Diffusion of Colour in the Simple Exclusion Process

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Abstract

We prove a diffusion scaling limit for the macroscopic densities of coloured particles performing the simply excluded random walk, and relate this to the limiting behaviour of a test particle in equilibrium.

1. Introduction

In a large system of interacting particles it is often of interest to study the asymptotic motion of one or more tagged particles in the system. One of the simplest ways to approach this is to consider a system of coloured, but mechanically identical, particles.

Lebowitz and Spohn have shown [L–S] that if, for the system in equilibrium, a rescaled test particle converges to Brownian motion (self–diffusion), then the nonequilibrium system has a hydrodynamical limit. Furthermore, if several test particles converge to independent Brownian motions (propagation of chaos), then colour density profiles have the hydrodynamic limit.

Both properties have been verified for the symmetric simple exclusion process in equilibrium by Kipnis and Varadhan [K–V]. In particular, their results imply that when the system, disregarding colour, is in equlibrium, the colour density profiles evolve asymptotically independently according to a linear diffusion equation.

In this paper we again consider the symmetric simple exclusion process, but not necessarily in equilibrium. Instead of proving propagation of chaos, we prove directly the hydrodynamical limit for the evolving colour profiles. These are shown to evolve asymptotically according to a coupled nonlinear diffusion equation. The method we use is an adaptation of the method developed by Varadhan for the non–gradient Ginzburg–Landau model [V2], which, in turn, is a refinement of the entropy method of Guo, Papanicolaou and Varadhan [G–P–V].

The model has two characteristics which conspire to make the proof of the hydrodynamic limit difficult. First, the system is of *non–gradient* type. In this situation the usual idea of replacing microscopic information by averages with respect to a slowly varying local equilibrium does not suffice. Lower order perturbations actually contribute to the limit, and one has to somehow show that the correct state is the one which minimises the local entropy production. In particular, we found it necessary to prove the spectral gap for the process. This estimate is in Section .

Second, the generator of the process is *degenerate* in the sense that considered as a speed change model the rates often vanish. Because of this, many of the standard estimates involving the Dirichlet form (see [G–P–V] or [K–O–V]) are not available. We get around this problem by using a kind of penalisation. These estimates are contained in Sections and .

For simplicity, the theorem has been stated and proved for the system with only two colours. The generalisation to any finite number of colours is immediate.

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2. Notation and results

The symmetric simple exclusion process is described as follows: For each positive integer N, let T_N be the periodic lattice $\{(1,\ldots,N)^d$, with addition modulo N$\}$. We think of $\frac{1}{N}T_N = \{\frac{u}{N} : u \in T_N\}$ as embedded in T^d, the standard d–torus. Particles are distributed on T_N and perform a simply excluded random walk: Each particle is held for an exponential (mean $1/N^2$) time, then chooses a nearby site according to a symmetric law $p(v)$ with finite range. If the site is empty the particle jumps there; if not, the jump is suppressed. We assume, for simplicity, that $p > 0$ for the jump of length 1 in any direction.

In this paper we consider the process in which two species of mechanically identical particles are distinguished, say blue particles and white particles. A configuration can be represented by $\eta \in X_N = \{-1, 0, 1\}^{T_N}$, where

$$\eta(u) = \begin{cases} 1 & \text{if there is a blue particle at u} \\ 0 & \text{if there is no particle at u} \\ -1 & \text{if there is a white particle at u} \end{cases}$$

or by the two functions $\beta(u) = \mathbf{1}_{\eta(u)=1}$ and $\omega(u) = \mathbf{1}_{\eta(u)=-1}$. The system is a Markov process with state space X_N and generator

$$\mathcal{L}_N f = N^2 \sum_{u,v \in T_N} p(v) a(u, u+v) D^{u,u+v} f(\eta)$$

where $a(u, u+v) = \mathbf{1}_{\eta(u)\eta(u+v)=0}$ represents the exclusion and

$$D^{u,u+v} f(\eta) = f(\eta^{u,u+v}) - f(\eta)$$

where

$$\eta^{u,u+v}(x) = \begin{cases} \eta(x) & \text{if } x \neq u \text{ or } u+v \\ \eta(u+v) & \text{if } x = u \\ \eta(u) & \text{if } x = u+v. \end{cases}$$

Let π^N be the measure on X_N which gives mass $\frac{1}{3^{N^d}}$ to each configuration. $\mathcal{L}_N$ is symmetric with respect to π^N and the Dirichlet form is given by

$$\mathcal{D}_N(f) = -\frac{1}{N^2} E^{\pi^N}[f \mathcal{L}_N f] = \frac{1}{2} E^{\pi^N}[\sum_{u,u+v} p(v) a(u, u+v)(D^{u,u+v} f)^2].$$

Let $\delta_{u/N}$ be the measure giving mass 1 to the point $\frac{u}{N} \in T^d$, and define profile measures,

$$\mu_\beta^N(\eta) = \frac{1}{N^d} \sum_{u \in T_N} \beta(u) \cdot \delta_{u/N} \quad \text{ and } \quad \mu_\omega^N(\eta) = \frac{1}{N^d} \sum_{u \in T_N} \omega(u) \cdot \delta_{u/N}.$$

Definition 1. *A sequence of densities f_N is said to have macroscopic profile $\rho_\beta(\theta), \rho_\omega(\theta)$ if for all $\delta > 0$ and smooth test functions ϕ and ψ,*

$$\lim_{N \to \infty} P^{f_N \pi^N}[\,|\int_{T^d} \phi(\theta) \mu_\beta^N(d\theta) - \int_{T^d} \phi(\theta) \rho_\beta(\theta) d\theta\,| \\ + |\int_{T^d} \psi(\theta) \mu_\omega^N(d\theta) - \int_{T^d} \psi(\theta) \rho_\omega(\theta) d\theta\,| > \delta] = 0$$

and we write $f_N \sim \rho_\beta(\theta), \rho_\omega(\theta)$.

The main result is that the colour density profiles have a hydrodynamic limit.

Theorem 2.1. *Suppose the process has initial distribution $f_N^0 \pi^N$ where*

$$f_N^0 \sim {\rho_\beta}^0(\theta), {\rho_\omega}^0(\theta)$$

where ${\rho_\beta}^0 + {\rho_\omega}^0 < 1$. Then at a later time t it has distribution $f_N^t \pi^N$ with $f_N^t \sim \rho_\beta(t,\theta), \rho_\omega(t,\theta)$ where $\rho_\beta(t,\theta), \rho_\omega(t,\theta)$ is the solution of

$$\begin{aligned}
\frac{\partial}{\partial t}\begin{pmatrix} \rho_\beta \\ \rho_\omega \end{pmatrix} &= \frac{1}{2}\nabla\cdot\begin{pmatrix} \frac{\rho_\omega}{\rho_\beta+\rho_\omega}D_s + \frac{\rho_\beta}{\rho_\beta+\rho_\omega}D & \frac{\rho_\beta}{\rho_\beta+\rho_\omega}(D-D_s) \\ \frac{\rho_\omega}{\rho_\beta+\rho_\omega}(D-D_s) & \frac{\rho_\beta}{\rho_\beta+\rho_\omega}D_s + \frac{\rho_\omega}{\rho_\beta+\rho_\omega}D \end{pmatrix}\nabla\begin{pmatrix} \rho_\beta \\ \rho_\omega \end{pmatrix} \\
\rho_\beta(0,\theta) &= {\rho_\beta}^0(\theta) \\
\rho_\omega(0,\theta) &= {\rho_\omega}^0(\theta). \qquad (2.1)
\end{aligned}$$

Here D is the $d \times d$ matrix with entries $D_{ij} = 2\sum_v p(v)v_i v_j$, and $D_s = D_s(\rho_\beta + \rho_\omega)$ is the self–diffusion covariance. Both are described below.

Define the colourblind macroscopic profile $\rho = \rho_\beta + \rho_\omega$. Under the assumptions of the theorem, it is well known that $\rho(t,\theta)$ evolves according to the heat equation,

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\nabla \cdot D\nabla\ \rho, \qquad \rho(0,\theta) = \rho^0(\theta). \qquad (2.2)$$

See, for example, [K–O–V].

To describe D_s we construct a new process: On Z^d infinitely many particles perform the simply excluded random walk. The particles are indistinguishable except for one test particle. If the state of the system is described by the position x of the test particle, and the environment $\zeta \in \{0,1\}^{Z^d - \{0\}}$ as seen from the test particle, the generator of the (x,ζ) process is given by

$$\begin{aligned}
\tilde{\mathcal{L}}f(x,\zeta) &= 2\sum_{v\neq 0} p(v)(1-\zeta(v))(f(x+v,\tau_{-v}\zeta) - f(x,\zeta)) \\
&\quad + \sum_{u,u+v\neq 0} p(v)(f(x,\zeta^{u,u+v}) - f(x,\zeta)).
\end{aligned}$$

The ζ process alone is reversible with respect to π_ρ, the product measure with constant density ρ. Let P_ρ^0 be the measure generated by starting x at 0 and the ζ process in π_ρ. Kipnis and Varadhan have proved the following central limit theorem for the test particle.

Theorem 2.2. *The rescaled test particle, $\frac{1}{N}x(N^2 t)$, converges in the Skorohod space to a diffusion with covariance*

$$(l, D_s(\rho)l) = \lim_{t\to\infty} \frac{E^{P_\rho^0}(l\cdot x(t))^2}{t}.$$

Based on this we can give a heuristic derivation of Theorem 0.1: For large N a particle at θ acts as if it were a test particle in Z^d with the other particles distributed according to $\pi_{\rho(\theta)}$. In addition, it is subject to a drift, b, caused by the evolution of the entire system towards equilibrium. Thus in the limit one expects it to converge to a diffusion with generator

$$\mathcal{L} = \frac{1}{2}D_s\nabla\cdot\nabla + b\cdot\nabla.$$

Macroscopic profiles should evolve according to the forward equations

$$\frac{\partial\rho}{\partial t} = \mathcal{L}^*\rho \tag{2.3}$$

$$\frac{\partial\rho_\beta}{\partial t} = \mathcal{L}^*\rho_\beta \tag{2.4}$$

$$\frac{\partial\rho_\omega}{\partial t} = \mathcal{L}^*\rho_\omega. \tag{2.5}$$

Since ρ also satisfies the heat equation (0.2), (0.3) gives

$$\frac{1}{2}\nabla\cdot\nabla D_s\rho - \nabla\cdot b\rho = \frac{1}{2}\nabla\cdot D\nabla\rho,$$

which forces

$$b = \frac{1}{2}\frac{\nabla((D_s - D)\rho)}{\rho}.$$

Then (0.4) and (0.5) become

$$\frac{\partial \rho_\beta}{\partial t} = \frac{1}{2}\nabla\cdot(\nabla(D_s\rho_\beta) - \frac{\nabla((D_s-D)\rho)}{\rho}\rho_\beta) \tag{2.6}$$
$$\frac{\partial \rho_\omega}{\partial t} = \frac{1}{2}\nabla\cdot(\nabla(D_s\rho_\omega) - \frac{\nabla((D_s-D)\rho)}{\rho}\rho_\omega), \tag{2.7}$$

which is equivalent to (0.1). Notice that if ρ is constant then these equations are linear:

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho_\beta\\ \rho_\omega\end{pmatrix} = \frac{1}{2}\nabla\cdot\begin{pmatrix}D_s(\rho) & 0\\ 0 & D_s(\rho)\end{pmatrix}\nabla\begin{pmatrix}\rho_\beta\\ \rho_\omega\end{pmatrix}.$$

Remark on ergodicity. Consider the "hyperplane" of $\rho_\beta N$ blue and $\rho_\omega N$ white particles, where $\rho_\beta, \rho_\omega, \rho_\beta + \rho_\omega \in [0,1]$ and $\rho_\beta N, \rho_\omega N$ are integers, and let $u^N_{\rho_\beta,\rho_\omega}$ denote the uniform measure on this set. The hyperplane is preserved by the process, and it is reversible with respect to $u^N_{\rho_\beta,\rho_\omega}$. If the model is not the one dimensional nearest neighbour, and if $\rho_\beta + \rho_\omega < 1$, then the process is ergodic with respect to $u^N_{\rho_\beta,\rho_\omega}$. For the 1–d nearest neighbour case this is not true. Furthermore, in this case, the self–diffusion constant vanishes. Nevertheless, Theorem 0.1 is true as stated. In this case the heuristic derivation above can easily be made into a rigorous proof. In all the other cases $D_s > 0$, and it is hard to make the heuristic derivation rigorous, so we use a completely different method. We will assume in the following that $p(2) > 0$ for the 1–d model.

Remark on uniqueness. From the form of the equations (0.6), (0.7) we see that the system of equations is coupled only through the total density $\rho = \rho_\beta + \rho_\omega$. Since ρ is evolving according to the heat equation this immediately gives us uniqueness of positive solutions to the system (0.6), (0.7), and therefore to (0.1).

3. Homogenisation formula

Let $M(a,b) = E^{\pi^N}[\exp(a\beta(u) + b\omega(u))]$. Then

$$h(\rho_\beta, \rho_\omega) = \rho_\beta \log \rho_\beta + \rho_\omega \log \rho_\omega + (1 - \rho_\beta - \rho_\omega) \log(1 - \rho_\beta - \rho_\omega)$$

is the convex conjugate of $\log M(a,b)$ and we have conjugate coordinates

$$\lambda_\beta = \log \frac{\rho_\beta}{1-\rho} = \frac{\partial h}{\partial \rho_\beta} \quad \text{and} \quad \lambda_\omega = \log \frac{\rho_\omega}{1-\rho} = \frac{\partial h}{\partial \rho_\omega}.$$

Then

$$\nabla \begin{pmatrix} \rho_\beta \\ \rho_\omega \end{pmatrix} = (h'')^{-1} \nabla \begin{pmatrix} \lambda_\beta \\ \lambda_\omega \end{pmatrix} = \begin{pmatrix} \rho_\beta(1-\rho_\beta)I & -\rho_\beta\rho_\omega I \\ -\rho_\beta\rho_\omega I & \rho_\omega(1-\rho_\omega)I \end{pmatrix} \nabla \begin{pmatrix} \lambda_\beta \\ \lambda_\omega \end{pmatrix}.$$

The $2d \times 2d$ matrix h'' is known as the compressibility. In this notation (0.1) becomes

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho_\beta \\ \rho_\omega \end{pmatrix} = \frac{1}{2} \nabla \cdot A \nabla \begin{pmatrix} \lambda_\beta \\ \lambda_\omega \end{pmatrix},$$

where

$$A = \begin{pmatrix} \frac{\rho_\beta\rho_\omega}{\rho} D_s + \frac{\rho_\beta{}^2(1-\rho)}{\rho} D & \frac{-\rho_\beta\rho_\omega}{\rho} D_s + \frac{\rho_\beta\rho_\omega(1-\rho)}{\rho} D \\ \frac{-\rho_\beta\rho_\omega}{\rho} D_s + \frac{\rho_\beta\rho_\omega(1-\rho)}{\rho} D & \frac{\rho_\beta\rho_\omega}{\rho} D_s + \frac{\rho_\omega{}^2(1-\rho)}{\rho} D \end{pmatrix}$$

The advantage here is that A is symmetric. The associated quadratic form on $R^d \times R^d$ is

$$((a,b), A(a,b)) = \frac{\rho_\beta\rho_\omega}{\rho}((a-b), D_s(a-b)) + \rho(1-\rho)((\frac{\rho_\beta}{\rho}a + \frac{\rho_\omega}{\rho}b), D(\frac{\rho_\beta}{\rho}a + \frac{\rho_\omega}{\rho}b)).$$

There is a variational formula for D_s. For the (x,ζ) process define the Dirichlet form

$$\tilde{\mathcal{D}}_\rho(f) = E^{dx\otimes\pi_\rho}[\sum_{v\neq 0} p(v)(1-\zeta(v))(f(x+v, \tau^{-v}\zeta) - f(x,\zeta))^2$$

$$+ \frac{1}{2} \sum_{u,u+v\neq 0} p(v)(f(x,\zeta^{u,u+v}) - f(x,\zeta))^2],$$

where dx is the measure on Z^d which gives each site mass 1. The next theorem is a consequence of results in [K–V]. It also appears in [S1].

Theorem 3.1. *The self–diffusion covariance $D_s(\rho)$ is given by the variational formula*

$$(l, D_s(\rho)l) = 2 \inf_{\psi=\psi(\zeta)} \tilde{\mathcal{D}}_\rho(l \cdot x + \psi).$$

The main result of this section is a variational formula for A. We start with some notation which will be used here and later. Let $\pi_{\rho_\beta,\rho_\omega}$ be the product measure on $X = \{-1, 0, 1\}^{Z^d}$ with

$$E^{\pi_{\rho_\beta,\rho_\omega}}[\beta(u)] = \rho_\beta \quad \text{and} \quad E^{\pi_{\rho_\beta,\rho_\omega}}[\omega(u)] = \rho_\omega.$$

A *cylinder function* is a function on X which depends only on $\{\eta(u)\}_{u\in F}$ for some finite set $F \subset Z^d$. Let $S = \{v \in R^d : p(v) > 0\}$. For a cylinder function g the function $\mathbf{f}_g : X \to R^S$ with coordinates

$$f_g^v = D^{0,v} \sum_{u\in Z^d} \tau^u g$$

is well defined. Any function $\mathbf{f} : X \to R^S$ which can be written $\mathbf{f} = \mathbf{f}_g$ for some cylinder function g will be called *exact*. Let $\mathcal{E}_0$ denote the set of exact functions. Define

$$\mathcal{H} = \{\mathbf{f} : E^{\pi_{\rho_\beta,\rho_\omega}}[\sum_v p(v)a(0,v)(f^v)^2] < \infty\} \tag{3.1}$$

and $\mathcal{T} \subset \mathcal{H}$ to be the closure in $\mathcal{H}$ of the linear span of functions of the form $\sum_u (a\beta(u) + b\omega(u))\psi_u^v(\zeta)$, where ψ_u^v are colourblind functions. Let $\mathcal{T}^\perp$ be the orthogonal complement of $\mathcal{T}$ in $\mathcal{H}$, and define the operator $D(\mathbf{f}) = \{D^{0,v} f^v\}_{v\in S}$. The subspace $\mathcal{T}$ is preserved by the evolution in the sense that $D\mathcal{T} \subset \mathcal{T}$. Since D is symmetric we also have $D\mathcal{T}^\perp \subset \mathcal{T}^\perp$. For a cylinder function g define $Q_g(\rho_\beta, \rho_\omega, a, b)$ to be the function

$$E^{\pi_{\rho_\beta,\rho_\omega}}\Big[\sum_v p(v)a(0,v)\big(v\cdot a\,(\beta(v) - \beta(0)) + v\cdot b\,(\omega(v) - \omega(0)) + f_g^v\big)^2\Big]. \tag{3.2}$$

Theorem 3.2. *The quadratic form $((a,b), A(a,b))$ is given by the variational formula*

$$((a,b), A(a,b)) = \inf_g Q_g(a,b).$$

Where Q_g is given by (0.9) and the infimum is taken over cylinder functions g.

Proof. Define, on $(X, \pi_{\rho_\beta, \rho_\omega})$, the functions

$$\chi(v) = \frac{\rho_\omega}{\rho}\beta(v) - \frac{\rho_\beta}{\rho}\omega(v) \quad \text{and} \quad \zeta(v) = \beta(v) + \omega(v).$$

These are orthogonal, and in fact,

$$E[\chi(u)f(\eta)] = 0 \tag{3.3}$$

for any function f which does not depend on the colour at the site u, but only on the presence of a particle there, for such a function can be written

$$\zeta(u)f_1(\eta) + (1 - \zeta(u))f_2(\eta)$$

where f_1 and f_2 are independent of $\eta(u)$. Also we have for any f_g and $u \in Z^d$,

$$E[(\zeta(u+v) - \zeta(u))\tau^u f_g^v] = 0 \tag{3.4}$$

for if the support of g contained in $[-m,m]^d$ then $D^{u,u+v}\sum_x \tau^x g$ is a finite sum so

$$E[(\zeta(u+v) - \zeta(u))D^{u,u+v}\sum \tau^x g]$$

$$= \sum E[(\zeta(u+v) - \zeta(u))D^{u,u+v}\tau^x g]$$

$$= \sum E[(\zeta(u+v) - \zeta(u))\tau^x g]$$

$$= E[\sum(\zeta(u+v-x) - \zeta(u-x))g] = 0$$

because the variables within the support of g cancel. Our third remark is that

$$a(u, u+v)(\zeta(u+v) - \zeta(u)) = (\zeta(u+v) - \zeta(u)) \tag{3.5}$$

Using (0.10), (0.11), (0.12) we can calculate

$$\inf_g E[\sum_v p(v)a(0,v)(v\cdot a(\beta(v)-\beta(0))+v\cdot b(\omega(v)-\omega(0))+f_g^v)^2]$$

$$=\inf_g E[\sum_v p(v)a(0,v)(v\cdot(a-b)(\chi(v)-\chi(0))$$

$$+v\cdot(\frac{\rho_\beta}{\rho}a+\frac{\rho_\omega}{\rho}b)(\zeta(v)-\zeta(0))+f_g^v)^2]$$

$$=\inf_g E[\sum_v p(v)a(0,v)(v\cdot(a-b)(\chi(v)-\chi(0))+f_g^v)^2]$$

$$+2\sum_v p(v)(v\cdot(\frac{\rho_\beta}{\rho}a+\frac{\rho_\omega}{\rho}b))^2\rho(1-\rho)$$

So it remains to show

$$\inf_g E[\sum_v p(v)a(0,v)(v\cdot(a-b)(\chi(v)-\chi(0))+f_g^v)^2]=\frac{\rho_\beta\rho_\omega}{\rho}((a-b),D_s(a-b))$$

Since $v\cdot(a-b)(\chi(v)-\chi(0))\in\mathcal{T}$, by the remarks prior to the Theorem the infimum over $\mathcal{T}\cap\mathcal{E}_0$ is the same as over $\mathcal{E}_0$. So the above is equal to

$$\inf_{\psi(\zeta)} E[\sum_v p(v)a(0,v)(v\cdot(a-b)(\chi(v)-\chi(0))+D^{0,v}(\sum_u \chi(u)\tau^u\psi))^2]$$

which, by direct calculation,

$$=\inf_{\psi(\zeta)} E[\sum_v p(v)a(0,v)(v\cdot(a-b)(\chi(v)-\chi(0))+\chi(v)\psi(\zeta^{0,v})$$

$$-\chi(0)\psi(\zeta)+\chi(0)\tau^v\psi(\zeta^{0,v})-\chi(v)\tau^v\psi(\zeta)+\sum_{u\neq 0,v}\chi(u)\tau^u D^{u,u+v}\psi)^2].$$

By using (0.12) and then (0.10) this can be reduced to

$$\inf_{\psi(\zeta)} E[\sum_v p(v)((1-\zeta(0))\chi^2(v)(\psi(\zeta^{0,v})-\tau^v\psi(\zeta)+v\cdot(a-b))^2$$

$$+(1-\zeta(v))\chi^2(0)(\tau^v\psi(\zeta^{0,v})-\psi(\zeta)-v\cdot(a-b)))^2$$

$$+\sum_v p(v)\sum_{u\neq 0,v}\chi^2(u)(\tau^u D^{u,u+v}\psi)^2].$$

We can take $\psi(\zeta)$ to be independent of $\zeta(0)$, so this is equal to

$$\frac{\rho_\beta\rho_\omega}{\rho}\inf_{\psi(\zeta)} E[\sum_v p(v)((1-\zeta(-v))(\tau^v\psi-\psi+v\cdot(a-b))^2$$

$$+(1-\zeta(v))(\tau^{-v}\psi-\psi-v\cdot(a-b)))^2+\sum_v p(v)\sum_{u\neq 0,v}(\tau^u D^{u,u+v}\psi)^2].$$

By Theorem 0.3, this is equal to $\frac{\rho_\beta\rho_\omega}{\rho}((a-b),D_s(a-b))$, as desired. □

4. Compactness

Let $\phi(\theta)$ and $\psi(\theta)$ be smooth test functions on the torus, T^d. We write a stochastic differential equation for $\frac{1}{N^d}\sum_u \phi(\frac{u}{N})\beta(u)+\psi(\frac{u}{N})\omega(u)$:

$$d(\frac{1}{N^d}\sum_u \phi(\frac{u}{N})\beta(u)+\psi(\frac{u}{N})\omega(u))$$

$$=-\frac{1}{N^d}\sum_{u,v}a(u,u+v)((\phi(\frac{u-v}{N})-\phi(\frac{u}{N}))(\beta(u+v)-\beta(u))$$

$$+(\psi(\frac{u-v}{N})-\psi(\frac{u}{N}))(\omega(u+v)-\omega(u)))(N^2p(v)dt+dM^{N^2p(v)}_{u,u+v})$$

$$=-\frac{1}{N^{d+1}}\sum_{u,v}a(u,u+v)(v\cdot\nabla\phi(\frac{u}{N})(\beta(u+v)-\beta(u))$$

$$+v\cdot\nabla\psi(\frac{u}{N})(\omega(u+v)-\omega(u)))(N^2p(v)dt+dM^{N^2p(v)}_{u,u+v})+o(1)$$

summing by parts once, where $M^{N^2p(v)}_{u,u+v}(t)=P^{N^2p(v)}_{u,u+v}(t)-N^2p(v)t$, $P^{N^2p(v)}_{u,u+v}$ independent Poisson processes with intensity $N^2p(v)$. Notice that it is not possible to sum by parts a second time, i.e. the model is *non–gradient.* If we define the following vector valued measures on the torus,

$$
\begin{aligned}
\gamma_\beta^N &= \frac{N}{N^d}\sum_{u,v} p(v)v\int_0^t a(u,u+v)(\beta_s(u+v)-\beta_s(u))ds\cdot\delta_{\frac{u}{N}}\\
\gamma_\omega^N &= \frac{N}{N^d}\sum_{u,v} p(v)v\int_0^t a(u,u+v)(\omega_s(u+v)-\omega_s(u))ds\cdot\delta_{\frac{u}{N}}\\
\nu_\beta^N &= \frac{1}{N^{d+1}}\sum_{u,v} v\int_0^t a(u,u+v)(\beta_s(u+v)-\beta_s(u))dM_{u,u+v}^{N^2p(v)}\cdot\delta_{\frac{u}{N}}\\
\nu_\omega^N &= \frac{1}{N^{d+1}}\sum_{u,v} v\int_0^t a(u,u+v)(\omega_s(u+v)-\omega_s(u))dM_{u,u+v}^{N^2p(v)}\cdot\delta_{\frac{u}{N}},
\end{aligned}
$$

then the stochastic differential equation can be written in integrated form

$$
\begin{aligned}
&(<\phi,\mu_\beta^N(t)> + <\psi,\mu_\omega^N(t)>) - (<\phi,\mu_\beta^N(0)> + <\psi,\mu_\omega^N(0)>)\\
&= - < (\nabla\phi,\gamma_\beta^N(t)) > - < (\nabla\psi,\gamma_\omega^N(t)) >\\
&- < (\nabla\phi,\nu_\beta^N(t)) > - < (\nabla\psi,\nu_\omega^N(t)) > + o(1). \qquad (4.1)
\end{aligned}
$$

Let Q_N be the measure on $D([0,T]\to X_N)$ given by the non–equilibrium process, with initial distribution $f_N^0\pi^N$, and let P_N be the measure given by the equilibrium process, with initial distribution π^N. These are related by a simple convexity inequality,

$$
E^{Q_N}[f] \le \frac{1}{N^d}\log E^{P_N}[\exp N^d f] + \frac{H(Q_N;P_N)}{N^d},
$$

called the entropy inequality. It is not hard to estimate

$$
\frac{H(Q_N;P_N)}{N^d} \le \log 3.
$$

Let $\mathcal{M}(T^d)$ denote the signed measures, and $H_n(T^d)$ the Sobolev space of order n. Q_N and P_N induce measures $\tilde{Q}_N$ and $\tilde{P}_N$ on $D([0,T]\to\mathcal{M}(T^d)^{4d+2})$, the set of $(\mu_\beta^N,\mu_\omega^N,\gamma_\beta^N,\gamma_\omega^N,\nu_\beta^N,\nu_\omega^N)$. If $k>d/2$ then $\mathcal{M}(T^d)\subset H_{-k}(T^d)$. If we prove tightness only in some $H_{-k}(T^d)$ we do not really lose any generality. The μ_β^N and μ_ω^N are probability measures so if they converge to measures whose densities are the solutions of (0.1) weakly in $H_{-k}(T^d)$, they also converge weakly in $\mathcal{M}(T^d)$. Before we start we need two estimates.

Lemma 1. *For ν^N either ν^N_β or ν^N_ω,*

$$E^{\tilde{Q}_N}[\sup_{0\le t\le T} <\phi,\nu^N(t)>^2] \le \frac{1}{N^d} CT \frac{1}{N^d}\sum_u \phi^2(\frac{u}{N}).$$

Proof.

$$E^{\tilde{Q}_N}[(\frac{1}{N^d}\sum_u \phi(\frac{u}{N})\int_0^T a(u,u+v)(\beta(u+v)-\beta(u))dM_{u,u+v})^2]$$

$$= \frac{1}{N^{2d}}\sum_u \phi^2(\frac{u}{N})\int_0^T E^{\tilde{Q}_N}[a(u,u+v)(\beta(u+v)-\beta(u))^2]dt$$

$$\le \frac{1}{N^d} CT \frac{1}{N^d}\sum_u \phi^2(\frac{u}{N}) \to 0 \quad as \quad N\to\infty,$$

so $E^{\tilde{Q}_N}[<\phi,\nu^N(T)>^2]\to 0$. By Doob's inequality, since $<\phi,\nu^N(t)>$ is a martingale,

$$E^{\tilde{Q}_N}[\sup_{0\le t\le T} <\phi,\nu^N(t)>^2] \le 4E^{\tilde{Q}_N}[<\phi,\nu^N(T)>^2],$$

which proves the lemma for ν^N_β. The proof for ν^N_ω is analogous. □

Lemma 2. *For γ^N either γ^N_β or γ^N_ω,*

$$E^{\tilde{P}_N}[\exp N^d\alpha \mid <\phi,\gamma^N(t)> - <\phi,\gamma^N(s)> \mid] \le 2\exp \mid t-s \mid \alpha^2 \sum_u \phi^2(\frac{u}{N})$$

Proof. By stationarity, if $t>s$,

$$E^{\tilde{P}_N}[\exp N^d\alpha \mid <\phi,\gamma^N(t)> - <\phi,\gamma^N(s)> \mid]$$

$$= E^{\tilde{P}_N}[\exp N^d\alpha \mid <\phi,\gamma^N(t-s)> \mid]$$

$$\le \exp(t-s)\Lambda_N(\phi)$$

by the spectral theorem and Feynman–Kac formula, where

$$\Lambda_N(\phi) = \sup_{densities\ f} \{N\alpha \sum_u \phi(\frac{u}{N}) E^{f\pi^N}[a(u,u+v)(\beta(u+v)-\beta(u))]$$

$$-\frac{N^2}{2}\mathcal{D}_N(\sqrt{f})\}$$

is the principal eigenvalue of $\mathcal{L}_N + \frac{N}{2}\alpha\sum_u \phi(\frac{u}{N})a(u,u+v)(\beta(u+v)-\beta(u))$. Now, for the expectation with respect to π^N,

$$E[a(u,u+v)(\beta(u+v)-\beta(u))f(\eta)]$$

$$= \tfrac{1}{2}E[a(u,u+v)(\beta(u+v)-\beta(u))(f(\eta^{u,u+v})-f(\eta))]$$

$$= \tfrac{1}{2}E[a(u,u+v)(\beta(u+v)-\beta(u))(\sqrt{f}(\eta^{u,u+v})+\sqrt{f}(\eta))(\sqrt{f}(\eta^{u,u+v})-\sqrt{f}(\eta))]$$

$$\leq \tfrac{1}{2}\sqrt{E[(\beta(u+v)-\beta(u))^2 2(f(\eta^{u,u+v})+f(\eta))]E[a(u,u+v)(D^{u,u+v}(\sqrt{f}))^2]}$$

$$\leq 2\sqrt{E^{\pi^N}[a(u,u+v)(D^{u,u+v}(\sqrt{f}))^2]},$$

so

$$\Lambda_N(\phi) \leq \sup_{densities\ f} \{\sum_u \alpha\phi(\frac{u}{N}) N\sqrt{E^{\pi^N}[a(u,u+v)(D^{u,u+v}(\sqrt{f}))^2]}$$

$$-\frac{N^2}{2}E^{\pi^N}[a(u,u+v)(D^{u,u+v}(\sqrt{f}))^2\}$$

$$\leq C\alpha^2 \sum_u \phi^2(\frac{u}{N})$$

which proves the lemma for γ_β^N. The proof for γ_ω^N is analogous. □

Theorem 4.1. *$\tilde{Q}_N$ is tight as a sequence of measures on*

$$D([0,T] \to (H_{-(d+2)}(T^d))^{4d+2}).$$

Proof. It suffices to prove

$$I.\ \lim_{l\to\infty}\overline{\lim_{N\to\infty}}\,\tilde{Q}_N(\sup_{0\le t\le T}\|v(t)\|_{H_{-(d+1)}}\ge l)=0$$

and, for every $\epsilon>0$,

$$II.\ \lim_{\delta\to 0}\overline{\lim_{N\to\infty}}\,\tilde{Q}_N(\sup_{\substack{0\le s<t\le T\\ |s-t|<\delta}}\|v(t)-v(s)\|_{H_{-(d+2)}}>\epsilon)=0$$

for v equals each of $i.\ \mu^N_\beta,\quad ii.\ \mu^N_\omega,\quad iii.\ \{\gamma^N_\beta\}_i,\quad iv.\ \{\gamma^N_\omega\}_i,\quad v.\ \{\nu^N_\beta\}_i,\quad vi.\ \{\nu^N_\omega\}_i$ (see, for example, [S–V]). For i and ii

$$<\phi,\mu^N>\le\int_{T^d}\phi(\theta)d\theta\le\sqrt{\int_{T^d}\phi^2(\theta)d\theta}$$

so I is obvious.

Next we prove I and II for v and vi. Consider the functions $f_n(\theta)=\exp 2\pi i n\cdot\theta$ for $n=(n_1,\ldots,n_d)$, n_j positive integers, which form an orthonormal basis of $L^2(T^d)$. The functions

$$f^k_n=\frac{f_n}{(1+n_1^2+\cdots+n_d^2)^{\frac{k}{2}}}$$

form an orthonormal basis of $H_{-k}(T^d)$. Using Lemma 1 we get

$$\begin{aligned}E^{\tilde{Q}_N}[\sup_{0\le t\le T}\|\nu^N(t)\|^2_{H_{-(d+1)}}] &= E^{\tilde{Q}_N}[\sum_n\sup_{0\le t\le T}<f^{d+1}_n\nu^N(t)>^2]\\ &\le \frac{1}{N^d}CT\sum_n\frac{1}{N^d}\sum_u(f^{d+1}_n(\frac{u}{N}))^2\\ &\le \frac{1}{N^d}CT\sum_n\frac{1}{(1+n_1^2+\cdots+n_d^2)^{d+1}}\to 0.\end{aligned}$$

I, II for v, vi follow from this by Tchebyshev's inequality.

Now we prove I and II for iii and iv. To make use of Lemma 2 we need the lemma of Garsia, Rodemich, and Rumsey ([S–V] p.47) which states that for any continuous function $f(t)$ if $\Psi(x)$ is a strictly increasing function with $\Psi(0)=0$ and $\lim_{x\to\infty}\Psi(x)=\infty$ and

$$B=\int_0^T\int_0^T\Psi(\frac{|f(t)-f(s)|}{|t-s|^{1/2}})dsdt$$

then for $0 \leq s < t \leq T$

$$| f(t) - f(s) | \leq 4 \int_0^{(t-s)} \Psi^{-1}(\frac{4B}{u^2}) \frac{du}{\sqrt{u}}.$$

We will take

$$\Psi(x) = \exp(N^d \alpha x) - 1.$$

By Lemma 2

$$E^{\tilde{P}_N}[B] \leq T^2(2 \exp(\alpha^2 \sum_u \phi^2(\frac{u}{N})) - 1).$$

For this Ψ it is not hard to show that for $(t-s) \leq 1$

$$4 \int_0^{(t-s)} \Psi^{-1}(\frac{4B}{u^2}) \frac{du}{\sqrt{u}} \leq \frac{8\sqrt{t-s}}{N^d \alpha} \log(4B + (t-s)^2).$$

Then we can calculate

$$E^{\tilde{P}_N}[\exp \frac{N^d \alpha}{8\sqrt{\delta}} \sup_{\substack{0 \leq s < t \leq T \\ |t-s| < \delta}} |< \phi, \gamma^N(t) > - < \phi, \gamma^N(s) >|]$$

$$\leq E^{\tilde{P}_N}[4B + \delta^2]$$

$$\leq 8T^2 \exp(\alpha^2 \sum_u \phi^2(\frac{u}{N}))$$

Integrating over α we get

$$E^{\tilde{P}_N}[\exp \frac{N^d}{C\delta} \sup_{\substack{0 \leq s < t \leq T \\ |t-s| < \delta}} \frac{|< \phi, \gamma^N(t) > - < \phi, \gamma^N(s) >|^2}{\frac{1}{N^d} \sum_u \phi^2(\frac{u}{N})}] \leq 8T^2.$$

From this, using the entropy inequality, we can get

$$E^{\tilde{Q}_N}[\sup_{\substack{0 \leq s < t \leq T \\ |t-s| < \delta}} |< \phi, \gamma^N(t) > - < \phi, \gamma^N(s) >|^2] \leq C\delta \frac{1}{N^d} \sum_u \phi^2(\frac{u}{N}).$$

We can estimate the $H_{-(d+1)}$ norm as before

$$E^{\tilde{Q}_N}[\sup_{\substack{0\le s<t\le T\\ |t-s|<\delta}} \|\gamma^N(t)-\gamma^N(s)\|^2_{H_{-(d+1)}}]$$

$$= E^{\tilde{Q}_N}[\sum_n \sup_{\substack{0\le s<t\le T\\ |t-s|<\delta}} |< f_n^{d+1}, \gamma^N(t) > - < f_n^{d+1}, \gamma^N(s) >|^2]$$

$$\le C\delta \sum_n \frac{1}{N^d}\sum_u (f_n^{d+1}(\frac{u}{N}))^2$$

$$\le C\delta \sum_n \frac{1}{(1+n_1^2+\cdots+n_d^2)^{d+1}} = C_d\delta < \infty.$$

I, II for iii, iv follow by Tchebyshev's inequality.

Finally to prove II for i and ii we use the stochastic differential equation and the result for iii, iv, v and vi. This completes the proof of compactness. □

5. Inequalities involving the Dirichlet form

Because of the exclusion, the operator $\mathcal{L}_N$ is degenerate, and we need some inequalities to keep things under control. These exploit the following simple fact: The exchange between sites x and y can be performed by a configuration independent sequence of excluded nearest neighbour exchanges, as long as two sites adjacent to x are unoccupied.

We start with some notation. For any l let $B_l(u)$ be the box of side length $2l+1$ centred at u. Let $\rho^l_\beta(u) = \frac{1}{(2l+1)^d}\sum_{x\in B_l(u)}\beta(u)$ and $\rho^l_\omega(u) = \frac{1}{(2l+1)^d}\sum_{x\in B_l(u)}\omega(u)$ be the densities of blue and white particles in that box. Also let $\rho^l = \rho^l_\beta + \rho^l_\omega$. For the uncoloured process the macroscopic profile ρ evolves according to the heat equation.

We have assumed for the initial profiles ${\rho_\beta}^0(\theta) + {\rho_\omega}^0(\theta) = \rho^0(\theta) < 1$. Fix $\delta > 0$ such that $\rho^0(\theta) < 1 - \delta$. Then at a later time we still have $\rho(t, \theta) < 1 - \delta$. Let A_k be the set of configurations where there are one or fewer unoccupied sites in $B_k(0)$. It then follows from results in [K–O–V] that

$$\overline{\lim_{k\to\infty}}\ \overline{\lim_{N\to\infty}}\ E^{Q_N}[\int_0^T \frac{1}{N^d} \sum_u \tau^u \mathbf{1}_{A_k} dt] = 0. \tag{5.1}$$

Lemma 3. *For any function f on X_N define $\bar{f} = \sum_u \tau^u f$. If $z \in B_l(0)$ for some $k < l < N$ then there is a universal constant C such that*

$$E^{\pi^N}[\mathbf{1}_{X_N \setminus A_k}(D^{0,z}(\sqrt{\bar{f}}))^2] \leq C \frac{k^{2d}(l + k^d)^2}{N^d} \mathcal{D}_N(\sqrt{\bar{f}})$$

Proof. To each configuration we will associate a sequence of bonds, $b_1, \ldots, b_n$, where $b_i = u_i, u_i + v_i$, with the following properties

i. $\eta^{b_1 \cdots b_n} = \eta^{0,z}$

ii. $p(v_i) > 0$

iii. $a(u_i, u_i + v_i)(\eta^{b_1 \cdots b_{i-1}}) = 1$

iv. No single bond is used more than 10 times.

Step I: For a configuration in $X_N \setminus A_k$ there must be at least two empty sites in B_k. Choose the closest unoccupied sites x_1, x_2 to 0 in the norm $\mid v \mid = max\{v_1, \ldots, v_d\}$ (if there is more than one possibility choose the candidate closest in dimension 1, then dimension 2, etc.) Move x_1 to $(1, 0, \ldots, 0)$ by first moving it in dimension d, then in dimension $d - 1$, etc. Next move x_2 to $(1, 1, 0, \ldots, 0)$ in the same way, using bonds $b_{m_0+1}, \ldots, b_{m_1}$.

Step II: Once two sites adjacent to 0 are unoccupied it is possible to choose a configuration independent sequence of bonds $b_{m_1+1}, \ldots, b_{m_2}$ such that $\eta^{b_{m_1} \cdots b_{m_2}} = \eta^{0,z}$.

This sequence can be assumed to use bonds only in the "corridor" of width 2 around the line joining 0 to z by first joining $(0,\dots,0)$ to $(0,\dots,0,z_d)$, then $(0,\dots,0,z_d)$ to $(0,\dots,z_{d-1},z_d)$, etc. This corridor has maximum length $dN\epsilon$.

Step III: Move the empty sites back where they came from by reversing the order of exchanges in step I.

Of course the sequence of bonds is random, but all the bonds used lie inside $B_k +$ *the corridor*, and the chain is less than $10(dl+k^d)$ bonds long. We have

$$E^\pi[(D^{0,z}\sqrt{\bar f})^2\mathbf{1}_{X_N\setminus A_k}]$$

$$= E^\pi[(\sum_{i=1}^n a(u_i,u_i+v_i)(\eta^{b_1\cdots b_{i-1}})D^{u_i,u_i+v_i}\sqrt{\bar f}(\eta^{b_1\cdots b_{i-1}}))^2]$$

$$\le 10(dl+k^d)E^\pi[\sum_{i=1}^n a(u_i,u_i+v_i)(\eta^{b_1\cdots b_{i-1}})(D^{u_i,u_i+v_i}\sqrt{\bar f}(\eta^{b_1\cdots b_{i-1}}))^2]$$

$$= 10(dl+k^d)\sum_{y_1,y_2\in B_k}E^\pi[\mathbf{1}_{\substack{y_1=x_1\\ y_2=x_2}}\sum_{i=1}^n a(u_i,u_i+v_i)(\eta^{b_1\cdots b_{i-1}})(D^{u_i,u_i+v_i}\sqrt{\bar f}(\eta^{b_1\cdots b_{i-1}}))^2]$$

$$\le 10(dl+k^d)\sum_{y_1,y_2\in B_k}E^\pi[\sum_{i=1}^n a(u_i,u_i+v_i)(\eta^{b_1\cdots b_{i-1}})(D^{u_i,u_i+v_i}\sqrt{\bar f}(\eta^{b_1\cdots b_{i-1}}))^2]$$

$$= 10(dl+k^d)\sum_{y_1,y_2\in B_k}E^\pi[\sum_{i=1}^n a(u_i,u_i+v_i)(\eta)(D^{u_i,u_i+v_i}\sqrt{\bar f}(\eta))^2]$$

$$\le 10(dl+k^d)\sum_{y_1,y_2\in[0,\dots,k]^d}E^\pi[\sum_{y,y+v\in[0,\dots,k]^d\ +\ corridor}a(y,y+v)(D^{y,y+v}(\sqrt{\bar f}))^2]$$

$$\le 10(dl+k^d)k^{2d}E^\pi[\sum_{y,y+v\in[0,\dots,k]^d\ +\ corridor}a(y,y+v)(D^{y,y+v}(\sqrt{\bar f}))^2]$$

$$\le \frac{C10K^{2d}(dl+k^d)^2}{N^d}\mathcal{D}_N(\sqrt{f}),$$

which proves the lemma. $\square$

Now we consider the action of $\mathcal{L}_N$ on the *non-periodic* hyperplane with $N^d\rho_\beta$ blue particles and $N^d\rho_\omega$ white particles, with uniform measure $u^N_{\rho_\beta,\rho_\omega}$. The Dirichlet form is

$$\mathcal{D}_N(f) = E^{u^N_{\rho_\beta,\rho_\omega}}[\sum_{u,v} p(v)a(u,u+v)(D^{u,u+v}f)^2].$$

Lemma 4. *For $1 \le p < 2$ and $\rho_\beta + \rho_\omega < 1$*

$$\sum_{u,v} E^{u^N_{\rho_\beta,\rho_\omega}}[p(v) \mid D^{u,u+v}f \mid^p] \le C_p N^{d(\frac{2-p}{2})}(\mathcal{D}_N(f))^{\frac{p}{2}}$$

where the constant C_p depends only on $\rho_\beta + \rho_\omega$ and p.

Proof. One can make the exchange $D^{u,u+v}$ by a configuration independent sequence of exchanges as long as two sites adjacent to u are unoccupied. So we find the closest two unoccupied sites x_1 and x_2 and move them by a sequence of exchanges to sites adjacent to u. We do this according to the same rule as in the previous lemma. The difference here is that we are no longer guaranteed unoccupied sites in a small box $B_k(u)$. We have, with respect to the uniform measure $u^N_{\rho_\beta,\rho_\omega}$

$$E[\mid D^{u,u+v}f \mid^p]$$

$$= \sum_{y_1,y_2} E[\mathbf{1}_{\substack{y_1=x_1\\ y_2=x_2}} \mid \sum_{i=1}^n a(u_i,u_i+v_i)(\eta^{b_1,\dots,b_{i-1}})D^{u_i,u_i+v_i}f(\eta^{b_1,\dots,b_{i-1}}) \mid^p]$$

$$\le \sum_{y_1,y_2} n^{p-1}\sum_{i=1}^n E[\mathbf{1}_{\substack{y_1=x_1\\ y_2=x_2}} a(u_i,u_i+v_i)(\eta^{b_1,\dots,b_{i-1}}) \mid D^{u_i,u_i+v_i}f(\eta^{b_1,\dots,b_{i-1}}) \mid^p]$$

$$\le \sum_{y_1,y_2} n^{p-1}\sum_{i=1}^n (P[y_1=x_1,y_2=x_2])^{\frac{2-p}{2}}$$

$$\times\, (E[a(u_i,u_i+v_i)(\eta^{b_1,\dots,b_{i-1}})(D^{u_i,u_i+v_i}f(\eta^{b_1,\dots,b_{i-1}}))^2])^{\frac{p}{2}}.$$

The path $b_1, \dots, b_n$ is uniquely determined by y_1, y_2 so, since $u^N_{\rho_\beta, \rho_\omega}$ is exchangeable, this is equal to

$$\sum_{y_1,y_2} n^{p-1}(P[y_1 = x_1, y_2 = x_2])^{\frac{2-p}{2}} \sum_{i=1}^{n} (E[a(u_i, u_i + v_i)(\eta)(D^{u_i,u_i+v_i} f(\eta))^2])^{\frac{p}{2}}$$

$$\leq \sum_{k_1 \leq k_2} \sum_{\substack{|y_1|=k_1 \\ |y_2|=k_2}} n^{p-1}(P[|x_1|=k_1, |x_2|=k_2])^{\frac{2-p}{2}} \sum_{i=1}^{n} (E[a(u_i, u_i + v_i)(D^{u_i,u_i+v_i} f)^2])^{\frac{p}{2}}.$$

It is not hard to show that $P[| x_1 |= k_1, | x_2 |= k_2]$ is less than $\rho^{-\frac{k_2^d}{2}}$ so

$$E[| D^{u,u+v} f |^p] \leq \sum_{k_1 \leq k_2} \sum_{\substack{|y_1|=k_1 \\ |y_2|=k_2}} n^{p-1} \rho^{-(\frac{2-p}{4}) k_2^d} \sum_{i=1}^{n} (E[a(u_i, u_i + v_i)(D^{u_i,u_i+v_i} f)^2])^{\frac{p}{2}}.$$

Since each bond is used less than 10 times and the chain is less than Ck_2 bonds long this is

$$\leq C \sum_{k_1 \leq k_2} \sum_{\substack{|y_1|=k_1 \\ |y_2|=k_2}} k_2^{p-1} \rho^{-(\frac{2-p}{4}) k_2^d} \sum_{y,y+v \in B_{k_2}(u)} (E[p(v) a(y, y + v)(D^{y,y+v} f)^2])^{\frac{p}{2}}$$

$$\leq C \sum_{k_2} k_2^{2d+p-2} \rho^{-(\frac{2-p}{4}) k_2^d} \sum_{y,y+v \in B_{k_2}(u)} (E[p(v) a(y, y + v)(D^{y,y+v} f)^2])^{\frac{p}{2}}.$$

Then

$$\sum_{u,u+v} E[p(v) \mid D^{u,u+v} f \mid^p]$$

$$\leq C \sum_{u} \sum_{k_2} k_2^{2d+p-2} \rho^{-(\frac{2-p}{4}) k_2^d} \sum_{y,y+v \in B_{k_2}(u)} (E[p(v) a(y, y + v)(D^{y,y+v} f)^2])^{\frac{p}{2}}$$

$$\leq C \sum_{k_2} k_2^{3d+p-2} \rho^{-(\frac{2-p}{4}) k_2^d} \sum_{u,u+v} (E[p(v) a(u, u + v)(D^{u,u+v} f)^2])^{\frac{p}{2}}$$

$$\leq C \sum_{u,u+v} (E[p(v) a(u, u + v)(D^{u,u+v} f)^2])^{\frac{p}{2}}$$

$$\leq C N^{\frac{2-p}{2}} (\mathcal{D}_N(f_N))^{\frac{p}{2}}. \qquad \square$$

Now we estimate how much Dirichlet form is needed to perform a shift.

Lemma 5. *Let $l < N$ and G_l be a function which only depends on variables at sites in $B_l(0)$. Then for $\rho_\beta + \rho_\omega < 1$ there exists a constant $C(\rho_\beta + \rho_\omega)$ such that for all functions f*

$$E^{u^N_{\rho_\beta,\rho_\omega}}[f(\tau^{e_j}(G_l\mathbf{1}_{X_N\setminus A_l}) - (G_l\mathbf{1}_{X_N\setminus A_l}))]$$

$$\leq Cl^{\frac{3d}{2}}\|G_l\|_2\sqrt{\sum_{y,y+v\in B_l(0)} E^{u^N_{\rho_\beta,\rho_\omega}}[p(v)a(u,u+v)(D^{u,u+v}f)^2]},$$

and

$$E^{u^N_{\rho_\beta,\rho_\omega}}[f^2(\tau^{e_j}(G_l\mathbf{1}_{X_N\setminus A_l}) - (G_l\mathbf{1}_{X_N\setminus A_l}))]$$

$$\leq Cl^{\frac{3d}{2}}\sqrt{E[f^2(\tau^{e_j}G_l^2 + G_l^2)]}\sqrt{\sum_{y,y+v\in B_l(0)} E^{u^N_{\rho_\beta,\rho_\omega}}[p(v)a(u,u+v)(D^{u,u+v}f)^2]}.$$

Proof. For the expectation with respect to $u^N_{\rho_\beta,\rho_\omega}$

$$\begin{aligned} E[f(\tau^{e_j}(G_l\mathbf{1}_{X_N\setminus A_l}) - (G_l\mathbf{1}_{X_N\setminus A_l}))] &= E[(\tau^{e_j}f - f)\mathbf{1}_{X_N\setminus A_l}G_l] \\ &\leq \|G_l\|_2\sqrt{E[(\tau^{e_j}f - f)^2\mathbf{1}_{X_N\setminus A_l}]} \end{aligned}$$

and

$$E[f^2(\tau^{e_j}(G_l\mathbf{1}_{X_N\setminus A_l}) - (G_l\mathbf{1}_{X_N\setminus A_l}))]$$

$$= E[(\tau^{e_j}f^2 - f^2)\mathbf{1}_{X_N\setminus A_l}G_l]$$

$$= E[(\tau^{e_j}f - f)(\tau^{e_j}f + f)\mathbf{1}_{X_N\setminus A_l}G_l]$$

$$\leq \sqrt{E[f^2(\tau^{e_j}G_l^2 + G_l^2)]}\sqrt{E[(\tau^{e_j}f - f)^2\mathbf{1}_{X_N\setminus A_l}]}$$

so the problem is to estimate

$$E[(\tau^{e_j}f - f)^2\mathbf{1}_{X_N\setminus A_l}] \leq Cl^{3d}\sum_{y,y+v\in B_l(0)} E[p(v)a(u,u+v)(D^{u,u+v}f)^2].$$

Once two sites adjacent to 0 are unoccupied we can find a configuration independent sequence of bonds of length less than Cl^d which performs the desired shift. Because we are in the set $X_N \setminus A_l$ there are two unoccupied sites somewhere in $B_l(0)$. Choose the two unoccupied sites x_1 and x_2 closest to 0 and move them adjacent to 0. Perform the desired shift, then move them back where they came from. Thus we have a sequence of bonds $b_1, \ldots, b_n$ with $n \leq Cl^d$ such that $a(u_i, u_i + v_i)(\eta^{b_1 \cdots b_{i-1}}) = 1$ and , for all f, $f(\eta^{b_1 \cdots b_n}) = \tau^{e_j} f(\eta)$. Now we can estimate

$$E[(\tau^{e_j} f - f)^2 \mathbf{1}_{X_N \setminus A_l}]$$

$$= \sum_{y_1, y_2 \in B_l(0)} E[\mathbf{1}_{\substack{y_1 = x_1 \\ y_2 = x_2}} (\sum_{i=1}^{n} a(u_i, u_i + v_i)(\eta^{b_1 \cdots b_{i-1}}) D^{u_i, u_i + v_i} f(\eta^{b_1 \cdots b_{i-1}}))^2]$$

$$\leq \sum_{y_1, y_2 \in B_l(0)} n \sum_{i=1}^{n} E[a(u_i, u_i + v_i)(\eta^{b_1 \cdots b_{i-1}})(D^{u_i, u_i + v_i} f(\eta^{b_1 \cdots b_{i-1}}))^2]$$

$$= \sum_{y_1, y_2 \in B_l(0)} n \sum_{i=1}^{n} E[a(u_i, u_i + v_i)(\eta)(D^{u_i, u_i + v_i} f(\eta))^2]$$

$$\leq Cl^{3d} \sum_{y, y+v \in B_l(0)} E[p(v) a(u, u+v)(D^{u, u+v} f)^2]. \qquad \square$$

If we are not guaranteed two empty sites in a small box B_l we can prove a slightly weaker version of the previous lemma.

Lemma 6. *Let $l < N$ and G_l be a function which only depends on variables at sites in $B_l(0)$. For $1 \leq p < 2$ and $\rho_\beta + \rho_\omega < 1$ there exists a constant $C_p(\rho_\beta + \rho_\omega)$ such that for all f*

$$\sum_u E^{u^N_{\rho_\beta, \rho_\omega}}[f(\tau^{u+e_i} G_l - \tau^u G_l)] \leq C_p l^{d(\frac{p-1}{p})} N^{\frac{d}{2}} \|G_l\|_q \sqrt{D_N(f)}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Consider a fixed u.

$$\begin{aligned} E[f(\tau^{u+e_i}G_l - \tau^u G_l)] &= E[(\tau^{u+e_i}f - \tau^u f)G_l] \\ &\leq \|G_l\|_q (E[|\ \tau^{u+e_i}f - \tau^u f\ |^p])^{\frac{1}{p}}. \end{aligned}$$

We can find the closest two unoccupied sites to u, x_1 and x_2, and move them adjacent to u as in the previous lemma. The difference is that we may have to look far for them. We get

$$E[|\ \tau^{u+e_i}f - \tau^u f\ |^p]$$

$$= \sum_{y_1,y_2\in B_l(0)} E[\mathbf{1}_{\substack{y_1=x_1\\ y_2=x_2}}\ |\sum_{i=1}^{n} a(u_i,u_i+v_i)(\eta^{b_1\cdots b_{i-1}})D^{u_i,u_i+v_i}f(\eta^{b_1\cdots b_{i-1}})\ |^p].$$

We can estimate this as in Lemma 4 to be

$$\leq \sum_{k_1\leq k_2}\sum_{\substack{|y_1|=k_1\\ |y_2|=k_2}} n^{p-1}\rho^{-(\frac{2-p}{4})(k_2)^d}\sum_{i=1}^{n}(E[a(u_i,u_i+v_i)(D^{u_i,u_i+v_i}f)^2])^{\frac{p}{2}}.$$

The chain is less than $C(l^d+k_2)$ bonds long so this is

$$\leq C\sum_{k_1\leq k_2}\sum_{\substack{|y_1|=k_1\\ |y_2|=k_2}}(l^d+k_2)^{p-1}\rho^{-(\frac{2-p}{4})(k_2)^d}\sum_{y,y+v\in B_{k_2}(u)}(E[a(y,y+v)(D^{y,y+v}f)^2])^{\frac{p}{2}}$$

$$\leq C\sum_{k_2}(k_2)^{2d-1}(l^d+k_2)^{p-1}\rho^{-(\frac{2-p}{4})(k_2)^d}\sum_{y,y+v\in B_{k_2}(u)}(E[a(y,y+v)(D^{y,y+v}f)^2])^{\frac{p}{2}}.$$

So we have

$$\sum_u E[f(\tau^{u+e_i}G_l - \tau^u G_l)]$$

$$\leq C\|G_l\|_q\sum_u\Big(\sum_{k_2}(k_2)^{2d-1}(l^d+k_2)^{p-1}\rho^{-(\frac{2-p}{4})(k_2)^d}\sum_{y,y+v\in B_{k_2}(u)}(E[a(y,y+v)(D^{y,y+v}f)^2])^{\frac{p}{2}}\Big)^{\frac{1}{p}}$$

$$\leq C\|G_l\|_q\sum_{k_2}(k_2)^{\frac{2d-1}{p}}(l^d+k_2)^{\frac{p-1}{p}}\rho^{-(\frac{2-p}{4p})(k_2)^d}\sum_{y,y+v}(E[a(y,y+v)(D^{y,y+v}f)^2])^{\frac{1}{2}}$$

$$\leq C_p l^{d(\frac{p-1}{p})}N^{\frac{d}{2}}\|G_l\|_q\sqrt{D_N(f)}$$ □

6. Local equilibrium distributions

In this section we show that the microscopic densities of blue and white particles do not fluctuate on a macroscopic scale. The particular result we need is that for any continuous function Q on $T^d \times [0,1] \times [0,1]$,

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{k\to\infty}}\ \overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}}\ \frac{\lambda_N}{N^d} = 0$$

where

$$\lambda_N = \sup_{densities f_N} \{E^{f_N \pi^N}[\sum_u (Q(\frac{u}{N}, \rho_\beta^k(u), \rho_\omega^k(u)) - Q(\frac{u}{N}, \rho_\beta^{N\epsilon}(u), \rho_\omega^{N\epsilon}(u))$$

$$- m\tau^u \mathbf{1}_{A_k})] - N^2 \mathcal{D}_N(\sqrt{f_N})\}.$$

An average over a block of side length $N\epsilon$ is just an average of averages over small blocks of side length k, and Q is uniformly continuous. Furthermore the result is known for functions only depending on $\rho = \rho_\beta + \rho_\omega$. Therefore it is enough to prove the following

Lemma 7.

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{k\to\infty}}\ \overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}}\ \sup_{z\in B_{N\epsilon}(0)} \frac{\tilde{\lambda}_N}{N^d} = 0$$

where

$$\tilde{\lambda}_N = \sup_{densities f_N} \{E^{f_N \pi^N}[\sum_u (\mid \rho_\beta^k(u+z) - \rho_\beta^k(u) \mid) - m\tau^u \mathbf{1}_{A_k})] - N^2 \mathcal{D}_N(\sqrt{f_N})\}$$

Proof. Since the first term in the supremum is $O(N^d)$ the lemma is equivalent to

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{k\to\infty}}\ \overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}}\ \sup_{\substack{\mathcal{D}_N(\sqrt{f_N}) \le C_1 N^{d-2} \\ E^{f_N \pi^N}[m \sum_u \tau^u \mathbf{1}_{A_k}] \le C_2 N^d}}\ \sup_{z\in B_{N\epsilon}(0)}$$

$$E^{f_N \pi^N}[\frac{1}{N^d} \sum_u \mid \rho_\beta^k(u+z) - \rho_\beta^k(u) \mid] = 0.$$

Define

$$\mathcal{B}^N_{C_1,C_2} = \{f_N : \mathcal{D}_N(\sqrt{f_N}) \le C_1 N^{d-2},\ E^{f_N \pi^N}[m \sum_u \tau^u \mathbf{1}_{A_k}] \le C_2 N^d\}.$$

We can project the density f_N onto $B_k(u+z) \times B_k(u)$ and average the result over all u. Write $\bar{f} = \frac{1}{N^d}\sum_u \tau^u f^k_N$ and

$$\bar{\mathcal{B}}^{N,k}_{C_1,C_2} = \{\bar{f} : f_N \in \mathcal{B}^N_{C_1,C_2}\}.$$

Then we need

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{k\to\infty}}\ \overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}} \sup_{f\in\bar{\mathcal{B}}^{N,k}_{C_1,C_2}} E^{f\pi}[\mid \rho^k_\beta(z) - \rho^k_\beta(0) \mid] = 0$$

where $\pi = \pi^{(2k+1)^d} \times \pi^{(2k+1)^d}$ is the projection of π^N. If $f \in \bar{\mathcal{B}}^{N,k}_{C_1,C_2}$ we have $E^{f\pi}[\mathbf{1}_{A_k}] \le \frac{C_1}{m} \to 0$. Since the integrand is less than 2 we have

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{k\to\infty}}\ \overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}} \sup_{f\in\bar{\mathcal{B}}^{N,k}_{C_1,C_2}} E^{f\pi}[\mathbf{1}_{A_k} \mid \rho^k_\beta(z) - \rho^k_\beta(0) \mid] = 0.$$

Then we need

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{k\to\infty}}\ \overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}} \sup_{f\in\bar{\mathcal{B}}^{N,k}_{C_1,C_2}} E^{f\pi}[\mathbf{1}_{X_k\setminus A_k} \mid \rho^k_\beta(z) - \rho^k_\beta(0) \mid] = 0.$$

Let $\mathcal{A}_{k,\epsilon}$ be the limit points of $\bar{f}$ as $N, \mid z \mid \to \infty$ with $\mid z \mid \le N\epsilon$. We need to show

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{k\to\infty}}\ \overline{\lim_{\epsilon\to 0}} \sup_{f\in\mathcal{A}_{k,\epsilon}} E^{f\pi}[\mathbf{1}_{X_k\setminus A_k} \mid \rho^k_\beta(z) - \rho^k_\beta(0) \mid] = 0.$$

So we need to study the set $\mathcal{A}_{k,\epsilon}$. Define the following functionals:

$$\mathcal{I}^0_k(f) = E^\pi[\sum_{u,u+v\in B_k(0)} p(v)a(u,u+v)(D^{u,u+v}(\sqrt{f}))^2]$$

$$\mathcal{I}^z_k(f) = \mathcal{I}^0_k(\tau^z f)$$

$$\mathcal{I}^{0,z}_k(f) = E^\pi[(D^{0,z}(\sqrt{f}))^2].$$

If f_N is a density with $\mathcal{D}_N(\sqrt{f_N}) \leq C_1 N^{d-2}$ then, by convexity,

$$\mathcal{I}_k^0(\bar{f}) = \mathcal{I}_k^z(\bar{f}) \leq \frac{k^d}{N^d}\mathcal{D}_N(\sqrt{f_N}) \leq C_1\frac{k^d}{N^2}.$$

By Lemma 3

$$\mathcal{I}_k^{0,z}(\mathbf{1}_{X_k\setminus A_k}\bar{f}) \leq C_3\frac{k^{2d}(N\epsilon+k^d)^2}{N^d}\mathcal{D}_N(\sqrt{f_N}) \leq C_3\frac{k^{2d}(N\epsilon+k^d)^2}{N^2}.$$

For $f \in \mathcal{A}_{k,\epsilon}$,

$$\mathcal{I}_k^z(f) = 0$$

$$\mathcal{I}_k^{0,z}(\mathbf{1}_{X_k\setminus A_k}f) \leq C_3 k^{2d}\epsilon^2$$

If we let $\mathcal{A}_k$ be the limit points of $\mathcal{A}_{k,\epsilon}$ as $\epsilon \to 0$ then for $f \in \mathcal{A}_k$ we have $\mathcal{I}_k^0(f) = \mathcal{I}_k^z(f) = \mathcal{I}_k^{0,z}(\mathbf{1}_{X_k\setminus A_k}f) = 0$ and we need

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{k\to\infty}}\ \sup_{f\in\mathcal{A}_k} E^{f\pi}[\mathbf{1}_{X_k\setminus A_k} \mid \rho_\beta^k(z) - \rho_\beta^k(0) \mid] = 0.$$

Clearly the supremum is achieved over f concentrated on $X_k \setminus A_k$. For such an f we have $\mathcal{I}_k^0(f) = \mathcal{I}_k^z(f) = \mathcal{I}_k^{0,z}(f) = 0$ which means that the density f is unchanged by exchanging any two variables $\eta(u)$ and $\eta(u+v)$ with $u, u+v \in B_k(0)$ or $B_k(z)$ and $p(v) > 0$, or by exchanging $\eta(0)$ and $\eta(z)$. From this we deduce that the density is unchanged by any permutation of the variables $\{\eta(x)\}_{x\in B_k(0)\times B_k(z)}$. Therefore the measures $f\pi$ are mixtures of trinomial distributions, so we can estimate the above by

$$\overline{\lim_{k\to\infty}} \sup_{0\leq i\leq 2(2k+1)^d} E^{\pi_{\frac{i}{2(2k+1)^d}}}[\mid \rho_\beta^k(z) - \rho_\beta^k(0) \mid],$$

where $\pi_{\frac{i}{2(2k+1)^d}}$ is a trinomial with $P[\beta(u)] = \frac{i}{2(2k+1)^d}$. This can easily be estimated by Schwarz's inequality to be

$$\leq \overline{\lim_{k\to\infty}} \sup_{0\leq i\leq 2(2k+1)^d} \frac{1}{(2k+1)^d}\left(\frac{i}{2(2k+1)^d}\left(1-\frac{i}{2(2k+1)^d}\right)\right) = 0. \qquad \square$$

7. Second order perturbations

We start this section with a general fact about finite state Markov processes. Let $\mathcal{L}$ be the generator of an ergodic Markov process on a finite state space, reversible with respect to v, and V a potential with $E^v[V]=0$. Let λ_N be the principal eigenvalue of $\mathcal{L}+\frac{1}{N}V$, and define

$$\sigma_V^2=\lim_{t\to\infty}E^v[(\frac{1}{\sqrt{t}}\int_0^t V(\eta_s)ds)^2].$$

It is fairly straightforward to show that

$$\overline{\lim_{N\to\infty}}\, N^2\lambda_N\le\frac{\sigma_V^2}{2}. \tag{7.1}$$

This fact provides the basis for the next theorem. Let $V(\theta,s,\{\eta(x)\}_{x\in B_l(0)})$ be a function on $T^d\times[0,T]\times\{-1,0,1\}^{B_l}$, smooth in θ, with the property that the principal eigenvalue of

$$\mathcal{L}_N+N\sum_u V(\theta,s,\{\eta(x)\}_{x\in B_l(u)})$$

is bounded by BN^d, and such that $V=0$ if all sites in $B_l(u)$ are occupied. Let $Q(\theta,s,\rho_\beta,\rho_\omega)$ be a continuous function on $T^d\times[0,T]\times[0,1]\times[0,1]$. Define

$$\sigma_k^2(\rho_\beta,\rho_\omega)=\tfrac{1}{(2(k-l)+1)^d}\lim_{t\to\infty}E[(\frac{1}{\sqrt{t}}\int_0^t\sum_{v\in B_{k-l}(0)}V(\theta_0,s_0,\{\eta_s(x)\}_{x\in B_l(v)})ds)^2]$$

for ρ_β,ρ_ω taking the discrete values $\frac{n}{(2k+1)^d},\frac{m}{(2k+1)^d}$ with $\rho_\beta+\rho_\omega<1$. Here η_s is the *non-speeded up* process, i.e. with generator $\hat{\mathcal{L}}_{2k+1}\overset{\text{def}}{=}\frac{1}{(2k+1)^2}\mathcal{L}_{2k+1}$ on the lattice of side length $2k+1$ starting with uniform measure $u^k_{\rho_\beta,\rho_\omega}$.

Theorem 7.1. *If* $Q(\rho_\beta,\rho_\omega)\ge\overline{\lim}_{\substack{k\to\infty\\ \rho_\beta{}^k,\rho_\omega{}^k\to\rho_\beta,\rho_\omega}}\sigma_k^2(\rho_\beta{}^k,\rho_\omega{}^k)$, *then*

$$\overline{\lim_{m\to\infty}}\,\overline{\lim_{k\to\infty}}\,\overline{\lim_{\epsilon\to0}}\,\overline{\lim_{N\to\infty}}\,\frac{1}{N^d}\log E^{P_N}[\exp\int_0^T F_{N,\epsilon,k,m}(s)ds]\le 0$$

where

$$F_{N,\epsilon,k,m}(s) = \sum_u NV(\frac{u}{N}, s, \{\eta(x)\}_{x\in B_l(u)}) - \tfrac{1}{2}Q(\frac{u}{N}, s, \rho_\beta^{N\epsilon}(u), \rho_\omega^{N\epsilon}(u)) - \tau^u m\mathbf{1}_{A_k}$$

Proof. Since P_N is the stationary process with reversible initial distribution π^N an application of the Feynman-Kac formula and the spectral theorem allows us to estimate

$$\frac{1}{N^d}\log E^{P_N}[\exp\int_0^T F_{N,\epsilon,k,m}(s)ds] \le \int_0^T \frac{\Lambda_{N,\epsilon,k,m}(s)}{N^d}ds$$

where $\Lambda_{N,\epsilon,k,m}(s)$ is the principal eigenvalue of $\mathcal{L}_N + F_{N,\epsilon,k,m}(s)$. The assumption on V implies that $\frac{\Lambda_{N,\epsilon,k,m}(s)}{N^d} \le B$ so by the dominated convergence theorem all we need is

$$\varlimsup_{m\to\infty}\varlimsup_{k\to\infty}\varlimsup_{\epsilon\to 0}\varlimsup_{N\to\infty}\frac{\Lambda_{N,\epsilon,k,m}(s)}{N^d} \le 0$$

for each fixed s. $\Lambda_{N,\epsilon,k,m}(s)$ is given by a variational formula

$$\Lambda_{N,\epsilon,k,m}(s) = \sup_{f_N}\{E^{f_N\pi^N}[F_{N,\epsilon,k,m}(s)] - N^2\mathcal{D}_N(\sqrt{f_N})\}.$$

Dropping the s we can write $\Lambda_{N,\epsilon,k,m} \le \Lambda^1_{N,k,1-\gamma} + \Lambda^2_{N,\epsilon,k,m,\gamma}$ where

$$\Lambda^1_{N,k,1-\gamma} = \sup_{f_N}\{E^{f_N\pi^N}[\sum_u NV(\frac{u}{N}, \{\eta(x)\}_{x\in B_l(u)}) - \tfrac{1}{2}Q(\frac{u}{N}, \rho_\beta^k(u), \rho_\omega^k(u))]$$

$$- N^2(1-\gamma)\mathcal{D}_N(\sqrt{f_N})\},$$

$$\Lambda^2_{N,\epsilon,k,m,\gamma} = \sup_{f_N}\{E^{f_N\pi^N}[\sum_u Q(\frac{u}{N}, \rho_\beta^k(u), \rho_\omega^k(u)) - Q(\frac{u}{N}, \rho_\beta^{N\epsilon}(u), \rho_\omega^{N\epsilon}(u))$$

$$- \tau^u m\mathbf{1}_{A_k}] - N^2\gamma\mathcal{D}_N(\sqrt{f_N})\}.$$

We have shown in Lemma 7 that for any $\gamma > 0$

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{k\to\infty}}\ \overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}}\ \frac{\Lambda^2_{N,\epsilon,k,m,\gamma}}{N^d} = 0.$$

Next we write $\Lambda^1_{N,k,1-\gamma} \leq \Lambda^3_{N,k,1-\gamma} + \Lambda^4_{N,k}$ where

$$\Lambda^3_{N,k,1-\gamma} = \sup_{f_N}\{E^{f_N\pi^N}[\sum_u \tfrac{1}{(2(k-l)+1)^d} \sum_{v\in B_{k-l}(0)} NV(\frac{u}{N}, \{\eta(x)\}_{x\in B_l(u+v)})$$

$$- \tfrac{1}{2}Q(\frac{u}{N}, \rho^k_\beta(u), \rho^k_\omega(u))] - N^2(1-\gamma)\mathcal{D}_N(\sqrt{f_N})\},$$

$$\Lambda^4_{N,k} = \sup_{f_N}\{E^{f_N\pi^N}[N\sum_u \tfrac{1}{(2(k-l)+1)^d} \sum_{v\in B_{k-l}(0)} V(\frac{u-v}{N}, \{\eta(x)\}_{x\in B_l(u)})$$

$$- V(\frac{u}{N}, \{\eta(x)\}_{x\in B_l(u)})]\}.$$

If we use a two term Taylor expansion of V,

$$\tfrac{1}{(2(k-l)+1)^d} \sum_{v\in B_{k-l}(0)} V(\frac{u-v}{N}) - V(\frac{u}{N})$$

$$= \tfrac{1}{(2(k-l)+1)^d} \sum_{v\in B_{k-l}(0)} \frac{v}{N}\frac{\partial V}{\partial\theta}(\frac{u}{N}) + \tfrac{1}{(2(k-l)+1)^d} \sum_{v\in B_l(0)} (\frac{v}{N})^2 \frac{1}{2}\frac{\partial^2 V}{\partial\theta^2}(\tilde{\theta}),$$

the first order terms cancel by symmetry, so we can estimate

$$\Lambda^4_{N,k} \leq \frac{(k-l)^3}{(k-l)^d}\frac{1}{N^2}N\|\frac{\partial^2 V}{\partial\theta^2}\|_\infty N^d = O(N^{d-1}).$$

It remains to show that

$$\overline{\lim_{\gamma\to 0}}\ \overline{\lim_{k\to\infty}}\ \overline{\lim_{N\to\infty}}\ \frac{\Lambda^3_{N,k,1-\gamma}}{N^d} \leq 0.$$

Now

$$\frac{\Lambda^3_{N,k,1-\gamma}}{N^d} \leq \frac{1}{N^d}\sum_u \sup_{f_N}\{E^{f_N\pi^N}[\tfrac{1}{(2(k-l)+1)^d} \sum_{v\in B_{k-l}(0)} NV(\frac{u}{N}, \{\eta(x)\}_{x\in B_l(u+v)})$$

$$- \tfrac{1}{2}Q(\frac{u}{N}, \rho^k_\beta(u), \rho^k_\omega(u))] - N^2(1-\gamma)\mathcal{D}_N(\sqrt{f_N})\}.$$

If we project f_N onto $B_k(u)$ this becomes

$$\frac{1}{N^d}\sum_u \sup_f\{E^{f\pi^k}[\tfrac{1}{(2(k-l)+1)^d}\sum_{v\in B_{k-l}(0)} NV(\frac{u}{N},\{\eta(x)\}_{x\in B_l(u+v)})$$

$$-\tfrac{1}{2}Q(\frac{u}{N},\rho^k_\beta(u),\rho^k_\omega(u))] - \frac{N^2(1-\gamma)}{(2k+1)^d}\mathcal{D}_k(\sqrt{f})\}.$$

Now we project onto the "hyperplane" of $\rho_\beta(2k+1)^d$ blue and $\rho_\omega(2k+1)^d$ white particles, with uniform measure $u^k_{\rho_\beta,\rho_\omega}$. Since there are only a finite number of such hyperplanes for each fixed k, the above is less than or equal to

$$\frac{1}{N^d}\sum_u \sup_{\rho_\beta,\rho_\omega}\sup_f\{E^{fu^k_{\rho_\beta,\rho_\omega}}[\tfrac{1}{(2(k-l)+1)^d}\sum_{v\in B_{k-l}(0)} NV(\frac{u}{N},\{\eta(x)\}_{x\in B_l(u+v)})$$

$$-\tfrac{1}{2}Q(\frac{u}{N},\rho_\beta,\rho_\omega)] - \frac{N^2(1-\gamma)}{(2k+1)^d}\mathcal{D}_k(\sqrt{f})\}.$$

$$\leq \sup_\theta\sup_{\rho_\beta,\rho_\omega}\sup_f\{E^{fu^k_{\rho_\beta,\rho_\omega}}[\tfrac{1}{(2(k-l)+1)^d}\sum_{v\in B_{k-l}(0)} NV(\theta,\{\eta(x)\}_{x\in B_l(v)})$$

$$-\tfrac{1}{2}Q(\theta,\rho_\beta,\rho_\omega)] - \frac{N^2(1-\gamma)}{(2k+1)^d}\mathcal{D}_k(\sqrt{f})\}.$$

$$\leq \sup_\theta\sup_{\rho_\beta,\rho_\omega}\{\frac{N^2(1-\gamma)}{(2k+1)^d}\lambda_N - \tfrac{1}{2}Q(\theta,\rho_\beta,\rho_\omega)\}$$

where λ_N is the principal eigenvalue of

$$\hat{\mathcal{L}}_{2k+1} + \frac{1}{N}\frac{(2k+1)^d}{(2(k-l)+1)^d(1-\gamma)}\sum_{v\in B_{k-l}(0)} V(\theta,\{\eta(x)\}_{x\in B_l(v)}).$$

By (0.15) ,

$$\overline{\lim_{\gamma\to 0}}\;\overline{\lim_{k\to\infty}}\;\overline{\lim_{N\to\infty}}\;\frac{\Lambda^3_{N,k,1-\gamma}}{N^d} \leq 0.$$

which proves the theorem. $\square$

8. Spectral gap

The estimates in this section concern functions of configurations on the *non-periodic* lattice of N^d sites. For the first estimate, consider the colourless configuration space $\{0,1\}^{T_N}$ with uniform measure u_ρ^N. The operator

$$\mathcal{L}_N^1(f) = \frac{1}{N^d}\sum_{x,y} D^{x,y} f \overset{\text{def}}{=} \frac{1}{N^d}\sum_{x,y}(f(\zeta^{x,y}) - f(\zeta))$$

is the generator of the process where exchanges occur between any two sites.

Lemma 8. *For any function f of mean 0,*

$$E^{u_\rho^N}[f^2] \leq \frac{1}{N^d} E^{u_\rho^N}[\sum_{x,y}(D^{x,y} f)^2].$$

Proof. $L^2(u_\rho^N)$ has dimension $\begin{pmatrix} N^d \\ \rho N^d \end{pmatrix}$. For integers $n \leq \rho N^d$, a polynomial of degree n is a function of the form $\sum_{j=0}^n a_j \prod_{i=1}^j \zeta(u_{ij})$. Let $\mathcal{U}_n$ be the subspace generated by polynomials of degree $\leq n$. If $\mathcal{V}_n$ is the orthogonal complement of $\mathcal{U}_{n-1}$ in $\mathcal{U}_n$ then

$$L^2 = \mathcal{V}_0 \oplus \cdots \oplus \mathcal{V}_{\rho N^d},$$

and each $\mathcal{V}_n$ has dimension $\begin{pmatrix} N^d \\ n \end{pmatrix} - \begin{pmatrix} N^d \\ n-1 \end{pmatrix}$. Applying $\mathcal{L}_N^1$ to $\prod_{i=1}^n \zeta(u_i)$

$$\mathcal{L}_N^1(\prod_{i=1}^n \zeta(u_i)) = -n\frac{N^d - (n-1)}{N^d}(\prod_{i=1}^n \zeta(u_i)) + \sum_{j=1}^n(\rho - \frac{(n-1)}{N^d})\prod_{\substack{i=1 \\ i\neq j}}^n \zeta(u_i),$$

so that if $f \in \mathcal{V}_n$

$$\mathcal{L}_N^1(f) = -n\frac{N^d - (n-1)}{N^d} f + g$$

where $g \in \mathcal{U}_{n-1}$. We can find $h \in \mathcal{U}_{n-1}$ such that

$$(\mathcal{L}_N^1 + n\frac{N^d - (n-1)}{N^d})h = -g.$$

Then $f + h$ is an eigenvector with eigenvalue $-n\frac{N^d-(n-1)}{N^d}$. Let $\mathcal{W}_n = \{f + h, f \in \mathcal{V}_n\}$. Since $h \in \mathcal{U}_{n-1}$ the $\mathcal{W}_n$ still span L^2, and since $\mathcal{L}_N^1$ is self–adjoint the $\mathcal{W}_n$

are orthogonal. Thus $\mathcal{W}_n$ are the eigenspaces, and the eigenvalues are $0, -1, -2 + \frac{1}{N^d}, \ldots, -\rho N^d + \frac{\rho N^d - 1}{N^d}$. $\square$

Next we consider our simple exclusion process, restricted to $\{0,1\}^{T_N}$.

Lemma 9. *There exists a constant C such that for $0 \le \rho \le 1$, and any function f of mean 0,*

$$E^{u_\rho^N}[f^2] \le CN^2 E^{u_\rho^N}[\sum_{u,u+v} p(v)(D^{u,u+v}f)^2].$$

Proof. Connect sites x and y by a chain of bonds $b_i = u_i, u_i + v_i; i = 1, \ldots, n$ so that $\zeta^{x,y} = \zeta^{b_1 \cdots b_n}$, in the manner described in Lemma 4. We move x to y first along dimension 1, then dimension 2, etc. It is not hard to see that any fixed bond is used for at most $d(\frac{N}{2})^{d+1}$ pairs x, y. Then

$$\begin{aligned}
\frac{1}{N^d}\sum_{x,y} E[(D^{x,y}f(\zeta))^2] &\le \frac{1}{N^d}\sum_{x,y} E[(\sum_{i=1}^{n} D^{u_i,u_i+v_i} f(\zeta^{b_1\cdots b_{i-1}}))^2] \\
&\le \frac{1}{N^d}\sum_{x,y} dN \sum_{i=1}^{n} E[(D^{u_i,u_i+v_i} f(\zeta^{b_1\cdots b_{i-1}}))^2] \\
&= \frac{1}{N^d}\sum_{x,y} dN \sum_{i=1}^{n} E[(D^{u_i,u_i+v_i} f(\zeta))^2] \\
&\le CN^2 \sum_{u,u+v} E[p(v)(D^{u,u+v}f(\zeta))^2].
\end{aligned}$$

Using the previous lemma we are done. $\square$

Lemma 10. *There exists a constant C such that for any $0 \le \rho \le 1$, $y \in T_N$, and f with $E^{u_\rho^N}[f \mid \zeta(y) = 1] = 0$,*

$$E^{u_\rho^N}[f^2 \mid \zeta(y) = 1] \le CN^2 E^{u_\rho^N}[\sum_{u,u+v \neq x,y} p(v)(D^{u,u+v}f)^2 \mid \zeta(y) = 1].$$

Proof. We do the same thing as in the previous lemma except that whenever a bond containing y comes up we go around it. For example, replace $(y - e_i, y), (y, y + e_i)$ by

$(y-e_i, y-e_i-e_{i+1}), (y-e_i-e_{i+1}, y-e_{i+1}), (y-e_{i+1}, y-e_{i+1}+e_i), (y-e_{i+1}+e_i, y+e_i)$. The only effect is to increase the size of the constant C by a fixed amount. $\square$

Now we come to the main result of this section. The generator $\mathcal{L}_N$ of the coloured simple exclusion process preserves the linear subspace $\mathcal{T} \subset L^2(u^N_{\rho_\beta,\rho_\omega})$ consisting of functions of the form $f = \sum_u (a\beta(u) + b\omega(u))F_u(\zeta)$. We now estimate the spectral gap of $\mathcal{L}_N$ as an operator on $\mathcal{T}$.

Lemma 11. *For $0 < \rho_\beta + \rho_\omega < 1$ there exists a constant $C(\rho_\beta + \rho_\omega)$ such that for any function of the form $f = \sum_u (a\beta(u) + b\omega(u))F_u(\zeta)$ with mean 0,*

$$E^{u^N_{\rho_\beta,\rho_\omega}}[f^2] \le CN^2 E^{u^N_{\rho_\beta,\rho_\omega}}[\sum_{u,u+v} p(v)a(u,u+v)(D^{u,u+v}f)^2].$$

Proof. Rewrite $\sum_u (a\beta(u) + b\omega(u))F_u$ as

$$\sum_u ((a-b)\chi(u) + (\frac{\rho_\beta}{\rho}a + \frac{\rho_\omega}{\rho}b)\zeta(u))F_u = f_\chi + f_\zeta.$$

Then for the expectation with respect to $u^N_{\rho_\beta,\rho_\omega}$,

$$E[(f_\chi + f_\zeta)^2] = E[f_\chi^2] + E[f_\zeta^2]$$

and

$$E[\sum_{u,u+v} p(v)a(u,u+v)(D^{u,u+v}(f_\chi + f_\zeta))^2]$$

$$= E[\sum_{u,u+v} p(v)a(u,u+v)(D^{u,u+v}f_\chi)^2] + E[\sum_{u,u+v} p(v)(D^{u,u+v}f_\zeta)^2],$$

so by Lemma 9 we only need to prove the estimate for f of the form $f = \sum_u \chi(u)F_u(\zeta)$. Replacing F_u by $F_u - F$ for any function F does not affect such an f, so we can take $F_u(\zeta) = 0$ if $\zeta(u) = 0$, and $\sum_u F_u(\zeta) = 0$. Calculation yields

$$E[(\sum_y \chi(y)F_y)^2] = (\frac{\rho_\beta\rho_\omega}{\rho^2} + \frac{\rho_\beta\rho_\omega}{\rho(\rho N^d - 1)})\sum_y E[F_y^2]$$

and

$$E[\sum_{u,u+v} p(v)a(u,u+v)(D^{u,u+v}(\sum_y \chi(y)F_y))^2]$$

$$= (\frac{\rho_\beta\rho_\omega}{\rho^2} + \frac{\rho_\beta\rho_\omega}{\rho(\rho N^d-1)})E[\sum_{u,u+v} p(v)(\sum_{y\neq u,u+v}(D^{u,u+v}F_y)^2$$

$$+ 2(1-\zeta(u+v))(F_{u+v}(\zeta^{u,u+v}) - F_u(\zeta))^2)].$$

Here we have used repeatedly that $\sum_u F_u = 0$ and that for any colourblind function $g(\zeta)$, $E[\chi^2(u)g] = \frac{\rho_\beta\rho_\omega}{\rho^2}E[\zeta(u)g]$, and, for $u \neq v$, $E[\chi(u)\chi(v)g] = -\frac{\rho_\beta\rho_\omega}{\rho(\rho N^d-1)}E[\zeta(u)\zeta(v)g]$. The problem is now reduced to showing

$$\sum_y E[F_y^2] \leq CN^2 E[\sum_{u,u+v} p(v)(\sum_{y\neq u,u+v}(D^{u,u+v}F_y)^2$$

$$+ 2(1-\zeta(u+v))(F_{u+v}(\zeta^{u,u+v}) - F_u(\zeta))^2)].$$

By Lemma 10 we have

$$E[F_y^2] - (E[F_y])^2 \leq CN^2 E[\sum_{u,u+v\neq y} p(v)(D^{u,u+v}F_y)^2]$$

so the problem is to estimate

$$\sum_y (E[F_y])^2 = \frac{1}{2N^d}\sum_{x,y}(E[F_x - F_y])^2$$

$$\leq CN^2 E[\sum_{u,u+v} p(v)(\sum_{y\neq u,u+v}(D^{u,u+v}F_y)^2 + 2(1-\zeta(u+v))(F_{u+v}(\zeta^{u,u+v}) - F_u(\zeta))^2)].$$

It is easy to see that

$$\frac{1}{2N^d}\sum_{x,y}(E[F_x - F_y])^2 \leq CN^2 \sum_{u,u+v}(E[p(v)(F_{u+v} - F_u)])^2,$$

where $C = \max_{v\in S}\{\frac{1}{p(v)}\}$. Fix $u, u+v$. To each configuration we will associate a sequence of bonds $b_1, \ldots, b_n$ such that $\zeta^{b_1\cdots b_n}(u+v) = 0$. Pick the unoccupied site x_u

nearest to $u+v$ and choose $b_1, \ldots, b_n$ as in *Step I* of Lemma 3. No b_i need have u or $u+v$ as an endpoint except b_n. Now

$$E[F_{u+v} - F_u] = E[F_{u+v}(\zeta^{u,u+v}) - F_u(\zeta)]$$

$$= E[\sum_{i=1}^{n}(D^{b_i}F_u(\zeta^{b_1\cdots b_{i-1}}) - D^{b_i}F_{u+v}((\zeta^{u,u+v})^{b_1\cdots b_{i-1}}))$$

$$+ (1-\zeta^{b_1\cdots b_n}(u+v))(F_{u+v}((\zeta^{u,u+v})^{b_1\cdots b_n}) - F_u(\zeta^{b_1\cdots b_n}))]$$

$$= \sum_x \sum_{i=1}^{n} E[\mathbf{1}_{x=x_u}(D^{b_i}F_u(\zeta^{b_1\cdots b_{i-1}}) - D^{b_i}F_{u+v}((\zeta^{u,u+v})^{b_1\cdots b_{i-1}})$$

$$+ (1-\zeta^{b_1\cdots b_n}(u+v))(F_{u+v}((\zeta^{u,u+v})^{b_1\cdots b_n}) - F_u(\zeta^{b_1\cdots b_n})))]$$

$$\leq \sum_x \sum_{i=1}^{n} \sqrt{P[x=x_u]}\{\sqrt{E[(D^{b_i}F_u(\zeta^{b_1\cdots b_{i-1}}))^2]} + \sqrt{E[(D^{b_{i+1}}F_{u+v}((\zeta^{u,u+v})^{b_1\cdots b_i}))^2]}$$

$$+ \sqrt{E[(1-\zeta^{b_1\cdots b_n}(u+v))(F_{u+v}((\zeta^{u,u+v})^{b_1\cdots b_n}) - F_u(\zeta^{b_1\cdots b_n})^2]}\}$$

$$\leq \sum_{|x|=M} \sqrt{P[x=x_u]}\{CM^{\frac{1}{2}}\sqrt{M^d E[\sum_{b\in B_M(u)}(D^bF_u(\zeta))^2]}$$

$$+ CM^{\frac{1}{2}}\sqrt{M^d E[\sum_{b\in B_M(u)}(D^bF_{u+v}(\zeta))^2]}$$

$$+ \sqrt{M^d E[(1-\zeta(u+v))(F_{u+v}(\zeta^{u,u+v}) - F_u(\zeta)^2]}\}$$

$$\leq C\sum_M M^{2d-\frac{1}{2}}\rho^{-\frac{(M-1)^d}{2}}\{\sqrt{E[\sum_{b\in B_M(u)}(D^bF_u(\zeta))^2]} + \sqrt{E[\sum_{b\in B_M(u)}(D^bF_{u+v}(\zeta))^2]}$$

$$+ \sqrt{E[(1-\zeta(u+v))(F_{u+v}(\zeta^{u,u+v}) - F_u(\zeta))^2]}\}.$$

Therefore

$$N^2\sum_{u,u+v}(E[(F_{u+v} - F_u)])^2$$

$$\leq CN^2 \sum_{u,u+v} (\sum_M M^{2d-\frac{1}{2}} \rho^{-\frac{(M-1)^d}{2}} \{ \sqrt{E[\sum_{b \in B_M(u)} (D^b F_u(\zeta))^2]}$$

$$+ \sqrt{E[\sum_{b \in B_M(u)} (D^b F_{u+v}(\zeta))^2]} + \sqrt{E[(1-\zeta(u+v))(F_{u+v}(\zeta^{u,u+v}) - F_u(\zeta))^2]} \})^2$$

$$\leq CN^2 \sum_{u,u+v} 3 \sum_M \frac{M^{2d-\frac{1}{2}} \rho^{-\frac{(M-1)^d}{2}}}{\sum_m m^{2d-\frac{1}{2}} \rho^{-\frac{(m-1)^d}{2}}} \{ E[\sum_{b \in B_M(u)} (D^b F_u(\zeta))^2]$$

$$+ E[\sum_{b \in B_M(u)} (D^b F_{u+v}(\zeta))^2] + E[(1-\zeta(u+v))(F_{u+v}(\zeta^{u,u+v}) - F_u(\zeta))^2] \}$$

$$\leq CN^2 E[\sum_{u,u+v} p(v) (\sum_{y \neq u,u+v} (D^{u,u+v} F_y)^2 + 2(1-\zeta(u+v))(F_{u+v}(\zeta^{u,u+v}) - F_u(\zeta))^2)],$$

as desired. □

9. Exactness of $D^{0,v}$

In this section we consider the space $(X, \pi_{\rho_\beta, \rho_\omega})$ defined in the preamble to Theorem 0.4. A chain of bonds $b_1, \ldots, b_n;\ b_i = u_i, u_i + v_i\ , u_i \in Z^d, p(v_i) > 0$ will be called *closed* if $\eta^{b_1 \cdots b_n} = \eta$ for all $\eta \in X$. For any function $F : X \to R$ and any closed chain of bonds

$$\sum_{i=1}^n D^{u_i, u_i+v_i} F(\eta^{b_1 \cdots b_{i-1}}) = 0.$$

Let g be a cylinder function. Since $D^{u_i,u_i+v_i} \sum_u \tau^u g = D^{u_i,u_i+v_i} \sum_{u \in B_l} \tau^u g$ for some large box B_l we also have

$$\sum_{i=1}^n \tau^{u_i} f_g^{v_i}(\eta^{b_1 \cdots b_{i-1}}) = \sum_{i=1}^n D^{u_i,u_i+v_i} \sum_u \tau^u g(\eta^{b_1 \cdots b_{i-1}}) = 0. \tag{9.1}$$

Definition 2. *A function* $\mathbf{f} : X \to R^S$ *is closed if for every closed chain of bonds*

$$\sum_{i=1}^n \tau^{u_i} f^{v_i}(\eta^{b_1 \cdots b_{i-1}}) = 0.$$

From (0.16) we see that all exact forms are closed. For c and d in R^d, the forms $v \cdot c(\beta(v) - \beta(0))$ and $v \cdot d(\omega(v) - \omega(0))$ are closed but not exact. Let $1 < p < 2$ be fixed and define $L^p = \{\mathbf{f} : \sum_{v \in S} E^{\pi_{\rho_\beta,\rho_\omega}}[| f^v |^p] < \infty\}$. Let

$$\mathcal{C} = \{\mathbf{f} \in \mathcal{H} \cap L^p : \mathbf{f} \text{ closed}\},$$

and $\mathcal{E}$ be the closure of $\mathcal{E}_0$ in $\mathcal{H} \cap L^p$. The main result of this section is

Theorem 9.1.

$$\mathcal{T} \cap \mathcal{C} = \{v \cdot c(\beta(v) - \beta(0))\} \oplus \{v \cdot d(\omega(v) - \omega(0))\} \oplus \mathcal{T} \cap \mathcal{E}.$$

Proof. Let $\mathbf{f} \in \mathcal{T} \cap \mathcal{C}$. Since the weak closure of a linear subspace is the same as the strong closure we need only find c, d and a sequence g_n of cylinder functions such that

$$v \cdot c(\beta(v) - \beta(0)) + v \cdot d(\omega(v) - \omega(0)) + D^{0,v} \sum_u \tau^u g_n \to f^v$$

weakly in L^p and $\mathcal{H}$. For any set of sites A let $\mathcal{F}_A$ be the σ–field generated by $\{\eta(u),\ u \in A\}$. Let $f^v_{2N} = E[f^v \mid \mathcal{F}_{B_{2N}(0)}]$. Then $\mathbf{f_{2N}} \in \mathcal{T}$ and for any closed chain of bonds in $B_{2N}(0)$,

$$\sum_{i=1}^n \tau^{u_i} f^{v_i}_{2N}(\eta^{b_1 \cdots b_{i-1}}) = 0.$$

Therefore there exists g_{2N} of mean 0 on $B_{2N}(0)$ with $D^{u,u+v} g_{2N} = \tau^u f^v_{2N}$ for $u, u+v$ in $B_{2N}(0)$. Since $D^{u,u+v}$ maps $\mathcal{T}$ into itself and $\mathcal{T}^\perp$ into itself, we must have $g_{2N} \in \mathcal{T}$. If we let $\tilde{f}^v_{2N} = f^v_{2N} \mathbf{1}_{X_{2N} \setminus A_{2N}}$ and $\tilde{g}_{2N} = g_{2N} \mathbf{1}_{X_{2N} \setminus A_{2N}}$ we still have $D^{u,u+v} \tilde{g}_{2N} = \tau^u \tilde{f}^v_{2N}$ and $\tilde{g}_{2N} \in \mathcal{T}$. Then we can apply the spectral gap estimate of Section to get:

$$\begin{aligned} E[(\tilde{g}_{2N})^2] &\leq CN^2 E[\sum_{u,u+v} p(v) a(u, u+v)(\tau^u \tilde{f}^v_{2N})^2] \\ &\leq CN^{d+2}. \end{aligned} \tag{9.2}$$

For $N \leq m \leq 2N$ let $g_m = E[\tilde{g}_{2N} \mid \mathcal{F}_{B_m(0)}]$. We claim that there exists m_N, $N \leq m_N \leq 2N$ such that the boundary term

$$\frac{1}{N^{d-1}} \sum_{\substack{u \in B_{m_N}(0) \\ u+v \notin B_{m_N}(0)}} E[(D^{u,u+v} g_{m_N})^2] = O(N^2). \tag{9.3}$$

To prove this: For any $g \in L^2$ and set of sites A let $g^A = E[g \mid \mathcal{F}_A]$. If $u \in A$ and $u+v \notin A$ then the norms of g^A and $g^{A\setminus u}$ are related by a simple formula,

$$E[(g^A)^2] = E[(g^{A\setminus u})^2] + \frac{1}{2} E[(D^{u,u+v} g^A)^2].$$

Applying this repeatedly we get

$$B^d E[g_{2N}^2] = B^d E[g_{2N-B}^2] + \frac{1}{2} \sum_{\substack{u \in B_{2N}(0) \\ u+v \notin B_{2N}(0)}} E[(D^{u,u+v} g_{2N})^2]$$

where $B = \mid S \mid$. Now applying this repeatedly we get

$$B^d E[g_{2N}^2] = B^d E[g_N^2] + \sum_{i=1}^{N/B} \frac{1}{2} \sum_{\substack{u \in B_{N+Bi}(0) \\ u+v \notin B_{N+Bi}(0)}} E[(D^{u,u+v} g_{N+Bi})^2].$$

By (0.17) this gives

$$\sum_{i=1}^{N/B} \frac{1}{2} \sum_{\substack{u \in B_{N+Bi}(0) \\ u+v \notin B_{N+Bi}(0)}} E[(D^{u,u+v} g_{N+Bi})^2] \leq O(N^{d+2}).$$

But there are N/B terms, so one of them must be $O(N^{d+1})$ which establishes (0.18).

Now consider the formal function

$$\xi_N = \frac{1}{(2m_N+1)^d} \sum_{u \in Z^d} \tau^{-u} g_{m_N}.$$

ξ_N does not make sense, but $D^{0,v}\xi_N$ does. It is

$$\frac{1}{(2m_N+1)^d} \sum_{u,u+v \in B_{m_N}(0)} \tau^{-u} D^{u,u+v} g_{m_N}. \tag{9.4}$$

$$+\frac{1}{(2m_N+1)^d} \sum_{\substack{u \in B_{m_N}(0) \\ u+v \notin B_{m_N}(0)}} \tau^{-u} D^{u,u+v} g_{m_N}. \tag{9.5}$$

First we deal with (0.19). It is equal to

$$\frac{1}{(2m_N+1)^d}\sum_{u,u+v\in B_{m_N}(0)}\tau^{-u}E[\tau^u\tilde{f}^v_{2N}\mid\mathcal{F}_{B_{m_N}(0)}]$$

$$=\frac{1}{(2m_N+1)^d}\sum_{u,u+v\in B_{m_N}(0)}E[f^v\mathbf{1}_{X_{2N}\setminus A_{2N}}\mid\mathcal{F}_{B_{m_N}(u)}]$$

$$=\frac{1}{(2m_N+1)^d}\sum_{u,u+v\in B_{m_N-N\epsilon}(0)}E[f^v\mid\mathcal{F}_{B_{m_N}(u)}] \tag{9.6}$$

$$-\frac{1}{(2m_N+1)^d}\sum_{u,u+v\in B_{m_N-N\epsilon}(0)}E[f^v\mathbf{1}_{A_{2N}}\mid\mathcal{F}_{B_{m_N}(u)}] \tag{9.7}$$

$$+\frac{1}{(2m_N+1)^d}\sum_{\substack{u\in B_{m_N-N\epsilon}(0)\\ u+v\in B_{m_N}(0)\setminus B_{m_N-N\epsilon}(0)}}E[f^v\mathbf{1}_{X_{2N}\setminus A_{2N}}\mid\mathcal{F}_{B_{m_N}(u)}] \tag{9.8}$$

The third term, (0.23), is clearly $O(m_N^{-1})$. We can estimate (0.22) as follows:

$$E[\mid\frac{1}{(2m_N+1)^d}\sum_{u,u+v\in B_{m_N-N\epsilon}(0)}E[f^v\mathbf{1}_{A_{2N}}\mid\mathcal{F}_{B_{m_N}(u)}]\mid^p]$$

$$\leq\frac{1}{(2m_N+1)^d}\sum_{u,u+v\in B_{m_N-N\epsilon}(0)}E[\mid f^v\mathbf{1}_{A_{2N}}\mid^p]$$

$$=E[\mid f^v\mathbf{1}_{A_{2N}}\mid^p]\to 0$$

since $P[A_{2N}]\to 0$. Now we consider (0.21). Since $f^v\in L^p$ for some $p>1$ and $B_{N\epsilon}(0)\subset\bigcap_{u\in B_{m_N-N\epsilon}(0)}B_{m_N}(u)$

$$\frac{1}{(2m_N+1)^d}\sum_{u,u+v\in B_{m_N-N\epsilon}(0)}E[f^v\mid\mathcal{F}_{B_{m_N}(u)}]\to f^v$$

in L^p by the martingale convergence theorem. Thus we have shown that (0.19) converges to f^v in L^p. Replacing f^v by $a(0,v)f^v$ and p by 2 in the above proof we see that (0.19) also converges to f^v in $\mathcal{H}$.

Finally, we deal with (0.20);

$$D^{0,v}\frac{1}{(2m_N+1)^d}\sum_{\substack{u\in B_{m_N}(0)\\ u+v\notin B_{m_N}(0)}}\tau^{-u}g_{m_N}\stackrel{\text{def}}{=}D^{0,v}G_N.$$

By (0.18) it is bounded in L^2, so there is a weakly convergent subsequence. We can write

$$D^{0,v}G_N=(\beta(v)-\beta(0))G_N^{\beta,v}+(\omega(v)-\omega(0))G_N^{\omega,v}$$

where $G_N^{\beta,v}$ and $G_M^{\omega,v}$ do not depend on $\eta(0)$ or $\eta(v)$. Then

$$(\beta(v)-\beta(0))G_N^{\beta,v}+(\omega(v)-\omega(0))G_N^{\omega,v}\to(\beta(v)-\beta(0))G^{\beta,v}+(\omega(v)-\omega(0))G^{\omega,v}$$

weakly in L^2. Therefore $G_N^{\beta,v}\to G^{\beta,v}$ and $G_N^{\omega,v}\to G^{\omega,v}$ weakly in L^2. Suppose that $u', u'+v'$ are not equal to 0 or v.

$$\begin{aligned}
D^{u',u'+v'}D^{0,v}G_N &= D^{0,v}D^{u',u'+v'}G_N\\
&= D^{0,v}\frac{1}{(2m_N+1)^d}\sum_{\substack{u\in B_{m_N}(0)\\ u+v\notin B_{m_N}(0)}}\tau^{-u}D^{u'+u,u'+v'+u}g_{m_N}\\
&= D^{0,v}\frac{1}{(2m_N+1)^d}\sum_{\substack{u\in B_{m_N}(0)\\ u+v\notin B_{m_N}(0)}}\tau^{-u}E[\tau^{u'+u}\tilde{f}_{2N}^{v'}\mid\mathcal{F}_{B_{m_N}(0)}]
\end{aligned}$$

$\to 0$ in L^p as $N\to\infty$. Thus $G^{\beta,v}$ and $G^{\omega,v}$ are constants. Since $L^p\cap\mathcal{H}$ is weaker than L^2 we have shown that (0.20) converges weakly in $L^p\cap\mathcal{H}$ to $G^{\beta,v}(\beta(v)-\beta(0))+G^{\omega,v}(\omega(v)-\omega(0))$ for some constants $G^{\beta,v}$ and $G^{\omega,v}$. This means that we have found cylinder functions g_N such that

$$D^{0,v}\sum_u\tau^u g_N\stackrel{w-L^p\cap\mathcal{H}}{\rightarrow}f^v+G^{\beta,v}(\beta(v)-\beta(0))+G^{\omega,v}(\omega(v)-\omega(0)).$$

But $D^{0,v}\sum_u\tau^u g_N\in\mathcal{C}$ and $G^{\beta,v}(\beta(v)-\beta(0))+G^{\omega,v}(\omega(v)-\omega(0))$ is only in $\mathcal{C}$ if $G^{\beta,v}=v\cdot c$ and $G^{\omega,v}=v\cdot d$ for some c and d in R^d. This proves the theorem. □

10. Central limit theorem variances I

For ϕ and ψ in R^d let

$$\nu_i(u,\phi,\psi) = \phi_i(\beta(u+e_i) - \beta(u)) + \psi_i(\omega(u+e_i) - \omega(u)), \tag{10.1}$$

and $\nu(u,\phi,\psi) = \sum_{i=1}^d \nu_i(u,\phi,\psi)$. Write $\Phi = (\phi,\psi)$. For the discrete points $\rho_\beta, \rho_\omega = \frac{n}{(2k+1)^d}, \frac{m}{(2k+1)^d}$; n and m integers, and $\rho_\beta + \rho_\omega < 1$ define $(\Phi, B_k(\rho_\beta,\rho_\omega)\Phi)$ to be the central limit theorem variance of V in the stationary process on the hyperplane with initial measure $u^k_{\rho_\beta,\rho_\omega}$.

$$(\Phi, B_k(\rho_\beta,\rho_\omega)\Phi) = \frac{1}{(2(k-1)+1)^d} \lim_{t\to\infty} E[(\frac{1}{\sqrt{t}}\int_0^t \sum_{u\in B_{k-1}(0)} \nu(u,\Phi)ds)^2]$$

Define, for sequences ${\rho_\beta}^k, {\rho_\omega}^k = \frac{n^k}{(2k+1)^d}, \frac{m^k}{(2k+1)^d}$,

$$(\Phi, B(\rho_\beta,\rho_\omega)\Phi) = \varlimsup_{\substack{k\to\infty \\ {\rho_\beta}^k, {\rho_\omega}^k \to \rho_\beta,\rho_\omega}} (\Phi, B_k({\rho_\beta}^k, {\rho_\omega}^k)\Phi).$$

Theorem 10.1.

$$(\Phi, B(\rho_\beta,\rho_\omega)\Phi) \le 4(\Phi, (h''Ah''(\rho_\beta,\rho_\omega))^{-1}\Phi).$$

Proof. The result can be rephrased as follows: For any M with

$$\varlimsup_{k\to\infty} (\Phi, B_k({\rho_\beta}^k, {\rho_\omega}^k)\Phi) > M \tag{10.2}$$

we want $(\Phi, (h''Ah'')^{-1}\Phi) \ge \frac{M}{4}$. We will prove this in the following manner: We will produce an $\mathbf{f} \in \mathcal{T} \cap \mathcal{C}$ satisfying

$$E^{\pi_{\rho_\beta,\rho_\omega}}[\sum_{i=1}^d f^{e_i}(\phi_i(\beta(e_i) - \beta(0)) + \psi_i(\omega(e_i) - \omega(0)))] = 2, \tag{10.3}$$

$$E^{\pi_{\rho_\beta,\rho_\omega}}[\sum_v p(v)a(0,v)(f^v)^2] \le 4M^{-1}. \tag{10.4}$$

By Theorem 0.7, f^v is the limit in $L^p \cap \mathcal{H}$ of

$$v \cdot c(\beta(v) - \beta(0)) + v \cdot d(\omega(v) - \omega(0)) + D^{0,v} \sum_u \tau^u g_n$$

for some c and d in R^d and sequence g_n of local functions. Since

$$E^{\pi_{\rho_\beta, \rho_\omega}}[(\phi_i(\beta(e_i) - \beta(0)) - \psi_i(\omega(e_i) - \omega(0)))D^{0,v} \sum_u \tau^u g_n] = 0$$

we get

$$\lim_{n\to\infty} E^{\pi_{\rho_\beta, \rho_\omega}}[\sum_{i=1}^{d}(c_i(\beta(e_i) - \beta(0)) + d_i(\omega(e_i) - \omega(0)))$$

$$\times (\phi_i(\beta(e_i) - \beta(0)) + \psi_i(\omega(e_i) - \omega(0)))]$$

$$= (2(h'')^{-1}(c, d), (\phi, \psi)) = 2.$$

Then (0.27) gives

$$E^{\pi_{\rho_\beta, \rho_\omega}}[\sum_v p(v)a(0, v)(v\cdot c(\beta(v)-\beta(0))+v\cdot d(\omega(v)-\omega(0))-D^{0,v}\sum_u \tau^u g)^2] \leq 4M^{-1}.$$

Therefore

$$\begin{aligned}
((\phi, \psi)(h''Ah'')^{-1}(\phi, \psi)) &= \sup_{x,y\in R^d} \frac{((h'')^{-1}(x, y), (\phi, \psi))}{((x, y), A(x, y))} \\
&\geq \frac{((h'')^{-1}(c, d), (\phi, \psi))}{((c, d), A(c, d))} \\
&\geq \frac{M}{4}.
\end{aligned}$$

So the problem is reduced to finding $\mathbf{f} \in \mathcal{T} \cap \mathcal{H}$ satisfying (0.26) and (0.27).

By (0.25) there exists a sequence g^k such that

$$(E^{u^k_{\rho_\beta{}^k, \rho_\omega{}^k}}[g^k \sum_{u\in B_{k-1}} \nu(u)])^2 > \frac{M(2k+1)^d}{2} \mathcal{D}_k(g^k)$$

By the remarks preceding Theorem 0.4, g^k can be taken to be of the form $g^k = \sum_u (a\beta(u) + b\omega(u))F_u(\zeta)$. Summing by parts and using the definition of the Dirichlet form, we have, for the expectation with respect to $u^k_{\rho_\beta{}^k, \rho_\omega{}^k}$,

$$(E[\sum_i \sum_{u,u+e_i \in B_{k-1}(0)} D^{u,u+e_i} g^k \nu_i(u)])^2$$

$$> M(2k+1)^d E[\sum_{u,u+v \in B_k(0)} p(v)a(u,u+v)(D^{u,u+v}g^k)^2]. \tag{10.5}$$

Let $m < k-1$, $1 < p < 2$, and q be the conjugate exponent of p. We claim that there exists a sequence $y^k \in B_{k-1-m}(0)$ and a constant C_0 such that

$$(E[\sum_i \sum_{u,u+e_i \in B_m(y^k)} D^{u,u+e_i} g^k \nu_i(u)])^2$$

$$\geq (1 - C_0 k^{-\frac{1}{2q}})M(2m+1)^d E[\sum_{u,u+v \in B_m(y^k)} p(v)a(u,u+v)(D^{u,u+v}g^k)^2]. \tag{10.6}$$

C_0 is a fixed constant to be described. Suppose there is no such sequence y^k. We can average over $y \in B_{k-1-m}(0)$ and use Jensen's inequality to obtain

$$(E[\tfrac{1}{(2(k-1-m)+1)^d} \sum_{y \in B_{k-1-m}} \tfrac{1}{(2m+1)^d} \sum_i \sum_{u,u+e_i \in B_m(y^k)} D^{u,u+e_i} g^k \nu_i(u)])^2$$

$$< (1 - C_0 k^{-\frac{1}{2q}}) \frac{M}{(2k+1)^d} \mathcal{D}_k(g^k). \tag{10.7}$$

The missing boundary terms on the left hand side can be estimated by using Lemma 4 and Hölder's inequality,

$$E[\sum_i \sum_{\substack{u,u+e_i \in B_{k-1} \\ u \text{ or } u+e_i \notin B_{k-1-m}}} D^{u,u+e_i} g^k \nu_i(u)]$$

$$\leq C_1(E[\sum_{u,u+v \in B_k} p(v) \mid D^{u,u+v} g^k \mid^p])^{\frac{1}{p}} (E[\sum_i \sum_{\substack{u,u+e_i \in Bk-1 \\ u \text{ or } u+e_i \notin B_{k-1-m}}} \mid \nu_i(u) \mid^q])^{\frac{1}{q}}$$

$$\leq C_2(k^{d(\frac{2-p}{2})}(\mathcal{D}_k(g^k))^{\frac{p}{2}})^{\frac{1}{p}}(mk^{d-1})^{\frac{1}{q}} = C_2(\mathcal{D}_k(g^k))^{\frac{1}{2}}k^{\frac{d}{2}-\frac{1}{q}}. \tag{10.8}$$

Taking $C_0 > 2C_2$, (0.30), together with (0.31), contradict (0.28), establishing (0.29). By rescaling the g^k we can restate this as

$$\tfrac{1}{(2m+1)^d}E^{u^k_{\rho_\beta{}^k,\,\rho_\omega{}^k}}[\sum_i \sum_{u,u+e_i\in B_m(y^k)} D^{u,u+e_i}g^k\nu_i(u)] \geq 2 + O(k^{-\frac{1}{q}}) \tag{10.9}$$

$$\tfrac{1}{(2m+1)^d}E^{u^k_{\rho_\beta{}^k,\,\rho_\omega{}^k}}[\sum_{u,u+v\in B_m(y^k)} p(v)a(u,u+v)(D^{u,u+v}g^k)^2] \leq 4M^{-1}. \tag{10.10}$$

Let

$$g^{k,m} = \tau^{-y}E^{u^k_{\rho_\beta{}^k,\,\rho_\omega{}^k}}[g^k \mid \mathcal{F}_{B_m(y^k)}]$$

and let μ^k be the projection of $u^k_{\rho_\beta{}^k,\,\rho_\omega{}^k}$ onto $B_m(0)$. We have

$$\tfrac{1}{(2m+1)^d}E^{\mu^k}[\sum_i \sum_{u,u+e_i\in B_m(0)} D^{u,u+e_i}g^{k,m}\nu_i(u)] \geq 2 + O(k^{-\frac{1}{q}}) \tag{10.11}$$

$$\tfrac{1}{(2m+1)^d}E^{\mu^k}[\sum_{u,u+v\in B_m(0)} p(v)a(u,u+v)(D^{u,u+v}g^{k,m})^2] \leq 4M^{-1}. \tag{10.12}$$

If we replace $g^{k,m}$ by $\tilde{g}^{k,m} = \mathbf{1}_{X_k\setminus A_m}g^{k,m}$ then the second inequality is still true and, using Lemma 4 and Hölder's inequality, we can show that the error in (0.35) goes to zero as k, then $m \to \infty$. The $\tilde{g}^{k,m}$ are still of the form $g^k = \sum_u(a\beta(u) + b\omega(u))F_u(\zeta)$. Since we have neutralised overfull configurations by multiplying by $\mathbf{1}_{X_k\setminus A_m}$ we can apply Lemma 11 to μ^k. This gives

$$E^{\mu^k}[(\tilde{g}^{k,m})^2] \leq Cm^2E^{\mu^k}[\sum_{u,u+v\in B_m(0)} p(v)(D^{u,u+v}\tilde{g}^{k,m})^2] \leq 4Cm^2M^{-1},$$

i.e. $\tilde{g}^{k,m}$ is pointwise bounded independent of k. So a subsequence $\tilde{g}^{k,m} \to g^m$ with

$$\tfrac{1}{(2m+1)^d}E^{\pi_{\rho_\beta,\,\rho_\omega}}[\sum_i \sum_{u,u+e_i\in B_m(0)} D^{u,u+e_i}\tilde{g}^m\nu_i(u)] \geq 2 \tag{10.13}$$

$$\tfrac{1}{(2m+1)^d}E^{\pi_{\rho_\beta,\,\rho_\omega}}[\sum_{u,u+v\in B_m(0)} p(v)a(u,u+v)(D^{u,u+v}\tilde{g}^m)^2] \leq 4M^{-1}. \tag{10.14}$$

For $c < m$ let

$$\xi^{m,c} = \tfrac{1}{(2m+1)^d} \sum_{x \in B_{m-c}(0)} \tau^x g^m.$$

This has the property that if $u \in B_c(0)$,

$$E^{\pi_{\rho_\beta,\,\rho_\omega}}[\mid D^{0,v}\xi^{m,c} - \tau^u D^{u,u+v}\xi^{m,c} \mid]$$

$$= \tfrac{1}{(2m+1)^d} E^{\pi_{\rho_\beta,\,\rho_\omega}}[\mid D^{0,v}(\sum_{x \in B_{m-c}(0)} \tau^x g^m - \sum_{x \in B_{m-c}(u)} \tau^x g^m) \mid]$$

$$\leq \tfrac{1}{(2m+1)^d} \sum_{u \in B_m(0)} E^{\pi_{\rho_\beta,\,\rho_\omega}}[\mid D^{u,u+v} g^m \mid \mathbf{1}_{u \in B_{m-c}(0) \triangle B_{m-c}(u)}]$$

$$\leq (\tfrac{1}{(2m+1)^d} \sum_{u \in B_m(0)} E^{\pi_{\rho_\beta,\,\rho_\omega}}[\mid D^{u,u+v} g^m \mid^p])^{\frac{1}{p}} (\tfrac{1}{(2m+1)^d} \sum_{u \in B_m(0)} E^{\pi_{\rho_\beta,\,\rho_\omega}}[\mathbf{1}_{u \in B_{m-c}(0) \triangle B_{m-c}(u)}])^{\frac{1}{q}}$$

$$\leq (C(2m+1)^{\frac{dp}{2}} (E^{\pi_{\rho_\beta,\,\rho_\omega}}[\sum_{u,u+v \in B_m(0)} p(v)(D^{u,u+v} g^{k,m})^2])^{\frac{p}{2}})^{\frac{1}{p}} (Cm^{-1})^{\frac{1}{q}}. \tag{10.15}$$

What (0.38) shows is that for any closed chain of bonds inside $B_c(0)$

$$\sum_{i=1}^{n} \tau^{u_i} D^{0,v_i} \xi^{m,c}(\eta^{b_1 \cdots b_{i-1}}) = O(m^{-\frac{1}{q}}).$$

The $\xi^{m,c}$ also satisfy:

$$E^{\pi_{\rho_\beta,\,\rho_\omega}}[\sum_{i=1}^{d} D^{0,e_i} \xi^{m,c} \nu_i(0)] \geq 2 + O(m^{-\frac{1}{q}}) \tag{10.16}$$

$$E^{\pi_{\rho_\beta,\,\rho_\omega}}[\sum_{v} p(v) a(0,v) (D^{0,v} \xi^{m,c})^2] \leq 4M^{-1} \tag{10.17}$$

The second of these follows from Jensen's inequality. The first differs from (0.36) by

$$\tfrac{1}{(2m+1)^d} E^{\pi_{\rho_\beta,\,\rho_\omega}}[\sum_{\substack{u \in B_m(0) \setminus B_{m-c}(0) \\ u+e_i \in B_m(0)}} D^{u,u+e_i} \tilde{g}^m \nu_i(u)] = O(m^{-\frac{1}{q}})$$

Write $f^{m,c,v} = D^{0,v}\xi^{m,c}$.

$$E^{\pi_{\rho_\beta,\,\rho_\omega}}[\mid f^{m,c,v} \mid^p] = E^{\pi_{\rho_\beta,\,\rho_\omega}}[\mid \tfrac{1}{(2m+1)^d} \sum_{x \in B_{m-c}(0)} \tau^x D^{-x,-x+v} g^m \mid^p]$$

$$\tfrac{1}{(2m+1)^d} \sum_{x \in B_{m-c}(0)} E^{\pi_{\rho_\beta,\,\rho_\omega}}[| \, D^{-x,-x+v} g^m \, |^p] < C$$

So we can take a subsequence which converges to f^v weakly in L^p as m, then $c \to \infty$. From (0.38), (0.39), (0.40) we get that $\mathbf{f} \in \mathcal{T} \cap \mathcal{C}$ and satisfies (0.26) and (0.27), which completes the proof. □

11. Central limit theorem variances II

For notational purposes let $F(\rho_\beta, \rho_\omega) \in R^d \times R^d$ be given by

$$F = (F^\beta, F^\omega) = \frac{1}{2}(h'' A) \begin{pmatrix} \phi \\ \psi \end{pmatrix}. \tag{11.1}$$

Let $V^k = V_I^k - V_{II}^k$ where, for a cylinder function g,

$$\begin{aligned} V_I^k(u) = \sum_v p(v)(v \cdot \phi \tfrac{1}{(2k-1)^d} \sum_{y \in B_{k-1}(u)} a(y, y+v)(\beta(y+v) - \beta(y)) \\ + v \cdot \psi \tfrac{1}{(2k-1)^d} \sum_{y \in B_{k-1}(u)} a(y, y+v)(\omega(y+v) - \omega(y)) - \hat{\mathcal{L}}_{2k+1} g, \end{aligned} \tag{11.2}$$

and, for ν defined in (0.24),

$$V_{II}^k(u) = \tfrac{1}{(2k-1)^d} \sum_{y \in B_{k-1}(u)} \nu(y, F(\rho_\beta^k(u), \rho_\omega^k(u)). \tag{11.3}$$

Lemma 12. *For the expectation with respect to the non–speeded up process on B_k with initial distribution $u^k_{\rho_\beta{}^k,\,\rho_\omega{}^k}$,*

$$\overline{\lim_{\substack{k \to \infty \\ \rho_\beta{}^k, \rho_\omega{}^k \to \rho_\beta, \rho_\omega}}} \tfrac{1}{(2k+1)^d} \lim_{t \to \infty} E[(\frac{1}{\sqrt{t}} \int_0^t V^k(\eta_s) ds)^2] \le Q_g((\phi, \psi)) - ((\phi, \psi), A(\phi, \psi)).$$

Proof.

$$\lim_{t \to \infty} E[(\frac{1}{\sqrt{t}} \int_0^t V^k(\eta_s) ds)^2] = 2 < V^k, (\hat{\mathcal{L}}_{2k+1})^{-1} V^k >$$

$$= 2 < V_I^k, (\hat{\mathcal{L}}_{2k+1})^{-1} V_I^k > -4 < V_{II}^k, (\hat{\mathcal{L}}_{2k+1})^{-1} V_I^k > +2 < V_{II}^k, (\hat{\mathcal{L}}_{2k+1})^{-1} V_{II}^k > \tag{11.4}$$

We have already calculated in Section that

$$\overline{\lim_{\substack{k\to\infty \\ \rho_\beta{}^k,\rho_\omega{}^k\to\rho_\beta,\rho_\omega}}} \frac{1}{(2k+1)^d} 2 < V_{II}^k, (\hat{\mathcal{L}}_{2k+1})^{-1} V_{II}^k > \leq ((\phi,\psi), A(\phi,\psi)). \tag{11.5}$$

We now calculate the two other terms. Notice that

$$V_I = \hat{\mathcal{L}}_{2k+1}[\sum_{x\in B_k(0)} x\cdot\phi\beta(x) + x\cdot\psi\omega(x) + \sum_{x\in B_{k-l}(0)} \tau^x g]$$

so that

$$2 < V_I^k, (\hat{\mathcal{L}}_{2k+1})^{-1} V_I^k >$$

$$= E[\sum_{u,u+v\in B_k(0)} p(v)a(u,u+v)(D^{u,u+v}[\sum_{x\in B_k(0)} x\cdot\phi\beta(x) + x\cdot\psi\omega(x) + \sum_{x\in B_{k-l}(0)} \tau^x g])^2]$$

$$= E[\sum_{u,u+v\in B_k(0)} p(v)a(u,u+v)(v\cdot\phi(\beta(u+v)-\beta(u))+v\cdot\psi(\omega(u+v)-\omega(u))+D^{u,u+v}\sum_{x\in B_{k-l}(0)}\tau^x g)^2]$$

$$= E[\sum_{u,u+v\in B_k(0)} p(v)a(0,v)(v\cdot\phi(\beta(v)-\beta(0))+v\cdot\psi(\omega(v)-\omega(0))+D^{0,v}\sum_{x\in B_{k-l}(u)}\tau^x g)^2]$$

$$= E[\sum_{u,u+v\in B_{k-2l}(0)} p(v)a(0,v)(v\cdot\phi(\beta(v)-\beta(0))+v\cdot\psi(\omega(v)-\omega(0))+D^{0,v}\sum_{x\in B_{k-l}(u)}\tau^x g)^2]$$

$$+E[\sum_{\substack{u,u+v\in B_k(0) \\ u \text{ or } u+v\notin B_{k-2l}(0)}} p(v)a(0,v)(v\cdot\phi(\beta(v)-\beta(0))+v\cdot\psi(\omega(v)-\omega(0))+D^{0,v}\sum_{x\in B_{k-l}(u)}\tau^x g)^2].$$

The second term is a sum of $k^{d-1}2l$ terms each of order l^d, so in the limit, when divided by k^d, it is negligible. If $u\in B_{k-2l}(0)$ then

$$D^{0,v}\sum_{x\in B_{k-l}(u)}\tau^x g = D^{0,v}\sum_{x\in B_{k-l}(0)}\tau^x g$$

so in the first term all the summands are the same. We get

$$\overline{\lim_{\substack{k\to\infty \\ \rho_\beta{}^k,\rho_\omega{}^k\to\rho_\beta,\rho_\omega}}} \frac{1}{(2k+1)^d} 2 < V_I^k, (\hat{\mathcal{L}}_{2k+1})^{-1} V_I^k > = Q_g((\phi,\psi)). \tag{11.6}$$

It remains to calculate the cross term

$$\frac{1}{(2k+1)^d} 4 < V_{II}^k, (\hat{\mathcal{L}}_{2k+1})^{-1} V_I^k > .$$

$$= \frac{4}{(2k+1)^d} E[(\sum_{y \in B_k} \sum_{i=1}^{d} F_i^{\beta} (\beta(u+e_i) - \beta(u)) + F_i^{\omega} (\omega(u+e_i) - \omega(u)))$$

$$\times (\sum_{x \in B_k} x \cdot \phi\beta(x) + x \cdot \phi\omega(x) + \tau^x g)]$$

$$= 2 \sum_{i=1}^{d} \frac{1}{(2k+1)^d} \sum_{y \in B_k} E[(F_i^{\beta} (\beta(u+e_i) - \beta(u)) + F_i^{\omega} (\omega(u+e_i) - \omega(u)))$$

$$\times (\phi_i(\beta(u+e_i) - \beta(u)) + \psi_i(\omega(u+e_i) - \omega(u)) + D^{u,u+e_i} \sum_{x \in B_k} \tau^x g)].$$

Since $E[(\beta(u+e_i) - \beta(u)) \sum_{x \in B_{k-l_0}} \tau^x g] = 0$ we get

$$\varlimsup_{\substack{k \to \infty \\ \rho_\beta{}^k, \rho_\omega{}^k \to \rho_\beta, \rho_\omega}} \frac{1}{(2k+1)^d} 2 < V_{II}^k, (\hat{\mathcal{L}}_{2k+1})^{-1} V_I^k >= 2((\phi, \psi), A(\phi, \psi)). \tag{11.7}$$

The proof is completed by plugging (0.45), (0.46), and (0.47) into (0.44). $\square$

We would like to apply Theorem 0.6 using the results of Lemma 12. Unfortunately, V_{II}^k does not satisfy the condition that $\mathcal{L}_N + \sum_u V_{II}^k(u)$ have principal eigenvalue bounded by BN^d. Therefore we introduce cutoff functions. Recall that A_l is the set of configurations having one or fewer occupied sites in the box B_l. Define

$$\nu^l(u, \phi, \psi) = \sum_{i=1}^{d} \phi_i(\beta(u+e_i)\tau^{u+e_i} \mathbf{1}_{X_N \setminus A_l} - \beta(u)\tau^u \mathbf{1}_{X_N \setminus A_l})$$

$$+ \psi_i(\omega(u+e_i)\tau^{u+e_i} \mathbf{1}_{X_N \setminus A_l} - \omega(u)\tau^u \mathbf{1}_{X_N \setminus A_l}) \tag{11.8}$$

and

$$V_{II}^{k,l}(u). = \tfrac{1}{(2k-l)^d} \sum_{y \in B_{k-l}(u)} \nu^l(y, F(\rho_\beta^k(u), \rho_\omega^k(u)) \tag{11.9}$$

Using Lemma 6 we can show that for $\rho < 1$ there exists a constant C depending only on ρ such that

$$\overline{\lim_{\substack{k\to\infty \\ \rho_\beta{}^k,\rho_\omega{}^k\to\rho_\beta,\rho_\omega}}} \tfrac{1}{(2k+1)^d} < (V_{II}^{k,l} - V_{II}^k), (\hat{\mathcal{L}}_{2k+1})^{-1}(V_{II}^{k,l} - V_{II}^k) > \leq Cl^{2d(\frac{p-1}{p})}\rho^{\frac{l^d}{2}}, \quad (11.10)$$

so that for large l, Lemma 12 is essentially unchanged by substituting $V_{II}^{k,l}$ for V_{II}^k. Now let $V_l(u) = \frac{1}{(2k-1)^d}\sum_{y\in B_{k-l-1}} \nu^l(y, \phi(u/N), \psi(u/N))$, and let λ_N^l be the principal eigenvalue of $\mathcal{L}_N + N\sum_u V_l(u)$. Using Lemma 5, we can show that there exists a constant B such that

$$\lambda_N^l \leq Bl^{4d}N^d. \quad (11.11)$$

Therefore, for each l our cutoff functions satisfy the conditions of Theorem 0.6.

The next result is needed in the proof of Lemma 13, but has independent interest.

Theorem 11.1. *The self-diffusion covariance $D_s(\rho)$ is continuous for $0 < \rho < 1$.*

Proof. We actually prove that $(\Phi, A\Phi)$ is continuous in ρ_β and ρ_ω. Since $(\Phi, A\Phi)$ is the infimum over cylinder functions g of $Q_g(\Phi)$, it is upper semicontinuous. In Section we showed that

$$4(\Phi, (h''Ah'')^{-1}(\rho_\beta, \rho_\omega)\Phi) \geq \overline{\lim_{\substack{k\to\infty \\ \rho_\beta{}^k,\rho_\omega{}^k\to\rho_\beta,\rho_\omega}}} (\Phi, B_k(\rho_\beta{}^k, \rho_\omega{}^k)\Phi).$$

where B_k are continuous. By Lemma 12 we have equality here. Therefore $(\Phi, (h''Ah'')^{-1}\Phi)$ is upper semicontinuous. But $(h'')^{-1}$ is continuous so $(\Phi, A^{-1}\Phi)$ is upper semicontinuous. Then $(\Phi, A\Phi) = \sup_x(\Phi, x) - (x, A^{-1}x)$ is lower semicontinuous. $\square$

Define

$$\tilde{V}_{II}^{k,l}(u) = \tfrac{1}{2N\epsilon+1} \sum_{\substack{x=u+he_i \\ |h|\leq N\epsilon}} \tfrac{1}{(2k-1)^d} \sum_{y\in B_{k-1}(x)} \nu^l(y, F(\rho_\beta^{N\epsilon}(u), \rho_\omega^{N\epsilon}(u)). \quad (11.12)$$

Lemma 13.

$$\overline{\lim_{l\to\infty}}\ \overline{\lim_{m\to\infty}}\ \overline{\lim_{k\to\infty}}\ \overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}}\ \frac{1}{N^d}\log E^{P_N}[\exp N^d G_{N,\epsilon,k,m,l}] \le 0$$

where

$$G_{n,\epsilon,k,m,l} = \int_0^T \frac{1}{N^d}\sum_u N(V_{II}^{k,l} - \tilde{V}_{II}^{k,l})(\frac{u}{N}, s, \{\eta(x)\}_{x\in B_{N\epsilon}(u)}) - m\tau^u \mathbf{1}_{A_k} ds.$$

Proof. We can rearrange $\sum_u N(V_{II}^{k,l} - \tilde{V}_{II}^{k,l})$ to get $\sum_u \nu^l(u, J(u))$ where $J \in R^{2d}$ is given by

$$J(u) = \tfrac{1}{(2k-1)^d}\sum_{y\in B_{k-1}(u)} \tfrac{1}{2N\epsilon+1}\sum_{\substack{x=y+he_i\\ |h|\le N\epsilon}} (F(\frac{x}{N}, s, \rho_\beta^{N\epsilon}(x), \rho_\omega^{N\epsilon}(x)) - F(\frac{u}{N}, s, \rho_\beta^k(u), \rho_\omega^k(u))).$$

By Feynman-Kac formula and spectral theorem

$$\log E^{P_N}[\exp(\int_0^T N\sum_u \nu^l(u, J(u)) - m\tau^u\mathbf{1}_{A_k} ds)]$$

is majorized by

$$\sup_{f_N} E^{\pi^N}[f_N N\sum_u a\nu^l(u, J(u)) - m\tau^u\mathbf{1}_{A_k}] - N^2\mathcal{D}_N(\sqrt{f_n}).$$

The functional $J(u)$ only involves empirical densities $\rho_\beta^{N\epsilon}(u), \rho_\omega^{N\epsilon}(u), \rho_\beta^k(u)$ and $\rho_\omega^k(u)$. Since $N \gg k \gg l$, $J(u)$ is unaffected by any exchange between sites in $B_{2l}(u)$. Therefore we can apply Lemma 5 to obtain

$$E[f_N\nu^l(u, J(u)) \le Cl^{2d}\sum_{y,y+v\in B_{2l}(u)}\sqrt{E[a(y,y+v)(D^{y,y+v}(\sqrt{f_n}))^2]}\sqrt{E[\mid J(u)\mid^2 f_N]}.$$

Then we can estimate the eigenvalue

$$\sup_{f_N} E^{\pi^N}[f_N N\sum_u a\nu^l(u, J(u)) - m\tau^u\mathbf{1}_{A_k}] - N^2\mathcal{D}_N(\sqrt{f_n}).$$

$$\le \sup_{f_N}\sum_u CNl^{2d}\sum_{y,y+v\in B_{2l}(u)}\sqrt{E^{\pi^N}[a(y,y+v)(D^{y,y+v}(\sqrt{f_n}))^2]}\sqrt{E^{\pi^N}[\mid J(u)\mid^2 f_N]}$$

$$- \frac{N^2}{2(2l+1)^d} E^{\pi^N}[a(y,y+v)(D^{y,y+v}(\sqrt{f_n}))^2] - \frac{N^2}{2}\mathcal{D}_N(\sqrt{f_n}) - mE^{\pi^N}[\tau^u \mathbf{1}_{A_k}]$$

$$\leq \sup_{f_N} \sum_u C l^{6d} E^{f_N \pi^N}[\mid J(u) \mid^2 - m\tau^u \mathbf{1}_{A_k}] - \frac{N^2}{2}\mathcal{D}_N(\sqrt{f_n}).$$

Now by Theorem 0.9, F is uniformly continuous so we can apply Lemma 7 to get

$$\overline{\lim_{l\to\infty}}\,\overline{\lim_{m\to\infty}}\,\overline{\lim_{k\to\infty}}\,\overline{\lim_{\epsilon\to 0}}\,\overline{\lim_{N\to\infty}} \frac{1}{N^d} \sup_{f_N} \sum_u C l^{6d} E^{f_N \pi^N}[|J(u)|^2 - m\tau^u \mathbf{1}_{A_k}] - \frac{N^2}{2}\mathcal{D}_N(\sqrt{f_n}) \leq 0$$

as desired. □

12. Hydrodynamic limit

In this section we apply the variance calculations of Sections and and Theorem 0.6 to prove the hydrodynamic limit.

Let $\tilde{Q}$ be any limit point of $\tilde{Q}_N$ (cf.p.9). We first prove a preliminary estimate.

Lemma 14.

$$E^{\tilde{Q}}[\int_{T^d}\int_0^\infty (\nabla \begin{pmatrix} \lambda_\beta \\ \lambda_\omega \end{pmatrix}, A\nabla \begin{pmatrix} \lambda_\beta \\ \lambda_\omega \end{pmatrix}) dt d\theta] \leq 8\log 3.$$

Proof. We first remark that μ_β, μ_ω have densities ρ_β, ρ_ω almost surely with respect to $\tilde{Q}$, so the integrand above is well defined. For test functions $\phi, \psi : T^d \to R^d$ and ν^l as in (0.48) define the functional

$$F_{N,\epsilon,k,m,l}(\phi,\psi) = \int_0^T \frac{1}{N^d}\sum_u N\nu^l(u, \phi(\frac{u}{N}), \psi(\frac{u}{N}))$$

$$- 2((\phi(\frac{u}{N}), \psi(\frac{u}{N})), (h''Ah'')^{-1}(\rho_\beta^{N\epsilon}(u), \rho_\omega^{N\epsilon}(u))(\phi(\frac{u}{N}), \psi(\frac{u}{N})))$$

$$- Cl^{2d(\frac{p-1}{p})}(\rho^{N\epsilon}(u))^{\frac{l^d}{2}} - m\tau^u \mathbf{1}_{A_k} ds.$$

Let $\{\phi^i\}_{i=1,2,\ldots}$, $\{\psi^j\}_{j=1,2\ldots}$ be countable dense subsets of $C^\infty(T^d \to R^d)$. By Theorem 0.8, (0.50), (0.51), Theorem 0.6, and the entropy inequality,

$$\overline{\lim_{l\to\infty}}\,\overline{\lim_{m\to\infty}}\,\overline{\lim_{k\to\infty}}\,\overline{\lim_{\epsilon\to 0}}\,\overline{\lim_{N\to\infty}} E^{\tilde{Q}_N}[\max_{i,j\leq M} F_{N,\epsilon,k,m,l}(\phi^i,\psi^j)]$$

$$\leq \overline{\lim_{l\to\infty}}\ \overline{\lim_{m\to\infty}}\ \overline{\lim_{k\to\infty}}\ \overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}}\ \frac{1}{N^d}\log\sum_{i,j\leq M} E^{\tilde{P}_N}[\exp N^d F_{N,\epsilon,k,m,l}(\phi^i,\psi^j)] + \frac{H(Q_N;P_N)}{N^d}$$

$$\leq \log 3.$$

We write $F_{N,\epsilon,k,m,l} = F^1_{N,\epsilon} + F^2_{N,\epsilon,k,m,l}$ where

$$F^1_{N,\epsilon} = \int_0^T \frac{1}{N^d}\sum_u N(\sum_{i=1}^d (\phi_i(\frac{u-e_i}{N}) - \phi_i(\frac{u}{N}))\beta(u) + (\psi_i(\frac{u-e_i}{N}) - \psi_i(\frac{u}{N}))\omega(u))$$

$$- 2((\phi(\frac{u}{N}),\psi(\frac{u}{N})),(h''Ah'')^{-1}(\rho^{N\epsilon}_\beta(u),\rho^{N\epsilon}_\omega(u))(\phi(\frac{u}{N}),\psi(\frac{u}{N})))ds,$$

and

$$F^2_{N,\epsilon,k,m,l} = \int_0^T \frac{1}{N^d}\sum_u N(\sum_{i=1}^d (\phi_i(\frac{u-e_i}{N}) - \phi_i(\frac{u}{N}))\beta(u) + (\psi_i(\frac{u-e_i}{N}) - \psi_i(\frac{u}{N}))\omega(u))\tau^u \mathbf{1}_{A_l}$$

$$- Cl^{2d(\frac{p-1}{p})}(\rho^{N\epsilon}(u))^{\frac{l^d}{2}} - m\tau^u \mathbf{1}_{A_k} ds.$$

By (0.14),

$$\overline{\lim_{l\to\infty}}\ \overline{\lim_{m\to\infty}}\ \overline{\lim_{k\to\infty}}\ \overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}}\ E^{\tilde{Q}_N}[\max_{i,j\leq M} F^2_{N,\epsilon,k,m,l}] = 0.$$

Therefore

$$\overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}}\ E^{\tilde{Q}_N}[\max_{i,j\leq M} F^1_{N,\epsilon}] \leq \log 3.$$

As functionals $F^1_{N,\epsilon}$ converge, as $N\to\infty$, to

$$F^1_\epsilon = \int_{T^d}\int_0^T -\nabla\cdot\phi(\theta)\rho_\beta(\theta) - \nabla\cdot\psi(\theta)\rho_\omega(\theta)$$

$$- 2((\phi(\theta),\psi(\theta))(h''Ah'')^{-1}(\frac{\mu_\beta(\theta-\epsilon,\theta+\epsilon)}{2\epsilon},\frac{\mu_\omega(\theta-\epsilon,\theta+\epsilon)}{2\epsilon})(\phi(\theta),\psi(\theta)))dsd\theta,$$

and the convergence is uniform, so

$$\overline{\lim_{\epsilon\to 0}}\, E^{\tilde{Q}}[\max_{i,j\leq M} F^1_\epsilon] \leq \log 3.$$

Now there is a fixed $\delta > 0$ such that $\rho(\theta) < 1-\delta$ almost everywhere with respect to $\tilde{Q}$ and the function $\rho_\beta, \rho_\omega \to ((\phi,\psi),(h''Ah'')^{-1}(\rho_\beta,\rho_\omega)(\phi,\psi))$ is continuous away from $\rho = 1$. Therefore F_ϵ^1 is uniformly bounded $\tilde{Q}$–a.e., and this allows us to take the limit inside the expectation. Then we can let $M \to \infty$ to get

$$E^{\tilde{Q}}[\sup_{\phi,\psi}\int_{T^d}\int_0^T -\nabla\cdot\phi\rho_\beta - \nabla\cdot\psi\rho_\omega - 2((\phi,\psi),(h''Ah'')^{-1}(\rho_\beta,\rho_\omega)(\phi,\psi))dsd\theta] \leq \log 3.$$

Since $h''\nabla\begin{pmatrix}\rho_\beta\\ \rho_\omega\end{pmatrix} = \nabla\begin{pmatrix}\lambda_\beta\\ \lambda_\omega\end{pmatrix}$ this is the variational form of the lemma. $\square$

The main result is

Theorem 12.1. *For test functions $\phi,\psi : T^d \to R$ the following holds $\tilde{Q}$–a.e.*

$$\int_{T^d}\phi(\theta)\rho_\beta(\theta,T)+\psi(\theta)\rho_\omega(\theta,t)d\theta - \int_{T^d}\phi(\theta)\rho_\beta(\theta,0)+\psi(\theta)\rho_\omega(\theta,0)d\theta$$

$$= \frac{1}{2}\int_{T^d}\int_0^T ((\nabla\phi(\theta),\nabla\psi(\theta)),(Ah'')(\rho_\beta(\theta,t),\rho_\omega(\theta,t))(\nabla\rho_\beta(\theta,t),\nabla\rho_\omega(\theta,t))dtd\theta.$$

Proof: By (0.13) we have that $\tilde{Q}$–a.e.

$$\int_{T^d}\phi(\theta)\rho_\beta(\theta,T)+\psi(\theta)\rho_\omega(\theta,T)d\theta - \int_{T^d}\phi(\theta)\rho_\beta(\theta,0)+\psi(\theta)\rho_\omega(\theta,0)d\theta$$

$$=< (\nabla\phi,\gamma_\beta(T)) > + < (\nabla\psi,\gamma_\omega(T)) > . \tag{12.1}$$

For test functions $\phi,\psi : T^d \to R^d$ and V_I^k, $\tilde{V}_{II}^{k,l}$ as in (0.42), (0.52) , and Q_g as in (0.9) consider the functional

$$G_{N,\epsilon,k,m,l}(\phi,\psi,g) = \int_0^T \frac{1}{N^d}\sum_u N(V_I^k(u,g,\phi(\frac{u}{N}),\psi(\frac{u}{N})) - \tilde{V}_{II}^{k,l}(u,\phi(\frac{u}{N}),\psi(\frac{u}{N})))$$

$$- (Q_g(\rho_\beta^{N\epsilon}(u),\rho_\omega^{N\epsilon}(u),\phi(\frac{u}{N}),\psi(\frac{u}{N}))$$

$$- ((\phi(\frac{u}{N}),\psi(\frac{u}{N})),A(\rho_\beta^{N\epsilon}(u),\rho_\omega^{N\epsilon}(u))(\phi(\frac{u}{N}),\psi(\frac{u}{N})))$$

$$- Cl^{2d(\frac{p-1}{p})}(\rho^{N\epsilon}(u))^{\frac{l^d}{2}} - m\tau^u\mathbf{1}_{A_k}dt.$$

Let $\{\phi^i\}_{i=1,2,\ldots}$, $\{\psi^j\}_{j=1,2\ldots}$ be countable dense subsets of $C^\infty(T^d \to R^d)$ and $\{g^h\}_{h=1,2,\ldots}$ an enumeration of the cylinder functions on X. By Lemmas 12, 13, (0.50), (0.51), and a slight modification of Theorem 0.6,

$$\overline{\lim_{l\to\infty}}\ \overline{\lim_{m\to\infty}}\ \overline{\lim_{k\to\infty}}\ \overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}}\ E^{\tilde{Q}_N}[\max_{i,j,h\le M} G_{N,\epsilon,k,m,l}(\phi^i,\psi^j,g^h)]$$

$$\le \overline{\lim_{l\to\infty}}\ \overline{\lim_{m\to\infty}}\ \overline{\lim_{k\to\infty}}\ \overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}}\ \frac{1}{N^d}\log\sum_{i,j,h\le M} E^{\tilde{P}_N}[\exp N^d G_{N,\epsilon,k,m,l}(\phi^i,\psi^j,g^h)]$$

$$+\frac{H(Q_N;P_N)}{N^d}$$

$$\le \log 3.$$

We can write $G_{N,\epsilon,k,m,l} = G^1_{N,\epsilon,k} + G^2_N + G^3_{N,\epsilon,k,m,l} + G^4_{N,\epsilon}$ where

$$G^1_{N,\epsilon,k} = \int_0^T \frac{1}{N^d}\sum_{u,v} p(v)\Big(\tfrac{1}{(2k-1)^d}\sum_{y\in B_{k-1}(u)} a(y,y+v)(v\cdot\phi(\frac{u}{N})(\beta(y+v)-\beta(y))$$

$$+v\cdot\psi(\frac{u}{N})(\omega(y+v)-\omega(y)))$$

$$-\sum_{i=1}^d F_i^\beta(\rho_\beta^{N\epsilon}(u),\rho_\omega^{N\epsilon}(u))\Big(\frac{\rho_\beta^k(u+N\epsilon e_i)-\rho_\beta^k(u-N\epsilon e_i)}{2N\epsilon+1}\Big)$$

$$+F_i^\omega(\rho_\beta^{N\epsilon}(u),\rho_\omega^{N\epsilon}(u))\Big(\frac{\rho_\omega^k(u+N\epsilon e_i)-\rho_\omega^k(u-N\epsilon e_i)}{2N\epsilon+1}\Big),$$

$$G^2_N = \int_0^T \frac{1}{N^d}\sum_u N\hat{\mathcal{L}}_N g(\frac{u}{N},t,\{\eta(x)\}_{x\in B_{l_0}(u)})dt,$$

$$G^3_{N,\epsilon,k,m,l} = \int_0^T \frac{1}{N^d}\sum_u -Cl^{2d(\frac{p-1}{p})}(\rho^{N\epsilon}(u))^{\frac{l^d}{2}} - m\tau^u\mathbf{1}_{A_k}$$

$$+\sum_{i=1}^{d} F_i^{\beta}(\rho_\beta^{N\epsilon}(u),\rho_\omega^{N\epsilon}(u))(\frac{{\rho_\beta}^{k,l}(u+N\epsilon e_i)-{\rho_\beta}^{k,l}(u-N\epsilon e_i)}{2N\epsilon+1})$$

$$+F_i^{\omega}(\rho_\beta^{N\epsilon}(u),\rho_\omega^{N\epsilon}(u))(\frac{{\rho_\omega}^{k,l}(u+N\epsilon e_i)-{\rho_\omega}^{k,l}(u-N\epsilon e_i)}{2N\epsilon+1})$$

$$G^4_{N,\epsilon}=\int_0^T\frac{1}{N^d}\sum_u -(Q_g(\rho_\beta^{N\epsilon}(u),\rho_\omega^{N\epsilon}(u),\phi(\frac{u}{N}),\psi(\frac{u}{N}))$$

$$-((\phi(\frac{u}{N}),\psi(\frac{u}{N})),A(\rho_\beta^{N\epsilon}(u),\rho_\omega^{N\epsilon}(u))(\phi(\frac{u}{N}),\psi(\frac{u}{N})))dt,$$

where ${\rho_\beta}^{k,l}=\frac{1}{(2k-1)^d}\sum_{y\in B_{k-1}}\beta(y)\tau^y\mathbf{1}_{A_l}$ and ${\rho_\omega}^{k,l}$ is the natural analogue. Notice that the summations in G^1 and G^3 have been rearranged. Using Itô's formula we can show that

$$\overline{\lim_{N\to\infty}}\,E^{Q_N}[G^2_N]=0.$$

By (0.14),

$$\overline{\lim_{l\to\infty}}\,\overline{\lim_{m\to\infty}}\,\overline{\lim_{k\to\infty}}\,\overline{\lim_{\epsilon\to0}}\,\overline{\lim_{N\to\infty}}\,E^{Q_N}[G^3_{N,\epsilon,k,m,l}]=0.$$

By Theorem 0.4,

$$\overline{\lim_{M\to\infty}}\,\overline{\lim_{\epsilon\to0}}\,\overline{\lim_{N\to\infty}}\,E^{Q_N}[\max_{h\le M}G^4_{N,\epsilon}]=0.$$

Therefore we have that

$$\overline{\lim_{k\to\infty}}\,\overline{\lim_{\epsilon\to0}}\,\overline{\lim_{N\to\infty}}\,E^{Q_N}[\max_{i,j\le M}G^1_{N,\epsilon,k}]\le\log 3.$$

As functionals $G^1_{N,\epsilon,k}$ converge uniformly as $N\to\infty$ to

$$G^1_{\epsilon,k}=<(\phi,\gamma_\beta(T))>+<(\psi,\gamma_\omega(T))>$$

$$-\int_{T^d}\int_0^T \sum_{i=1}^d F_i^\beta\left(\frac{\mu\beta(\theta-\epsilon,\theta+\epsilon)}{2\epsilon},\frac{\mu\omega(\theta-\epsilon,\theta+\epsilon}{2\epsilon}\right)\left(\frac{\rho_\beta(\theta+\epsilon e_i)-\rho_\beta(\theta-\epsilon e_i)}{2\epsilon}\right)$$

$$+F_i^\omega\left(\frac{\mu\beta(\theta-\epsilon,\theta+\epsilon)}{2\epsilon},\frac{\mu\omega(\theta-\epsilon,\theta+\epsilon)}{2\epsilon}\right)\left(\frac{\rho_\omega(\theta+\epsilon e_i)-\rho_\omega(\theta-\epsilon e_i)}{2\epsilon}\right)+o(k)$$

and we have

$$\overline{\lim_{k\to\infty}}\,\overline{\lim_{\epsilon\to 0}}\,E^{\tilde{Q}}[\max_{i,j\le M} G^1_{\epsilon,k}]\le \log 3.$$

Using Lemma 14 we apply the dominated convergence theorem to take the limits inside the expectation. Finally, letting $M\to\infty$ we get

$$E^{\tilde{Q}}[\sup_{\phi,\psi} < (\phi,\gamma_\beta(T)) > + < (\psi,\gamma_\omega(T)) >$$

$$-\frac{1}{2}\int_{T^d}\int_0^T ((h''A)(\rho_\beta(\theta),\rho_\omega(\theta))\begin{pmatrix}\phi(\theta)\\ \psi(\theta)\end{pmatrix},\begin{pmatrix}\nabla\rho_\beta(\theta)\\ \nabla\rho_\omega(\theta)\end{pmatrix})dtd\theta]\le\log 3.$$

Together with (0.53) this gives the desired result. □

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