$\int_0^T \int_{\mathbf{T}^d} |\nabla\sigma|^2 dxdt < \infty. \tag{18}$$

In the applications to particle systems where the problem arose (see [3] and the appendix), $a(x,t) = S(\rho(x,t))$ for a smooth matrix valued function S of $\rho \in [0,1]$ with bounded derivative and $S(\rho) \ge c(1-\rho)I$. In such cases, (7), (17) and (18) are identical, and require that $\int_0^T \int \frac{|\nabla\rho|^2}{\rho(1-\rho)} dxdt < \infty$.

The main point is that one only needs the finiteness of some quadratic norm involving only first derivatives of σ or a, and not second derivatives nor Lipschitz conditions on σ and b or anything like that.

2. Appendix. In this appendix we present an application of time reversal of diffusions to interacting particle systems.

For large integers N, consider $L \sim rN^d$, $0 \le r \le 1$, particles moving on $\mathbf{Z}^d/N\mathbf{Z}^d$ as symmetric random walks with simple exclusion. A particle at x attempts to jump to $x+z$ at rate $p(z)$, the jump being performed only if the target site is unoccupied. The jump law $p(\cdot)$ is taken to be symmetric, irreducible, finite range and have full rank covariance matrix D. We start with at most one particle per site and it remains so for all time. From any initial condition $(x_1, \ldots, x_L)$ we have a stochastic process $(x_1(t), \ldots, x_L(t))$. We rescale the process by defining

$$y_i(t) = N^{-1} x_i(N^2 t)$$

and introduce the empirical tagged particle process

$$R_{L,\omega} = N^{-d}\left[\delta_{y_1(\cdot)} + \cdots + \delta_{y_L(\cdot)}\right].$$

The marginals of the process $R_{N,\omega,t} = N^{-d}\left[\delta_{y_1(t)} + \cdots + \delta_{y_L(t)}\right]$ are commonly known as the empirical density field. We assume that the initial configuration is chosen such that $R_{N,\omega,t} \to \rho_0(x)dx$ weakly in probability for some given ρ_0 on $\mathbf{T}^d$ with $\int \rho_0 dx = r$. The hydrodynamic limit for symmetric simple exclusions is the statement that under these conditions we have $R_{N,\omega,t} \to \rho(t,x)dx$ weakly in probability as well, where $\rho(t,x)$ is obtained from $\rho_0(x)$ by solving the heat equation $\partial_t \rho = \frac{1}{2}\nabla \cdot D\nabla\rho$ with $\rho(0,x) = \rho_0(x)$. The empirical density field also satisfies a large deviation principle. If $q(t,x)$ is an arbitrary nonnegative function of $t \in [a,b]$ and $\theta \in \mathbf{T}^d$ with $\int q dx \equiv r$ then $R_{N,\omega,t}$ let

$$I_{[a,b]}(q(\cdot,\cdot)) = \int h(q(a,x), \rho_0(x))dx + \frac{1}{2}\inf_{b \in B_q} \int_a^b \int q(1-q)|b|^2 dx dt$$

where the infimum is over the class B_q of b satisfying

$$\partial_t q - \frac{1}{2}\Delta q = \nabla \cdot q(1-q)b$$

for which the last integral is finite. Here $h(a,b) = a\log(a/b) + (1-a)\log((1-a)/(1-b))$. If we start our process at a and observe it on the time interval $[a,b]$ then we have

$$P(R_{N,\omega,\cdot} \in A) \sim \exp\left\{ -N^d \inf_{\substack{\mu \in A \\ \mu(t,dx) = q(t,x)dx}} I_{[a,b]}(q(\cdot,\cdot)) \right\}$$

in the sense that $\limsup_{N\to\infty} N^{-d}\log$ of the left hand side is controlled above by that of the left hand side for all closed sets, and controlled below by that of the right hand side for all open sets.

We now recall the Onsager-Machlup time reversal relation. Set $\rho_0 \equiv r$ and define the quasipotential

$$V(f(\cdot)) = \inf_{\substack{q(0,x) = f(x) \\ q(-\infty,x) \equiv r}} I_{(-\infty,0]}(q(\cdot,\cdot)).$$

The following result is well known.

Theorem. *The infimum is obtained on the path $\bar\rho(x,t)=\rho(x,-t)$ where $\partial_t\rho=\frac{1}{2}\Delta\rho$ with $\rho(x,0)=f(x)$ and the minimum is $V(f(\cdot))=\int h(f(x))dx$.*

Proof. Let $H=\log\frac{\bar\rho}{1-\bar\rho}$. Then $b=\nabla H$ is an element of $B_{\bar\rho}$ and we can check that it is the minimiser. Hence

$$I_{(-\infty,0]}(\bar\rho)=\int_{-\infty}^{0}\int\frac{|\nabla\bar\rho|^2}{\bar\rho(1-\bar\rho)}dxdt=\int_{-\infty}^{0}\int\partial_t\bar\rho\log\frac{\bar\rho}{1-\bar\rho}dxdt.$$

Integrating by parts gives

$$I_{(-\infty,0]}(\bar\rho)=\int[\bar\rho\log\bar\rho+(1-\bar\rho)\log(1-\bar\rho)]dx\Big|_{-\infty}^{0}=\Phi(f).$$

Now suppose that q is any trajectory with $\int q(t,x)dx=\int f(x)dx$ for all $t\in(-\infty,0]$ with $q(-\infty,x)$ constant and $q(0,x)=f(x)$. Then for some $b\in B_q$,

$$\begin{aligned}I(q)-I(\bar\rho)&=\int_{-\infty}^{0}\int\frac{1}{2}q(1-q)|b|^2dxdt-\Phi(f)\\&=\int_{-\infty}^{0}\int[\frac{1}{2}q(1-q)|b|^2-\partial_tq\log\frac{q}{1-q}]dxdt\\&=\int_{-\infty}^{0}\int[\frac{1}{2}q(1-q)|b|^2-(\frac{1}{2}\Delta q+\nabla\cdot q(1-q)b)\log\frac{q}{1-q}]dxdt\\&=\int_{-\infty}^{0}\int\frac{1}{2}q(1-q)|b+\frac{\nabla q}{q(1-q)}|^2dxdt\geq 0.\end{aligned}$$

Next we consider the asymptotic behaviour of the empirical process $R_{n,\omega}$. It was shown in [KV], [Q], [R] that $R_{n,\omega}$ also has a law of large numbers with limit the diffusion process P_ρ on $\mathbf{T}^d$ with generator

$$\mathcal{A}_\rho=\frac{1}{2}\nabla\cdot S(\rho(t,x))\nabla+\frac{1}{2}\left[S(\rho(t,x)-D\right]\nabla\log\rho(t,x)\cdot\nabla. \tag{19}$$

Here $S(\rho)$, $0\leq\rho\leq 1$ is the self-diffusion matrix; the limiting covariance of a tagged particle in equilibrium with density ρ. It is known to have the following properties. Clearly $S(1)=0$ and $S(0)=D$. As long as we are not in the case of one dimension with only nearest neighbour jumps, then $S(\rho)>0$ for $\rho<1$. It is a smooth matrix valued function of ρ [LOV]. In dimensions $d\geq 3$ it is known that $S(\rho)\geq c(1-\rho)$ for some $c>0$. We need this for the large deviations of the empirical process and therefore they are at this time only valid if $d\geq 3$.

The argument to obtain (19) is remarkably similar to that for the reverse generator (4), (5). In equilibrium, ρ is a constant and we have $\mathcal{A}_\rho=\nabla\cdot S(\rho)\nabla$. We guess that in nonequilibrium we should have a diffusion whose principle part only feels the local density, and is therefore of the form $\mathcal{A}_\rho=\nabla\cdot S(\rho(t,x))\nabla+b\cdot\nabla$ for an appropriate drift b. On the other hand, consistency requires that the density itself should be a solution of the forward equation $\partial_t\rho=\mathcal{A}_\rho^*\rho$ and from this we obtain the drift b in (19).

Let Q be any subprobability measure on $D([a,b],\mathbf{T}^d)$ with marginals $q(t,x)$ satisfying $\int q dx \equiv r$. For each $b \in B_q$ define

$$\mathcal{A}_{q,b} = \mathcal{A}_q + (1-q)Db\cdot\nabla.$$

It is proved in [QV] that the martingale problem for $\mathcal{A}_{q,b}$ is well posed and so we have a diffusion process P_q^b which we think of as on $D([a,b],\mathbf{T}^d)$ instead of $C([a,b],\mathbf{T}^d)$, for purposes of comparison. The entropy $H(Q/P_q^b)$ turns out to be either finite for all $b \in B_q$ or infinite for all $b \in B_q$. In the former case we single out a special $b = b_Q$ by the condition

$$E^Q\left[\int_a^b \phi(t,x_t)\cdot dx_t\right] = E^{P_q^{b_Q}}\left[\int_a^b \phi(t,x_t)\cdot dx_t\right]$$

for all smooth ϕ on $[a,b]\times\mathbf{T}^d$. The rate function for the large deviations of the empirical process $R_{N,\omega}$ was shown in [QRV] to be

$$\mathcal{I}(Q) = \int h(q(a,x))dx + H(Q/P_q^{b_Q}) + \frac{1}{2}\int_a^b\int q(1-q)|b_Q|^2 dxdt.$$

By the contraction principle the quasipotential

$$V(f(\cdot)) = \inf_{\substack{q(0,x)=f(x)\\ q(-\infty,x)=\int f(x)dx}} \mathcal{I}_{(-\infty,0]}(Q).$$

The time reversal relation for the empirical process is

Theorem. *The infimum is obtained at the time reversed process $\bar{P}_\rho$.*

Proof. From the previous theorem and the contraction principle we know that the infimum is obtained on a process with marginals $\bar\rho$, so $\bar{P}_\rho$ is certainly a candidate. By the theorem on time reversal, the generator of $\bar{P}_\rho$ is

$$\bar{\mathcal{A}} = \frac{1}{2}\nabla\cdot S(\bar\rho(t,x)\nabla + \frac{1}{2}\left[S(\bar\rho(t,x)) + D\right]\nabla\log\bar\rho(t,x)\cdot\nabla.$$

Once again we take $b = \nabla H$ with $H = \log\frac{\bar\rho}{1-\bar\rho}$. The corresponding process $P_{\bar\rho}^b$ has generator $\mathcal{A}_{\bar\rho,b=\bar{\mathcal{A}}}$; in other words $P_{\bar\rho}^b = \bar{P}_\rho$. Hence the entropy $H(\bar{P}_\rho/P_{\bar\rho}^b) = 0$ and

$$\mathcal{I}(\bar{P}_\rho) = \frac{1}{2}\int_{-\infty}^0 \bar\rho(1-\bar\rho)|b|^2$$

is the minimum as in the previous theorem.

References.

[HP] Haussmann, U.G., Pardoux, E, *Time reversal of diffusions*, Ann. Prob. **14** 1188–1205 (1986).

[QV] Quastel, J., Varadhan, S.R.S., *Diffusion semigroups and diffusion processes corresponding to degenerate divergence form operators*, Comm. Pure Appl. Math. **50** 667–706 (1997).

[QRV] Quastel, J.; Rezakhanlou, F.; Varadhan, S. R. S. *Large deviations for the symmetric simple exclusion process in dimensions* $d \geq 3$, Probab. Theory Related Fields 113 (1999), no. 1, 1–84.

[KV] Kipnis, C., Varadhan, S.R.S., *Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusion*, Comm. Math. Phys. **106**, 1-19.

[Q] Quastel, J., *Diffusion of color in the simple exclusion process*, Comm. Pure Appl. Math., **45**, 623-679 (1992).

[MN] Millet, A.; Nualart, D.; Sanz, M. *Integration by parts and time reversal for diffusion processes*, Ann. Probab. 17 (1989), no. 1, 208–238.

[F1] Föllmer, H. *Time reversal on Wiener space*, Stochastic processes—mathematics and physics (Bielefeld, 1984), 119–129, Lecture Notes in Math., 1158, Springer, Berlin-New York, 1986.

[F2] Föllmer, H. *An entropy approach to the time reversal of diffusion processes*, Stochastic differential systems (Marseille-Luminy, 1984), 156–163, Lecture Notes in Control and Information Sci., 69, Springer, Berlin-New York, 1985.

[CF] Cattiaux, P., Fradon, M. *Entropy, reversible diffusion processes, and Markov uniqueness*, J. Funct. Anal. 138 (1996), no. 1, 243–272.

[P] Petit, F. *Time reversal and reflected diffusions*, Stochastic Process. Appl. 69 (1997), no. 1, 25–53.

[R] Rezakhanlou, F., *Propagation of chaos for symmetric simple exclusions*, Comm. Pure Appl. Math. 47 (1994), no. 7, 943–957.

[LOV] Landim, C., Olla, S., Varadhan, S.R.S., in preparation.