Exponential decay of entropy in the Random Transposition and Bernoulli-Laplace models

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Abstract

We give bounds on the exponential decay rate of entropy in the random transposition model and the Bernoulli-Laplace model which are independent of the number of sites and the number of particles. This is then used to give a bound on the time to stationarity in the total variation norm.

1 Introduction

This article concerns the decay of entropy and time to stationarity in two models of continuous time interacting random walks: The random transposition model and the Bernoulli-Laplace model.

Random transposition model. The state space S_n is the permutation group of n objects. The random transposition model is the Markov process uniquely characterized by an initial distribution on S_n and the Markov generator $\mathcal{L}_n$ given by

$$(\mathcal{L}_n f)(\sigma) = \frac{1}{n}\sum_{i,j=1}^{n}[f(\sigma^{ij}) - f(\sigma)]$$

where f is a function on S_n and $(\sigma^{ij})_i = \sigma_j$, $(\sigma^{ij})_j = \sigma_i$ and $(\sigma^{ij})_k = \sigma_k$ for $k \neq i$ or j. We can think of n distinct particles, numbered 1 through n, placed in n distinct sites numbered 1 through n. S_n is the set of configurations with σ_i the number of the particle in site i. At rate one, a particle chooses uniformly from

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the n sites and exchanges position with the particle at that site. The model is symmetric and irreducible and has the uniform distribution $\mu_n(\sigma) = \frac{1}{n!}$ as unique stationary distribution. The associated expectation will be denoted E_n.

Bernoulli-Laplace model. The Bernoulli-Laplace model has two parameters n and r, $1 \le r \le n$, the number of distinct sites and the number of identical particles. A site can be occupied by at most one particle. So the state space, denoted by $C_{n,r}$ is the space of all subsets of the n sites with r elements. For $\eta \in C_{n,r}$, denote by η_i the number of particles at the site i. The Bernoulli-Laplace model is the Markov process with the state space $C_{n,r}$ and with the Markov generator $\mathcal{L}_{n,r}$ given by

$$(\mathcal{L}_{n,r} f)(\eta) = \frac{1}{n} \sum_{i,j=1}^{n} \eta_i (1-\eta_j)[f(\eta^{ij}) - f(\eta)] = \frac{1}{n} \sum_{i<j} [f(\eta^{ij}) - f(\eta)]$$

where f is a function on $C_{n,r}$ and $(\eta^{ij})_i = \eta_j$, $(\eta^{ij})_j = \eta_i$ and $(\eta^{ij})_k = \eta_k$ for $k \neq i, j$. Each particle, at rate one, picks a site with uniform probability, and jumps there as long as it is unoccupied. The model is symmetric and irreducible and has the uniform distribution $\mu_{n,r}(\eta) = \binom{n}{r}^{-1}$ as unique stationary distribution. The associated expectations will be denoted $E_{n,r}$.

Let $\mathcal{L}$ be an ergodic Markov process on a finite state space S symmetric with respect to invariant measure μ. Let $P_t = e^{t\mathcal{L}}$ denote the associated semigroup. If ν is the initial distribution then $P_t \nu \to \mu$. We are interested in the speed of convergence. There are several choices of norm. Suppose that $\nu << \mu$ and $f = \frac{d\nu}{d\mu}$. The $V^{(p)}$ norm is defined

$$V^{(p)}(\nu, \mu) = E_\mu[|f-1|^p]^{1/p}$$

if $f \in L^p(\mu)$, and $+\infty$ otherwise. $\frac{1}{2}V^{(1)}$, the total variation norm, is most common, but $V^{(2)}$ is used as well. The relative entropy is defined by

$$H(\nu, \mu) = E_\mu [f \log f]$$

if $f \log f \in L^1(\mu)$, and $+\infty$ otherwise. If μ is the invariant measure we will use $H(f)$ or $H_\mu(f)$ for $H(f\mu, \mu)$.

The *time to stationarity*, $\tau^{(p)}$ is defined by

$$\tau^{(p)} = \inf\{t > 0 : \sup_{\nu} V^{(p)}(P_t\nu, \mu) \leq e^{-1}\}.$$

Often the bound $\tau^{(1)} \leq \tau^{(2)}$ is used. The Dirichlet form is defined by

$$D(f, g) = -E_{\mu}[f\mathcal{L}g], \quad D(f) \equiv D(f, f).$$

The logarithmic Sobolev constant α is defined by

$$\alpha = \sup\{H(f)/D(\sqrt{f}) : f \geq 0; E_{\mu}[f] = 1\}.$$

It is known [4] that $c_1\alpha \leq \tau^{(2)} \leq c_2\alpha$ where $c_1 = 0.5$ and $c_2 = 1 + \frac{1}{4}\log\log\frac{1}{\mu^*}$ where $\mu^* = \min_{x\in S}\mu(x)$.

On the other hand, if one is interested in convergence of $H(P_t\nu, \mu)$ let $f_t = dP_t\nu/d\mu$ and note that

$$\frac{d}{dt}H(f_t) = -D(f_t, \log f_t).$$

The entropy contant β is defined by

$$\beta = \sup\{H(f)/D(f, \log f) : f \geq 0; E_{\mu}[f] = 1\}.$$

Therefore

$$H(P_t\nu, \mu) \leq e^{-t/\beta}H(\nu, \mu).$$

The entropy constant was obtained for Poisson measure in [3]. From the general inequality,

$$(\sqrt{b} - \sqrt{a})^2 \leq \frac{1}{4}(b - a)(\log b - \log a),$$

valid for $a, b \geq 0$, we have $D(\sqrt{f}) \leq \frac{1}{4}D(f, \log f)$ and hence $\beta \leq 4\alpha$. Since $H \geq V^{(1)}$ it is easy to check that

$$\tau^{(1)} \leq \beta(1 + \log\log\frac{1}{\mu^*}).$$

The relative entropies and Dirichlet forms for the random transposition model and Bernoulli-Laplace models are given respectively by

$$H_n(f) = \frac{1}{n!}\sum_{\sigma\in S_n} f(\sigma)\log f(\sigma),$$

$$D_n(f,g) = \frac{1}{n!2n} \sum_{\sigma\in S_n} \sum_{i,j=1}^{n} [f(\sigma^{ij}) - f(\sigma)][g(\sigma^{ij}) - g(\sigma)]$$

$$H_{n,r}(f) = \frac{1}{\binom{n}{r}} \sum_{\eta\in C_{n,r}} f(\eta)\log f(\eta),$$

$$D_{n,r}(f,g) = \frac{1}{2n\binom{n}{r}} \sum_{\eta\in C_{n,r}} \sum_{1\le i<j\le n} [f(\eta^{ij}) - f(\eta)][g(\eta^{ij}) - g(\eta)].$$

Let α_n and β_n denote the logarithmic Sobolev constant and entropy constant of the random transposition model on S_n, and $\alpha_{n,r}$ and $\beta_{n,r}$ denote the logarithmic Sobolev constant and entropy constant of the Bernoulli-Laplace model on $C_{n,r}$. In [4], [14] it is shown that there exists $0 < c < \infty$ such that for any $n \ge 2$,

$$c^{-1}\log n \le \alpha_n \le c\log n,$$

and for any $r = 1, \ldots, n-1$,

$$c\log\frac{n^2}{r(n-r)} \le \alpha_{n,r} \le \frac{2}{\log 2}\log\frac{n^2}{r(n-r)}.$$

The upper bound on $\tau_n^{(2)}$ for the random transposition, and $\tau_{n,r}^{(2)}$ for the Bernoulli-Laplace models then read

$$\tau_n^{(2)} \le C(\log n)^2, \qquad \tau_{n,r}^{(2)} \le C\log\frac{n^2}{r(n-r)}\log\log\binom{n}{r}.$$

Our purpose in this paper is to bound the entropy constants for the two models of random walks. We obtain the bound which is independent of the numbers of sites and the numbers of particles. The method used is the martingale method of [15]. The main results of this article are

Theorem 1 (Random transposition model). *For any $n \ge 2$,*

$$1/2 \le \beta_n \le 1.$$

In particular, for any probability measure ν_n on S_n, for $n \ge 2$,

$$H(P_t\nu_n, \mu_n) \le e^{-t}H(\nu_n, \mu_n),$$

and for some $C < \infty$,

$$\tau_n^{(1)} \le (1 + \log\log n!) \simeq \log n.$$

Theorem 2 (Bernoulli-Laplace model). *For any $n \geq 2$ and any $1 \leq r \leq n-1$,*

$$1 \leq \beta_{n,r} \leq 2.$$

In particular, for any probability measure $\nu_{n,r}$ on $C_{n,r}$, for $n \geq 2$, and $1 \leq r \leq n-1$,

$$H(P_t \nu_{n,r}, \mu_{n,r}) \leq e^{-t/2} H(\nu_{n,r}, \mu_{n,r}),$$

and for some $C < \infty$,

$$\tau_{n,r}^{(1)} \leq 2(1 + \log\log \binom{n}{r}).$$

The main conclusion is that the commonly used bound $\tau^{(1)} \leq \tau^{(2)}$ gives the wrong order for the time to stationarity in the total variation sense in the random transposition model and in the Bernoulli-Laplace model at high or low densities.

The upper bounds on β_n and $\beta_{n,k}$ are proved in sections 2 and 3 respectively. The lower bounds are obtained from the general inequality [2],

$$\beta \geq \lambda/2$$

where λ is the Poincaré constant (= 1/spectral gap),

$$\lambda = \sup\{Var(f)/D(\sqrt{f})\},$$

where $Var(f) = E[(f - E[f])^2]$. The spectral gap can be computed for both models [7],[8],[16]. In the random transposition model $\lambda_n = 1$ and for the Bernoulli-Laplace model $\lambda_{n,r} = 2$.

2 Random transposition model

To prove the upper bound for β_n, using martingale methods as in [14] we will derive an inequlity involving β_{n+1} and β_n. First of all we give an entropy inequality as follows.

Lemma 1. *Let μ be the uniform distribution on $\{1, 2, \ldots, n\}$ and f a probability density function relative to μ. Then for all $n \geq 2$,*

$$H_\mu(f) \leq \frac{1}{2n^2} \sum_{x,y=1}^{n} (f(y) - f(x))(\log f(y) - \log f(x)).$$

Proof. Set

$$f(x_1, \cdots, x_n) = \sum_{i,j=1}^{n} (x_i - x_j)(\log x_i - \log x_j) - 2n \sum_{i=1}^{n} x_i \log x_i$$

where $x_1 \ldots, x_n \geq 0$. The lemma is equivalent to $f(x_1, \cdots, x_n) \geq 0$ for any $x_1, \ldots, x_n \geq 0$ with $x_1 + \cdots + x_n = n$. Since under the condition $x_1 + \cdots + x_n = n$,

$$f(x_1, \cdots, x_n) = -2n \sum_{i=1}^{n} \log x_i,$$

we need only prove that $\frac{1}{n} \sum_{i=1}^{n} \log x_i \leq 0$ for any $x_1, \ldots, x_n \geq 0$ with $x_1 + \cdots + x_n = n$ which is an obvious consequence of the concavity of the logarithm.

Proof of Theorem 1. Let f be a the probability density function relative to the uniform distribution μ_{n+1} on S_{n+1}. Let f_i be the marginal probability density function of σ_i and $f(\cdot|\sigma_i)$ be the conditional the probability density function given σ_i. Define

$$I_{1,i,j,k}(f) = E_{n+1}\big[f_i(\sigma_i) E_{n+1}\big[(f(\sigma^{jk}|\sigma_i) - f(\sigma|\sigma_i))(\log f(\sigma^{jk}|\sigma_i) - \log f(\sigma|\sigma_i))\big|\sigma_i\big]\big]$$

$$I_{1,i} = \sum_{i,j \neq k} I_{1,i,j,k} = E_{n+1}\big[\sum_{j,k \neq i} (f(\sigma^{jk}) - f(\sigma))(\log f(\sigma^{jk}) - \log f(\sigma))\big],$$

$$I_{2,i}(f) = E_{n+1}[f_i(\sigma_i) \log f_i(\sigma_i)],$$

$$I_l(f) = \frac{1}{n+1} \sum_{i=1}^{n+1} I_{l,i}(f), \quad l = 1, 2.$$

Then

$$H_{n+1}(f) = E_{n+1}[f_i(\sigma_i) E_{n+1}[f(.|\sigma_i) \log f(.|\sigma_i)|\sigma_i]] + I_{2,i} \ .$$

Since the uniform distribution on S_{n+1}, given σ_i, is the uniform distribution on S_n, from the definition of β_n we have that

$$H_{n+1}(f) \leq \frac{\beta_n}{2n} I_{1,i} + I_{2,i},$$

thus

$$H_{n+1}(f) \leq \frac{\beta_n}{2n} I_1 + I_2. \tag{2.1}$$

Note that

$$I_1(f) = 2(n-1)D(f, \log f). \tag{2.2}$$

Applying Lemma 1 to f_i we get

$$I_{2,i}(f) \leq \frac{1}{2(n+1)^2} \sum_{x,y=1}^{n+1} (f_i(y) - f_i(x))(\log f_i(y) - \log f_i(x)). \tag{2.3}$$

Since $f_i(x) = E_{n+1}[f(\sigma)|\sigma_i = x)]$ and the function: $(a,b) \to (a-b)(\log a - \log b)$ is convex for $a, b \geq 0$, Jensen's inequality implies (with $E[\cdot] = E_{n+1}[\cdot]$),

$$\begin{aligned}
&(f_i(y) - f_i(x))(\log f_i(y) - \log f_i(x)) \\
&= (E[f(\sigma)|\sigma_i = y)] - E[f(\sigma)|\sigma_i = x)])(\log E[f(\sigma)|\sigma_i = y)] - \log E[f(\sigma)|\sigma_i = x)]) \\
&= (E[f(\sigma^{xy})|\sigma_i = x)] - E[f(\sigma)|\sigma_i = x)])(\log E[f(\sigma^{xy})|\sigma_i = x)] - \log E[f(\sigma)|\sigma_i = x)]) \\
&\leq (n+1)E\big[(f(\sigma^{xy}) - f(\sigma))(\log f(\sigma^{xy}) - \log f(\sigma)))|\sigma_i = x\big]. \qquad (2.4)
\end{aligned}$$

Combining (2.3) and (2.4) we have

$$I_{2,i} \leq \frac{1}{2(n+1)} E_{n+1}\big(\sum_{x,y=1}^{n+1} (f(\sigma^{xy}) - f(\sigma))(\log f(\sigma^{xy}) - \log f(\sigma)))\big).$$

Therefore

$$I_2 \leq \frac{1}{(n+1)} D(f, \log f). \tag{2.5}$$

By the definition of the Entropy constant β_{n+1}, the inequalities (2.1), (2.2) and (2.5) now imply

$$\beta_{n+1} \leq \frac{(n-1)\beta_n}{n} + \frac{1}{(n+1)}. \tag{2.6}$$

The desired upper bound is got from (2.6) by an induction on $n \geq 2$. The initial check for $n = 2$ follows from the $n = 2$ case of Lemma 1 which gives $\beta_2 \leq 1/2$. Assuming that $\beta_n \leq 1$, the last inequality yields

$$\beta_{n+1} \leq \frac{n-1}{n} + \frac{1}{(n+1)} < 1.$$

3 Bernoulli-Laplace model

Lemma 2. *Let μ_ρ be the Bernoulli measure on $\{0,1\}$, $0<\rho<1$. Then for any probability density function f on $\{0,1\}$, relative to the Bernoulli measure,*

$$\begin{aligned} H_{\mu_\rho}(f) \equiv&(1-\rho)f(0)\log f(0)+\rho f(1)\log f(1) \\ \le&\rho(1-\rho)(f(0)-f(1))(\log f(0)-\log f(1)). \end{aligned} \tag{3.1}$$

Proof. If $(1-\rho)x+\rho y=1$, $x\ge 0, y\ge 0$, one can check easily that

$$\begin{aligned} f(x,y) =&\rho(1-\rho)(x-y)(\log x-\log y)-(1-\rho)x\log x-\rho y\log y \\ =&-(1-\rho)\log x-\rho\log y \\ \ge&0. \end{aligned}$$

Proof of Theorem 2. We will prove theorem 2 by induction. Let f be a the probability density function relative to the uniform distribution $\mu_{n,r}$ on $C_{n,r}$ and denote by f_i the marginal probability density function of σ_i and by $f(\cdot|\eta_i)$ the conditional the probability density function of η given η_i. Define

$$\begin{aligned} I_{1,i}(f) =&E_{n+1,r}\Bigg[f_i(\eta_i)E_{n+1,r}\Big[\beta_{n,r-\eta_i}\sum_{j,k\ne i}(f(\eta^{jk}|\eta_i)-f(\eta|\eta_i)) \\ &\times(\log f(\eta^{jk}|\eta_i)-\log f(\eta|\eta_i))|\eta_i\Big]\Bigg] \\ =&E_{n+1,r}\Bigg[\sum_{j,k\ne i}\beta_{n,r-\eta_i}(f(\eta^{jk})-f(\eta))(\log f(\eta^{jk})-\log f(\eta))\Bigg], \\ I_{2,i}(f) =&E_{n+1,r}\left[f_i(\eta_i)\log f_i(\eta_i)\right], \\ I_l(f) =&\frac{1}{n+1}\sum_{i=1}^{n+1}I_{l,i}(f), \quad l=1,2. \end{aligned}$$

Simple calculation yields

$$H_{n+1}(f)=E_{n+1,r}\left[f_i(\eta_i)E_{n+1,r}[f(\cdot|\eta_i)\log f(\cdot|\eta_i)|\eta_i]\right]+I_{2,i}. \tag{3.3}$$

Since if $\eta_i = 0$ or 1, then the inner expectation is with respect to $\mu_{n,r}$ or $\mu_{n,r-1}$, respectively, from the definition of $\beta_{n,r}$ and $I_{1,i}$ we have that

$$H_{n+1,r}(f) \leq \frac{1}{2n} I_{1,i} + I_{2,i},$$

and hence

$$H_{n+1,r}(f) \leq \frac{1}{2n} I_1 + I_2, \qquad 2 \leq r \leq n-1. \tag{3.4}$$

We have,

$$\begin{aligned} I_{1,i}(f) =& \beta_{n,r} E_{n+1,r}\big[\sum_{j,k \neq i} (f(\eta^{jk}) - f(\eta))(\log f(\eta^{jk}) - \log f(\eta)) \mid \eta_i = 0 \big] \\ &+ \beta_{n,r-1} E_{n+1,r}\big[\sum_{j,k \neq i} (f(\eta^{jk}) - f(\eta))(\log f(\eta^{jk}) - \log f(\eta)) \mid \eta_i = 1 \big], \end{aligned}$$

and so

$$I_1 = \frac{1}{n+1} \sum_{i=1}^{n+1} I_{1,i} = 2\Big((n-r)\beta_{n,r} + (r-1)\beta_{n,r-1} \Big) D_{n+1,r}(f, \log f). \tag{3.5}$$

Now applying Lemma 2 with $\rho = \frac{r}{n+1}$ to f_i we get

$$I_{2,i}(f) \leq \frac{r(n+1-r)}{(n+1)^2} (f_i(0) - f_i(1))(\log f_i(0) - \log f_i(1)). \tag{3.6}$$

Now

$$f_i(0) = E_{n+1,r}[f(\eta) | \eta_i = 0] = \frac{1}{n-r+1} \sum_{j \neq i} E_{n+1,r}\left[(1-\eta_j) f(\eta^{ij}) | \eta_i = 1 \right].$$

$(a,b) \to (a-b)(\log a - \log b)$ is convex for $a, b \geq 0$ so by Jensen's inequality,

$$\begin{aligned} &(f_i(0) - f_i(1))(\log f_i(0) - \log f_i(1)) \\ &\leq \frac{1}{n-r+1} \sum_{j \neq i} E_{n+1,r}\left[(1-\eta_j)(f(\eta^{ij}) - f(\eta))(\log f(\eta^{ij}) - \log f(\eta)) | \eta_i = 1 \right]. \end{aligned}$$

Summing over i and writing out the conditional expectation we get

$$\begin{aligned} &\sum_{i=1}^{n+1} (f_i(0) - f_i(1))(\log f_i(0) - \log f_i(1)) \\ \leq& \frac{n+1}{r(n+1-r)} \sum_{i \neq j} E_{n+1,r}\left[\eta_i (1-\eta_j)(f(\eta^{ij}) - f(\eta))(\log f(\eta^{ij}) - \log f(\eta)) \right]. \end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7) we have

$$I_2 \leq \frac{2}{n+1} D_{n+1,r}(f, \log f). \tag{3.8}$$

By the definition of the entropy constant and (3.4), (3.5) and (3.8)

$$\beta_{n+1} \leq \frac{(n-r)}{n}\beta_{n,r} + \frac{(r-1)}{n}\beta_{n,r-1} + \frac{2}{n+1}. \tag{3.9}$$

Now we prove the upper bound by (3.8) and an induction on $n \geq 2$. The initial case $n = 2, r = 1$ is a consequence of Lemma 2. Suppose $\beta_{k,j} \leq 2$ for any $2 \leq k \leq n$ and $2 \leq j \leq k-1$. Then

case (i), for any $2 \leq r \leq n-1$,

$$\frac{1}{2}\beta_{n+1,r} \leq \frac{n-r}{n} + \frac{r-1}{n} + \frac{1}{n+1} < 1;$$

case (ii), for $r = 1$ or $r = n$, by Lemma 1 we have $\beta_{n+1,1} \leq \frac{2}{n+1} < 2$.

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