Lattice gases, large deviations, and the incompressible Navier-Stokes equations

J. Quastel[1] and H.-T. Yau[2]

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Abstract. We study the incompressible limit for a class of stochastic particle systems on the cubic lattice $\mathbb{Z}^d$, $d = 3$. For initial distributions corresponding to arbitrary macroscopic L^2 initial data the distributions of the evolving empirical momentum densities are shown to have a weak limit supported entirely on global weak solutions of the incompressible Navier-Stokes equations. Furthermore explicit exponential rates for the convergence (large deviations) are obtained. The probability to violate the divergence free condition decays at rate at least $\exp\{-\epsilon^{-d+1}\}$ while the probability to violate the momentum conservation equation decays at rate $\exp\{-\epsilon^{-d+2}\}$ with an explicit rate function given by an H_{-1} norm.

1. Introduction

Although the Navier-Stokes equations for the flow of an incompressible fluid have been known and widely used for several centuries, fundamental issues concerning these equations remain unanswered. The equations are derived from a continuum formulation of conservation of mass and momentum. But one can ask the deeper question of how to obtain these equations directly from microscopic models, in particular from many body Hamiltonian systems. Then one can ask for the microscopic origin of the viscosity, the nonlinearity, and the time irreversibility. Fundamental open problems also exist at the continuum level which are not completely unrelated. Do solutions of the equations stay smooth, or do the nonlinearities succeed in overcoming the dissipative terms to create singularities in finite time for even smooth initial data? Connecting these problems is the question of whether one needs smoothness of solutions at the macroscopic scale in order to derive the equations from microscopic models.

At the present time we are very far from a rigorous derivation of fluid dynamics from Hamiltonian mechanics. This is because the ergodic properties of many body Hamiltonian systems are so poorly understood. The best that has been done is to derive the Euler equations from Hamiltonian systems under some weak ergodicity assumtions [OVY]. Another approach is to derive the Euler, and then the Navier-Stokes equations from the Boltzmann equation [C], [DEL], [BGL]. But the Boltzmann equation is not really a microscopic model; it should itself be derived as a scaling limit of a more basic model. Furthermore, the Boltzmann equation is obtained in a low density limit, so its fluid dynamic limit will necessarily be that of an ideal gas. And moreover, these results need as input stronger solutions than the best existence theorem [DL] can provide. In particular, at the time of writing, the program [BGL] of deriving the weak (Leray) solutions of the incompressible

[1] Department of Mathematics, University of California, Davis, CA 95616. Partially supported by U. S. National Science Foundation grant DMS-9504791.

[2] Courant Institute, 251 Mercer St., New York, NY 10012. Partially supported by U. S. National Science Foundation grant DMS-9403462 and David and Lucile Packard Foundation Fellowship.

Navier-Stokes equations from the renormalized (DiPerna-Lions) solutions of the Boltzmann equation remains incomplete.

An alternate approach is to substitute at the microscopic level various simplified and regularized versions for the Hamiltonian systems. One class of simplified models are the lattice gases or cellular automata, in which the particles are confined to a lattice and given deterministic motions and collision rules in order to make the system as close as possible to the Hamiltonian system. Such models have been used extensively for simulations of fluids. However, the discrete nature does not really simplify the analysis. The lattice structure tends to impose spurious conservation laws on the dynamics, and the problem of ergodicity remains.

One can consider instead the stochastic versions of the lattice gases. The stochasticity regularizes the model, causes the degenerate states, which are a source of such difficulty in the deterministic models, to become unstable. In a recent work [EMY] consider such a model in which the particles are confined to a d- dimensional lattice and have a finite number of possible velocities. The particles perform a continuous time asymmetric random walk, mimicking the free part of the motion in the Hamiltonian systems. In addition, there are two types of interaction: Hard core exclusions between particles of the same velocity, and stochastic collisions between particles of different velocities. In [EMY] it is shown that there exists a viscosity matrix so that if the incompressible Navier-Stokes equations have a smooth solution, then the empirical momentum densities converge to that solution. The viscosity matrix is given by variational formulas formally equivalent to the Green-Kubo formula, which have their origins in the work of Varadhan on the Ginzburg-Landau model [V1]. This serves as an important step in understanding the microscopic origin of the viscosity.

However, the critical objection to *all* previous results is that they require as an assumption the existence of smooth solutions of the Navier-Stokes equations. In fact it had become widely believed that regularity of solutions was essential in order to justify the separation of scales needed to derive the equations either from microscopic models or from the Boltzmann equation.

Smooth solutions of the incompressible Navier-Stokes equations are only known to exist for short periods of time. Furthermore, this time depends on the size and not the smoothness of the initial data. In three space dimensions, in the absence of extra symmetry assumptions, the only global existence theorem is Leray's (1934) result asserting the existence of global weak solutions for arbitrary square integrable initial data. A corresponding uniqueness result is only known for a smaller class of strong solutions.

In this article we present a result parallel to Leray's theorem. However, instead of starting from a regularization of the Navier-Stokes equations themselves, we obtain Leray's weak solutions directly as the incompressible scaling limit of the empirical velocity fields of the model from [EMY]. The key point here is that *we obtain weak solutions of the incompressible Navier-Stokes equations from a well defined microscopic model without any smoothness assumptions at the macroscopic scale.*

Our theorem is a weak law of large numbers for the evolving velocity fields, and the assertion is that the probability to be near a non-solution vanishes as a scaling parameter ϵ, relating the microscopic to macroscopic length scales, approaches zero. A natural question one can ask is, at what rate does the probability go to zero? Such questions are known as large deviations, and in fact in our model there are two different large deviation rates. Roughly what we show is that the probability to violate the divergence free condition decays at rate at least $\exp\{-\epsilon^{-d+1}\}$, while the probability to violate the momentum conservation equation decays at rate $\exp\{-\epsilon^{-d+2}\}$ with an explicit rate function given as the H_{-1} norm of the error.

In fact both the law of large numbers and the large deviations are obtained at once. The main tools used are the spectral gap [Q] and logarithmic Sobolev inequality [Y] for the Dirichlet form of the random walk part of the dynamics, the entropy method for hydrodynamic limits, martingale approach to large deviations [DV, Q], and a multiscale analysis. The use of the logarithmic Sobolev inequality and the multiscale analysis in hydrodynamic limits first appeared in [EMY] and then was used in [LOY]. In this paper we extend this approach to its full generality, and this provides us with an important 'two-block' estimate. This avoids the relative entropy method used in [EMY, LOY] which requires smoothness of solutions to the macroscopic equation. We also use a basic fact proved in [EMY] that, at the microscopic scale, the current can be decomposed into a gradient of the velocity field and a fluctuation term. As will be explained in Section 2, this decomposition is related to the Green-Kubo formula.

The incompressible scaling limit means to consider velocities of order ϵ, varying on spacial scales of order ϵ^{-1}, and on time scales of order ϵ^{-2}. This is the regime in which velocities are very small compared to the speed of sound. In fact the incompressible Navier-Stokes equations themselves have the scale invariance $x \to \epsilon^{-1}x$, $t \to \epsilon^{-2}t$, $u \to \epsilon u$ and so it is through this incompressible scaling that one expects them to arise from more basic models. In terms of our model, if one looks at data varying on spacial scales of order ϵ^{-1}, then on times of order ϵ^{-1} the velocity fields should satisfy the inviscid Euler equations. In fact these equations will have viscous corrections of order ϵ. Rescaling time again by ϵ^{-1} the viscous corrections become order 1 but the transport terms are now divergent of order ϵ^{-1}. Finally taking velocities of order ϵ the quadratic part of the transport terms becomes of the same order as the viscous and time derivative terms. The non-quadratic part is still divergent and leads directly to the incompressibility condition.

Although the microscopic model we consider makes sense in any dimension d, we have restricted ourselves in this article to $d = 3$. This is not only because $d = 3$ is the most physically relevant case. In higher dimensions one loses regularity, and although many of our results remain correct with appropriate modifications, new techniques need to be employed. On the other hand, *the restriction to dimensions $d \geq 3$ is intrinsic*. In dimensions $d \leq 2$ the scaling is not correct. In a box of side length $O(\epsilon^{-1})$ one has $O(\epsilon^{-d})$ particles. One wants to show that the distribution of these particles is approximately a Gibbs distribution with the given values of the conserved quantities; mass and momenta. Then by the central limit theorem the fluctuations of averages of local quantitites over the box are $O(\epsilon^{d/2})$. However in the incompressible limit one must expand these fields by $O(\epsilon^{-1})$ to obtain the macroscopic quantities. To obtain a deterministic limit the $O(\epsilon^{d/2})$ from the central limit theorem must overcome the $O(\epsilon^{-1})$ expansion in the incompressible limit. Therefore in $d \leq 2$ there is not a deterministic limit at this scale. Note that this phenomenon is due to the incompressible scaling and is not a vestige of the added noise. In particular, all derivations of fluid equations from either lattice gases or real Hamiltonian systems assume the correctness of the local equilibrium description, which is *not* in accord with the incompressible scaling in $d \leq 2$.

Finally a word on the organisation of the material. In Section 2 we introduce the model and state the main results. Section 3 contains a heuristic derivation of the incompressible Navier-Stokes equations from the microscopic model, assuming local equilibrium, and also sets up the expansions and incompressible limit which are used later in the proof. In Section 4, we begin the proof itself with the main apriori estimates on the velocity field. Section 5 contains a proof of the key local ergodic theorem, which is Theorem 5.2, using the multiscale analysis. We suggest that the reader skip everything in this section but the statement of the theorem on first reading. The upper bound large deviations are proved in Sections 6, 7 and 8, and the law of large numbers is derived as a consequence. In

Section 9, the driven systems are introduced, and it is shown that they converge to the incompressible Navier-Stokes equations with appropriate forcing terms. This is then used to derive the lower bound large deviations. Finally Section 10 contains some technical variance computations which are used earlier in the proof.

2. Statement of Main Results

The stochastic lattice gas models which will be considered in this paper are described as follows: Let $\Lambda_L \in \mathbb{Z}^d$, $d = 3$ be the cubic lattice $\{-L, ..., L\}^d$ with periodic boundary conditions. We put $L = \epsilon^{-1}$ and denote by e_j, $j = 1, ..., d$ the unit vectors in the positive coordinate directions. Let $\mathcal{V} \subset \boldsymbol{R}^d$ be a finite set representing the possible velocities.

On each site of the lattice at most one particle for each velocity is allowed. A configuration of particles on the lattice is denoted by $\eta = \{\eta_x,\ x \in \Lambda_L\}$ where $\eta_x = \{\eta(x,v),\ v \in \mathcal{V}\}$ and $\eta(x,v) \in \{0,1\}$, $x \in \mathbb{Z}^d$, $v \in \mathcal{V}$ is the number of particles with velocity v at x. The set of particle configurations is $X = \left(\{0,1\}^{\mathcal{V}}\right)^{\Lambda_L}$.

The dynamics consists of two parts: Asymmetric random walk with exclusion among particles of the same velocity and binary collisions between particles of different velocities.

We first describe the random walk part of the dynamics. Particles of velocity v perform a continuous time asymmetric random walk with simple exclusion. A particle at x waits a random time, exponentially distributed with mean one, then chooses a nearby site $x + y$ with probability $p(y,v)$ and jumps there as long as the site is not occupied by another particle of the same velocity. If there is a particle of the same velocity, the jump is suppressed and the particle waits for a new exponential time. All particles are doing this simultaneously, and since time is continuous ties do not occur. The jump law $p(y,v)$ should be finite range, irreducible and have mean velocity v:

$$\sum_y y p(y,v) = v. \tag{2.1}$$

For the sake of concreteness, we will take in this paper the jump law to be nearest neighbor with

$$p(e,v) = \gamma + e \cdot v/2 \tag{2.2}$$

for each standard basis vector e. We assume here that γ is large enough that all the $p(e,v)$ are non-negative. The generator of the random walk part of the dynamics is given by $\mathcal{L}^{ex}$ which acts on functions f on the configuration space X by

$$\mathcal{L}^{ex} f(\eta) = \sum_{x, v \in \mathcal{V}, |e|=1} \tilde{p}(x,e,v;\eta) \left(\nabla^v_{x,e} f\right)(\eta) \tag{2.3}$$

where e denotes unit vectors along coordinate directions,

$$\tilde{p}(x,e,v;\eta) = \eta(x,v)(1 - \eta(x+e,v))p(e,v) \tag{2.4}$$

is the jump rate from x to y, and $\nabla^v_{y,e}$ is the lattice gradient given by

$$\left(\nabla^v_{y,e} f\right)(\eta) = f(\eta^v_{y,e}) - f(\eta)$$

where

$$\eta^v_{y,e}(z,w) = \begin{cases} \eta(z,w) & \text{if } v \neq w \text{ or } z \neq x, x+e; \\ \eta(y,v) & \text{if } v = w \text{ and } z = y+e; \\ \eta(y+e,v) & \text{if } v = w \text{ and } z = y. \end{cases} \tag{2.5}$$

The sum in (2.3) is over all $x \in \Lambda_L$. Note that we will also use the symbol $\nabla\phi$ to denote the usual gradient of a function $\phi : T^d \to \boldsymbol{R}$. The lattice gradient will always be written with sub and super-indices to distinguish it from the usual gradient of a smooth function.

The collision part of the dynamics is described as follows. Particles of velocities v and w at the same site collide and produce two particles of velocities v' and w' at that site at rate $\tilde{r}(q)$ where q denotes the collision $(v, w) \mapsto (v', w')$. The set of such collisions will be denoted $\mathcal{Q}$. The generator of the collisions is given by

$$\mathcal{L}^c f(\eta) = \sum_{q \in \mathcal{Q},\ x} r(x, q; \eta)\, (Q^q_x f)\, (\eta) \tag{2.6}$$

where if $q = (v, w) \mapsto (v', w')$,

$$r(x, q; \eta) = \eta(x, v)\eta(x, w)(1 - \eta(x, v'))(1 - \eta(x, w'))\tilde{r}(q) \tag{2.7}$$

and Q^q_x is the collision operator given by

$$(Q^q_x f)\, (\eta) = f(Q^q_x \eta) - f(\eta) \tag{2.8}$$

where

$$Q^q_x \eta(y, u) = \begin{cases} \eta(y, u) & \text{if } y \neq x \text{ or } u \neq v, w, v', w' \\ 0 & \text{if } y = x \text{ and } u = v, w \\ 1 & \text{if } y = x \text{ and } u = v', w' \end{cases} \tag{2.9}$$

In order to show that the model is ergodic, and also in order to obtain the Navier-Stokes equations in their standard form, one has to make certain assumptions on the set $\mathcal{V}$ and the rates q: Let η and η' be configurations at a single site, say x; in other words $\eta, \eta' \in \{0, 1\}^{\mathcal{V}}$. We say η' can be reached from η through allowable collisions if there exists m and $\eta_1, \ldots, \eta_m$ and collisions q^i, $i = 1, \ldots, m-1$, such that $\eta_1 = \eta$, $\eta_m = \eta'$, $r(x, q^i; \eta_i) > 0$ and $Q^{q^i}_x \eta_i = \eta_{i+1}$. The first assumption is that η' can be reached from η through allowable collisions if and only if η and η' have the same momenta, ie. that $\sum_{v \in \mathcal{V}} v\eta(v) = \sum_{v \in \mathcal{V}} v\eta'(v)$. It is shown in [EMY2] that, under this assumption, the dynamics is ergodic when restricted to a finite volume, and given on the total mass and momentum in that volume. Note that we do not need to make any assumptions about conservation of energy since the goal here is only to derive the incompressible Navier-Stokes equations, in which the energy conservation equation does not appear. The second assumption is that $\mathcal{V}$ is invariant under reflections with respect to coordinate planes and permutations of the coordinate axes, and that

$$\sum_{v \in \mathcal{V}} v_1^4 - 3v_1^2 v_2^2 = 0, \tag{2.10}$$

where v_i is the ith coordinate of v. The second assumption is made only to ensure that the Navier-Stokes equations for the model are the standard ones. It has the following interpretation. One can define the ith moment of $\mathcal{V}$ to be the i-tensor $v^{(i)}_{\alpha_1, \ldots, \alpha_i} = \sum_{v \in \mathcal{V}} v_{\alpha_1} \cdots v_{\alpha_i}$. Because of the symmetry, $v^{(2)}$ is a constant multiple of the identity matrix. We will refer to this constant as $v^{(2)}$:

$$\sum_{v \in \mathcal{V}} v_\alpha v_\beta = v^{(2)} \delta_{\alpha\beta}, \tag{2.11}$$

where $\delta_{\alpha\beta}=0$ if $\alpha\neq\beta$ and $\delta_{\alpha\beta}=1$ if $\alpha=\beta$. Similarly we will refer to $v^{(4)}_{\alpha,\alpha,\beta,\beta}$ as $v^{(4)}_{12}$ if $\alpha\neq\beta$ and $v^{(4)}_{11}$ if $\alpha=\beta$.

$$\sum_{v\in\mathcal{V}} v_\alpha v_\alpha v_\beta v_\beta = \begin{cases} v^{(4)}_{12} & \text{if } \alpha\neq\beta, \\ v^{(4)}_{11} & \text{if } \alpha=\beta. \end{cases} \tag{2.12}$$

The interpretation of (2.10) is that the first four moments of $\mathcal{V}$ match those of a Gaussian.

We now give an example of a set of velocities $\mathcal{V}$ satisfying the assumptions. Let σ be an element of $\mathcal{T}$, the permutation group of $\{1,2,3\}$. If $v\in\boldsymbol{R}^3$ denote $\sigma v=(v_{\sigma 1},v_{\sigma 2},v_{\sigma 3})$. Let

$$\mathcal{V}=\{v:\sigma v=(\pm 1,\pm 1,\pm\omega) \text{ for some } \sigma\in\mathcal{T}\}$$

where ω is the positive solution of

$$\omega^4-6\omega^2-1=0.$$

The rate of collisions $q=(v,w)\mapsto(v',w')$ is given by

$$\tilde{r}(q)=\begin{cases} 1 & \text{if } v+w=v'+w'; \\ 0 & \text{otherwise.} \end{cases}$$

This is called Model II in [EMY2], where it is shown that is satisfies the above assumptions.

The full dynamics consist of the random walk part and the collision part running at the same time, and hence the generator is given by $\mathcal{L}=\mathcal{L}^{ex}+\mathcal{L}^c$. Let

$$I_0(\eta_x):=\sum_{v\in\mathcal{V}}\eta(x,v), \qquad I_\alpha(\eta_x):=\sum_{v\in\mathcal{V}} v_\alpha\eta(x,v), \quad \alpha=1,...,d, \tag{2.13}$$

where v_α is the αth coordinate of the vector v, and η_x denotes the configuration at site x. Note that our physical space is still the finite set $\{-L,\dots,L\}^d$ with periodic boundary conditions, so in light of the previous remarks about ergodicity, $\bar{I}_\alpha=(2L+1)^{-d}\sum_x I_\alpha(\eta_x),\ \alpha=0,...,d$ represent a complete set of conserved quantities and the set of invariant measures is the convex hull of the uniform distributions on configurations with fixed values of $\bar{I}_\alpha,\ \alpha=0,\dots,d$. In particular, the Bernoulli measures

$$\mu_\lambda=Z^{-1}_{L,\lambda}\exp\left\{\sum_{x\in\Lambda_L}\sum_{\alpha=0}^{d}\lambda_\alpha I_\alpha(\eta_x)\right\} \tag{2.14}$$

are invariant. λ is a parameter called the chemical potential, and the convex hull of these measures gives again the full set of invariant measures. One can check immediately that $\int f\mathcal{L}^c g\,d\mu=\int g\mathcal{L}^c f\,d\mu$, so $\mathcal{L}^c$ is symmetric with respect to any such μ. The particular case when $\lambda_0=r$ for some real number r, and $\lambda_\alpha=0$ for $\alpha=1,\dots,d$ will be denoted μ_r and will be used throughout as a reference measure. Any distribution can be written as $fd\mu_r$ and the meaning of the generator is that if $f_0 d\mu_r$ is the initial distribution of our process and $f(t)d\mu_r$ is the distribution at time t then $f(t)$ is obtained from f_0 by solving the evolution equation

$$\frac{\partial f}{\partial t}=\mathcal{L}^*f, \quad f(0)=f_0. \tag{2.15}$$

where $\mathcal{L}^*$ denotes the adjoint of $\mathcal{L}$ with respect to the reference measure. The Dirichlet form $\mathcal{D}(f) = -E^{\mu_r}[f\mathcal{L}f]$ is given by

$$\mathcal{D}(f) = \frac{1}{2}E^{\mu_r}\left[\gamma \sum_{\substack{x,\ v\in\mathcal{V}\\ e>0}} \left(\nabla^v_{x,e} f\right)^2 + \sum_{x,\ q\in\mathcal{Q}} r(x,q;\eta)\left(Q^q_x f\right)^2\right]. \tag{2.16}$$

We now discuss the incompressible limit. The INS equations have the following scale invariance: Suppose that u is a solution of

$$\begin{cases} \frac{\partial u}{\partial t} + u\cdot\nabla u = -\nabla p + \nabla D\nabla u, \quad x\in[0,\epsilon^{-1}]^d, \quad \text{periodic boundary conditions,}\\ \nabla\cdot u = 0, \qquad u\big|_{t=0} = u_0, \end{cases} \tag{2.17}$$

Let

$$x' = \epsilon x, \quad t' = \epsilon^2 t, \quad u' = \epsilon^{-1}u, \quad p' = \epsilon^2 p. \tag{2.18}$$

In the new variables

$$\begin{cases} \frac{\partial u'}{\partial t'} + u'\cdot\nabla' u' = -\nabla' p' + \nabla' D\nabla' u', \quad x\in[0,1]^d, \quad \text{periodic boundary conditions,}\\ \nabla'\cdot u' = 0, \qquad u'\big|_{t=0} = u'_0, \end{cases} \tag{2.19}$$

It is this scale invariance that allows the INS to be derived from more basic models in the incompressible limit. Corresponding to this rescaling we speed up the microscopic time by a factor of ϵ^{-2} to correspond to the macroscopic time scale,

$$\mathcal{L} \mapsto \epsilon^{-2}\mathcal{L}. \tag{2.20}$$

The initial state of the system is given by the local equilibrium distribution,

$$d\mu^\epsilon_\lambda = Z^{-1}_{\lambda(\cdot),\epsilon}\exp\left\{\sum_{x\in\Lambda_L}\left(\lambda_0(\epsilon x)I_0(\eta_x) + \sum_{\alpha=1}^d \lambda_\alpha(\epsilon x)I_\alpha(\eta_x)\right)\right\} \tag{2.21}$$

where, for some fixed real r, and given smooth functions $\lambda^{(2)}_0$ and $\lambda^{(1)}_\alpha$ on the torus,

$$\lambda_0(\epsilon x) = r + \epsilon^2\lambda^{(2)}_0(\epsilon x), \qquad \lambda_\alpha(\epsilon x) = \epsilon\lambda^{(1)}_\alpha(\epsilon x). \tag{2.22}$$

Corresponding to this initial distribution and the given dynamics we have a measure Q_ϵ on $D([0,T];X)$, the set of right continuous trajectories with left limits with values in the configuration space X, equipped with the Skorohod topology. For each configuration η the empirical momentum densities are the following measures on the torus T^d,

$$\nu_\alpha(dy) := \epsilon^{d-1}\sum_{x\in\Lambda_L}\delta_{\epsilon x}I_\alpha(\eta_x), \qquad \alpha = 1,...,d. \tag{2.23}$$

Here δ_y denotes the Dirac delta measure at $y\in T^d$. On the torus in $d=3$, the Sobolev space H_2 of functions with two square integrable derivatives is embedded in C, the space of continuous functions. Therefore the space of measures $\mathcal{M}$ is embedded in the dual H_{-2} of H_2 through the L^2 pairing. The Sobolev spaces are more convenient for our purposes and therefore we will consider the empirical momentum densities as elements of H_{-2}. The probability measure Q_ϵ on $D([0,T];X)$ is then carried by (2.23) to a probability measure P_ϵ on $L^2(0,T;H_{-2})$. In other words, P_ϵ is the distribution of the trajectories $\nu_\alpha(t,d\theta)$ considered as elements of $L^2(0,T;H_{-2})$. The elements here have of course d components and we will always mean the d-fold product of L^2. Let $u^0 \in L^2$ be given and suppose that the initial distributions $\lambda^{(1)}(x)$ are chosen so that for all smooth test functions ψ and each $\delta > 0$,

$$\lim_{\epsilon\to 0} \mu_\lambda^\epsilon \left(\left| \int \psi d\nu - \int \psi u^0 dx \right| \geq \delta \right) = 0. \tag{2.24}$$

If u^0 is sufficiently regular one can simply take $\lambda_\alpha^{(1)}(x) = C u_\alpha^0(x)$ with an appropriate constant C. In fact $C = (\theta^{(1)} v^{(2)})^{-1}$ where $v^{(2)}$ is given in (2.11) and

$$\theta^{(i)} = \left(\frac{d}{dr}\right)^i \theta \tag{2.25}$$

where θ is the density of particles in the equilibrium measure μ_r,

$$\theta(r) = e^r/(1-e^r) \tag{2.26}$$

For arbitrary $u^0 \in L^2$ one takes $\lambda_\alpha^{(1)}(x) = \lambda_{\alpha,\epsilon}^{(1)}(x) = C u_\alpha^0 * \phi_\epsilon$. Here $\phi_\epsilon(x) = \epsilon^d \phi(\epsilon^{-1}x)$ where ϕ is any positive smooth function of compact support and total mass 1. The convergence (2.24) is a simple application of the law of large numbers.

In [EMY2] it was shown under these conditions that *if the INS equations*

$$\begin{cases} \frac{\partial u}{\partial t} + cu\cdot\nabla u = -\nabla p + \nabla D \nabla u, & x \in T^d \\ \nabla\cdot u = 0, \qquad u\big|_{t=0} = u_0, \end{cases} \tag{2.27}$$

have a smooth solution, then the empirical momentum density fields converge weakly in P_ϵ probability to that solution. We use $\nabla D\nabla u$ to denote the vector with ith component

$$(\nabla D \nabla u)_i = \sum_{j,k,\ell=1}^{d} D_{j,k}^{i,\ell} \partial_j \partial_k u_\ell. \tag{2.28}$$

Note that for our model it is not known whether $D_{j,k}^{i,\ell} = 0$ for $i \neq \ell$. The constant c is given by

$$c = \theta^{(3)} v_{12}^{(4)} / (\theta^{(1)} v^{(2)})^2 \tag{2.29}$$

The symmetry of $\mathcal{V}$ implies that (2.29) is independent of α and β. The pressure is given by

$$p = \theta^{(2)} v^{(2)} \lambda_0^{(2)} + (\theta^{(1)} v^{(2)})^{-2} v_{12}^{(4)} \sum_{i=1}^{d} (u_i^{(1)})^2. \tag{2.30}$$

The viscosity matrix $D_{j,k}^{i,\ell}$ is given by variational formulas which are formally equivalent to the Green-Kubo formula

$$D_{j,k}^{i,\ell} = \gamma \delta_{i\ell} \delta_{jk} + \chi^{-1} \int_0^\infty dt \sum_x \left\langle \sigma_j^i \, e^{\mathcal{L}t} \tau_x \sigma_k^\ell \right\rangle. \tag{2.31}$$

Here τ_x is the shift by x on the lattice and

$$\chi = \left\langle \left(I_\alpha - \left\langle I_\alpha \right\rangle\right)^2 \right\rangle \tag{2.32}$$

is the susceptibility and the expectations $\langle \cdot \rangle$ are with respect to the reference invariant measure for our process μ_r. The currents w_j^i are given by $\mathcal{L}[I_i(\eta_x)] = \sum_{j=1}^d \nabla_j^- w_{x,j}^i$ where ∇_j^- is the lattice gradient acting on functions on the lattice by $\nabla_j^- f(n) = f(n) - f(n - e_j)$, and $w_{x,j}^i = \tau_x w_j^i$. σ_j^i are the parts of the current orthogonal to the constants of motion. More precisely, let $\tilde{\mathcal{G}}$ be the space of local functions satisfying $E^{\mu_\lambda}[g] = 0$ and $E^{\mu_\lambda}[g; \bar{I}_\alpha] =: E^{\mu_\lambda}[g\bar{I}_\alpha] - E^{\mu_\lambda}[g]E^{\mu_\lambda}[\bar{I}_\alpha] = 0$ for $\alpha = 0, \dots, d$. Then

$$\sigma_j^i = w_j^i - \sum_{\alpha=0}^d c_{j,\alpha}^i I_\alpha \tag{2.33}$$

where the coefficents $c_{j,a}^i$ are chosen so that $\sigma_j^i \in \tilde{\mathcal{G}}$.

To describe the variational formulas we have to construct a suitable Hilbert space. Let $\mu_{\ell,m}$ denote the canonical Gibbs measure on a box Λ_ℓ of side length ℓ with $\bar{I}_{\alpha,\ell} = \ell^{-d} \sum_{x \in \Lambda_\ell} I_\alpha(\eta_x) = m_\alpha$ for each $\alpha = 1, \dots, d$. Let $\mathcal{L}_{s,\ell}$ denote the symmetric part of the generator restricted to this box. More precisely, $\mathcal{L}_{s,\ell}$ is the operator corresponding to the Dirichlet form

$$\mathcal{D}(f) = \frac{1}{2} E^{\mu_r} \left[\gamma \sum_{\substack{x, x+e \in \Lambda_\ell, e>0, \\ v \in \mathcal{V}}} \left(\nabla_{x,e}^v f\right)^2 + \sum_{\substack{x \in \Lambda_\ell \\ q \in \mathcal{Q}}} r(x, q; \eta) \left(Q_x^q f\right)^2 \right]. \tag{2.34}$$

For local functions f let $\bar{f}_\ell = \sum_{x \in \Lambda_\ell'} \tau_x f$ be the sum of shifts of f in a slightly smaller box Λ_ℓ', so that all the shifts depend only on variables inside Λ_ℓ. If f is orthogonal to the conserved quantities we can define the variance of f by $\mathbf{V}_\ell(f, m) = \ell^{-d} \langle \bar{f}_\ell, -\mathcal{L}_{s,\ell}^{-1} \bar{f}_\ell \rangle_{\mu_{\ell,m}}$ and $\mathbf{V}(f) = \lim_{\ell \to \infty} \langle \mathbf{V}_\ell(f, \bar{I}_\ell) \rangle$. The scalar product $\langle\!\langle f, g \rangle\!\rangle$ is defined by polarizing the quadratic form $\mathbf{V}(f)$ and the Hilbert space $\bar{\mathcal{G}}$ is defined to be the completion of $\tilde{\mathcal{G}}$ with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. For $a = (a_j^\alpha)$ and $b = (b_j^\alpha)$, $j, \alpha = 1, \dots, d$, let

$$(a, Db) = \sum_{\alpha, i, \beta, j=1}^d D_{i,j}^{\alpha,\beta} a_i^\alpha b_j^\beta. \tag{2.35}$$

The variational formulas obtained in [EMY2] for the diffusion matrix and its inverse are

$$(a, Da) = \frac{\gamma}{\chi} \inf_{F \in \tilde{\mathcal{G}}} \langle\!\langle \sigma \cdot a + \mathcal{L}F, \sigma \cdot a + \mathcal{L}F \rangle\!\rangle, \tag{2.36}$$

and

$$\left(a, D^{-1} a\right) = \frac{1}{\gamma\chi} \inf_{F \in \tilde{\mathcal{G}}} \langle\!\langle \nabla I \cdot a + \mathcal{L}F, \nabla I \cdot a + \mathcal{L}F \rangle\!\rangle. \tag{2.37}$$

We now recall the concept of weak solutions of the INS equations. The H_1 inner product is given by

$$\langle \phi, \psi \rangle = \int (\nabla \phi, D \nabla \psi) dx. \tag{2.38}$$

Since D is strictly positive definite this definition of H_1 is equivalent to the standard one. But the precise form will be useful. Let

$$\begin{aligned} V &= \text{closure of } \left\{\psi \in C^\infty \;\middle|\; \nabla\cdot\psi = 0\right\} \text{ in } H_1; \\ H &= \text{closure of } \left\{\psi \in C^\infty \;\middle|\; \nabla\cdot\psi = 0\right\} \text{ in } L^2. \end{aligned} \tag{2.39}$$

Let V' be the dual space of V and denote by $\mathbf{P}$ the orthogonal projection $\mathbf{P} : L^2 \to H$. $\mathbf{P}$ is known as the Leray projector. Also let $C_{\mathrm{w}}(0,T;H)$ denote the subspace of $L^\infty(0,T;H)$ consisting of functions which are weakly continuous in the sense that $\int J(x)u(t,x)\,dx$ is continuous for all $J \in H$. Given $f \in L^2(0,T;V')$, *a weak solution u of the forced INS equations*

$$\begin{cases} \frac{\partial u}{\partial t} + c\mathbf{P}(u\cdot\nabla u) - \nabla D \nabla u = f, \\ \nabla\cdot u = 0, \qquad u\big|_{t=0} = u_0, \end{cases} \tag{2.40}$$

is a function $u \in L^2(0,T;V) \cap C_{\mathrm{w}}(0,T;H)$ satisfying $\frac{\partial u}{\partial t} \in L^1(0,T;V')$, $u(0) = u_0$ in the weak sense, and for almost every $t \in [0,T]$,

$$\int \sum_{\alpha=1}^{d} \frac{\partial u_\alpha}{\partial t} J_\alpha\,dx + c\int \sum_{i,\alpha=1}^{d} u_i \frac{\partial u_\alpha}{\partial x_i} J_\alpha\,dx + \int (\nabla u, D\nabla J)\,dx = \int fJ\,dx \tag{2.41}$$

for all $J \in V$. The definition can be reformulated as follows. Let

$$\mathcal{I}_0(u) = \|u\|^2_{L^\infty(0,T;H)} + \|u\|^2_{L^2(0,T;V)} + \|\partial_t u\|_{L^{\frac{4}{d}}(0,T,V')}. \tag{2.42}$$

It is well known that weak solutions of the INS equations satisfy $\mathcal{I}_0(u) < \infty$. Let $\mathcal{I}_1(u,f)$ be defined as follows: $\mathcal{I}_1(u,f) = \infty$ unless $\mathcal{I}_0(u) < \infty$ in which case

$$\mathcal{I}_1(u,f) = \frac{1}{4\chi}\left\|\frac{\partial u}{\partial t} + c\mathbf{P}(u\cdot\nabla u) - \nabla D\nabla u - f\right\|^2_{L^2(0,T;V')}. \tag{2.43}$$

By optimising over J in (2.41) one sees that weak solutions of the forced INS equations are exactly those $u \in L^2(0,T;V)$ satisfying $\mathcal{I}_1(u,f) = 0$. For our purposes we need a slightly regularized version $\mathcal{I}_2(u,f)$ of $\mathcal{I}_1(u,f)$. The class $\mathcal{S}$ consists of those u satisfying $\mathcal{I}_0(u) < \infty$ and, in addition,

$$\left\|\frac{\partial u}{\partial t}\right\|_{L^2(0,T;V')} < \infty, \quad \text{and} \quad \|\mathbf{P}(u\cdot\nabla u)\|_{L^2(0,T;V')} < \infty. \tag{2.44}$$

We define $\mathcal{I}_2(u,f) = \infty$ unless $\mathcal{I}_0(u) < \infty$ in which case

$$\mathcal{I}_2(u,f) = \liminf_{\substack{u_\epsilon \to u \\ u_\epsilon \in \mathcal{S}}} \mathcal{I}_1(u,f). \tag{2.45}$$

The limit inferior is taken over all sequences $u_\epsilon \in \mathcal{S}$ converging to u weakly in $L^2(0,T;H_{-2})$. One can show that $\mathcal{I}_2(u,f) \geq \mathcal{I}_1(u,f)$ with equality holding for $u \in \mathcal{S}$. But unlike $\mathcal{I}_1(u,f)$, we know that $\mathcal{I}_2(u,f)$ has the property that if $\mathcal{I}_2(u,f) = 0$ then there exist smooth functions u_ϵ such that $\mathcal{I}_1(u_\epsilon,f) \to 0$.

The main result of law of large numbers type is

Theorem 2.1. *Let P_ϵ be the distributions of the empirical momentum densities (2.23). Then P_ϵ are tight as a set of probability measures on the space $L^2(0,T;H_{-2})$ with the weak topology, and any weak limit P is supported entirely on weak solutions of the INS equations with $f=0$. These weak solutions satisfy in addition $\mathcal{I}_2(u,0)=0$.*

Next we consider the large deviations from this scaling limit. There are in fact two different large deviation rates: Off the hyperplane of divergence free measures, the decay rate is at least $\exp\left\{-\epsilon^{-d+1}\right\}$, while on that hyperplane the decay rate is $\exp\left\{-\epsilon^{-d+2}\right\}$. This corresponds to the fact that there is an initial layer in which the gradient part of the velocity field is removed. At the present time we are unable to pin down the exact rate of this faster process. On the other hand, for the second, slower part we have precise asymptotics.

The large deviation result we can obtain for the divergence part of the empirical velocity field is:

Theorem 2.2. *For each smooth function J and $\delta>0$,*

$$\limsup_{\epsilon\to 0}\epsilon^{d-1}\log P_\epsilon\left(\int_0^T\int\nabla J\cdot\nu\,dxdt\geq\delta\right)<0. \tag{2.46}$$

The large deviation result for the divergence free part of the velocity field is as follows. Recall that P_ϵ has initial distribution given by (2.21) , (2.22) corresponding to initial data u_0 for the INS. The static part of the rate function $\mathcal{I}_{\mathrm{stat}}$ is given by

$$\mathcal{I}_{\mathrm{stat}}(\tilde{u}_0(\cdot))=\frac{1}{2\theta^{(1)}v^{(2)}}\int\sum_{\alpha=1}^{d}\left(\tilde{u}_{0,\alpha}(x)-u_{0,\alpha}(x)\right)^2dx. \tag{2.47}$$

This is the relevant part of the initial entropy. The rate function is the sum of this static piece, plus a dynamic piece given by the functional $\mathcal{I}_2(u)=\mathcal{I}_2(u,f=0)$ defined in (2.45):

$$\mathcal{I}(u(\cdot,\cdot))=\mathcal{I}_{\mathrm{stat}}(u(\cdot,0))+\mathcal{I}_2(u(\cdot,\cdot)). \tag{2.48}$$

To observe the slower decay of the violation of the momentum conservation equation one must project out the divergence part of the velocity field. The Leray projector extends to an orthogonal projection $\mathbf{P}$ from H_{-2} to the closure in H_{-2} of smooth, divergence free functions. Let $\tilde{P}_\epsilon$ on $L^2(0,T;H_{-2})$ with the weak topology be the distributions of the divergence free part

$$u=\mathbf{P}\nu \tag{2.49}$$

of the empirical momentum densities.

Theorem 2.3. *$\tilde{P}_\epsilon$ have the large deviations property with rate function $\mathcal{I}$ and rate ϵ^{2-d}. In particular, $\mathcal{I}$ is lower semicontinuous on bounded sets of $\mathcal{I}_0$, and for each closed set $\mathcal{U}$ and open set $\mathcal{O}$,*

$$\limsup_{\epsilon\to 0}\epsilon^{d-2}\log\tilde{P}_\epsilon(\mathcal{U})\leq-\inf_{u(\cdot,\cdot)\in\mathcal{U}}\mathcal{I}(u(\cdot,\cdot)) \tag{2.50}$$

and

$$\liminf_{\epsilon\to 0}\epsilon^{d-2}\log\tilde{P}_\epsilon(\mathcal{O})\geq-\inf_{u(\cdot,\cdot)\in\mathcal{O}}\mathcal{I}(u(\cdot,\cdot)). \tag{2.51}$$

3. Incompressible Limit

The stochastic lattice gas described in Section 2 has $d+1$ conserved quantities,

$$\sum_x I_\alpha(\eta_x), \quad \alpha = 0, \dots, d. \tag{3.1}$$

The corresponding currents, $w^\alpha_{x,j}$, $\alpha = 0, \dots, d$ are defined by

$$\mathcal{L}\big[I_\alpha(\eta_x)\big] + \sum_{j=1}^{d} \nabla_j^- w^\alpha_{x,j} = 0, \tag{3.2}$$

where ∇_j^- is the lattice gradient acting on functions $f : \mathbb{Z}^d \to \boldsymbol{R}$ by $\nabla_j^- f(n) = f(n) - f(n - e_j)$. Likewise we will write $\nabla_j^+ f(n) = f(n+e_j) - f(n)$. The lattice gradients $\nabla^\pm \sim \epsilon\nabla$ where ∇ is the usual gradient acting on smooth functions on the torus. The lattice gradients will always be written with sub and super-scripts to distinguish them from the usual gradient. Note that the collision operator plays no role in (3.2) since $\mathcal{L}^c\big[I_\alpha(\eta_x)\big] = 0$. Explicitly, the currents are given by

$$\begin{aligned} w^0_{x,j} &= \sum_{v\in\mathcal{V}} (e_j \cdot v) b_{x,j}(v) + \gamma \nabla_j^+ I_0(\eta_x), \quad j = 1, \dots, d, \\ w^\alpha_{x,j} &= \sum_{v\in\mathcal{V}} (e_\alpha \cdot v)(e_j \cdot v) b_{x,j}(v) + \gamma \nabla_j^+ I_\alpha(\eta_x), \quad \alpha, j = 1, \dots, d. \end{aligned} \tag{3.3}$$

where

$$b_{x,j}(v) = \eta(x+e_j, v)\eta(x,v) - \frac{1}{2}\Big[\eta(x+e_j,v) + \eta(x,v)\Big] \qquad (3.4).$$

Let the initial state of the system be given by a local equilibrium of the form (2.21) with chemical potentials $\lambda(\epsilon x)$. We will now give a heuristic derivation of the hydrodynamic equations by assuming that the state at a later time t is still well approximated by such a local equilibrium with chemical potentials $\lambda(\epsilon x, t)$. The microscopic equations of motion for our system can be written

$$dI_\alpha(\eta_x) + \sum_{j=1}^{d} \nabla_j^- w^\alpha_{x,j} dt = dM^\alpha_x(t), \quad \alpha = 0, \dots, d, \tag{3.5}$$

where the $M^\alpha_x(t)$ are martingales. These should be thought of as the thermal noise in the system and it is not hard to show that they do not contribute to the macroscopic equations. We will need to compute averages with respect to these local equilibria. Define the velocity distribution at x by

$$f(x,v,\lambda) = E^{\mu_\lambda}[\eta(x,v)] = \frac{e^{\lambda_0(x) + \underline{\lambda}(x)\cdot v}}{1 + e^{\lambda_0(x) + \underline{\lambda}(x)\cdot v}}. \tag{3.6}$$

where $\underline{\lambda}$ denote the vector $(\lambda_1, \cdots, \lambda_d)$.

The mass and momentum densities are given by

$$\begin{aligned} \rho(\epsilon x, t) &:= E^{\mu_\lambda}[I_0(\eta_x)] = \sum_{v\in\mathcal{V}} f(x,v,\lambda) \\ u_\alpha(\epsilon x, t) &:= E^{\mu_\lambda}[I_\alpha(\eta_x)] = \sum_{v\in\mathcal{V}} (v\cdot e_\alpha) f(x,v,\lambda). \end{aligned} \tag{3.7}$$

The mass current is

$$\tilde{\pi}_{0,j}(\epsilon x,t) := E^{\mu_\lambda}[w^0_{x,j}] - \gamma \nabla_j^+ \rho(\eta_x) = \sum_{v\in\mathcal{V}} (v\cdot e_j)(f(x,v,\lambda)^2 - f(x,v,\lambda)) - \gamma\nabla_j^+\rho(\eta_x), \tag{3.8}$$

and the momentum currents $\pi_{\alpha,j}$, $\alpha = 1,\dots,d$ are given by

$$\tilde{\pi}_{\alpha j}(\epsilon x,t) := E^{\mu_\lambda}[w^\alpha_{x,j}] - \gamma \nabla_j^+ u_\alpha = \sum_{v\in\mathcal{V}} (e_\alpha\cdot v)(e_j\cdot v)[f(x,v,\lambda)^2 - f(x,v,\lambda)] - \gamma\nabla_j^+ u_\alpha. \tag{3.9}$$

From (3.7), λ can be solved in terms of the density ρ and the velocity u. Hence the currents π can be regarded as functions of the density and the velocity. At the Euler scale we therefore have closed equations for the density and velocity,

$$\begin{cases} \partial_t \rho + \sum_{j=1}^d \partial_j \pi_{0,j} = \epsilon\gamma\Delta\rho, \\ \partial_t u_\alpha + \sum_{j=1}^d \partial_j \pi_{\alpha,j} = \epsilon\gamma\Delta u_\alpha, \quad \alpha = 1,\dots,d, \end{cases} \tag{3.10}$$

where

$$\begin{aligned} \pi_{0,j}(\epsilon x,t) &= \sum_{v\in\mathcal{V}} (v\cdot e_j)(f(x,v,\lambda) - f(x,v,\lambda)^2) \\ \pi_{\alpha,j}(\epsilon x,t) &= \sum_{v\in\mathcal{V}} (e_\alpha\cdot v)(e_j\cdot v)[f(x,v,\lambda)^2 - f(x,v,\lambda)], \qquad \alpha = 1,\dots,d, \end{aligned} \tag{3.11}$$

are functions of the density and velocity.

Now we consider the incompressible limit. Rescaling time $t \to \epsilon^{-1}t$, the Euler equations become

$$\begin{cases} \partial_t \rho + \epsilon^{-1}\sum_{j=1}^d \partial_j \pi_{0,j} = \gamma\Delta\rho, \\ \partial_t u_\alpha + \epsilon^{-1}\sum_{j=1}^d \partial_j \pi_{\alpha,j} = \gamma\Delta u_\alpha, \quad \alpha = 1,\dots,d. \end{cases} \tag{3.12}$$

If we expand the chemical potentials

$$\lambda_\alpha = \lambda_\alpha^{(0)} + \epsilon\lambda_\alpha^{(1)} + \epsilon^2\lambda_\alpha^{(2)} + \cdots, \quad \alpha = 0,\dots,d, \tag{3.13}$$

then the mass density, momentum density and mass and momentum currents have the corresponding expansions

$$\begin{cases} \rho = \rho^{(0)} + \epsilon\rho^{(1)} + \epsilon^2\rho^{(2)} + \cdots, \\ u = u^{(0)} + \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \cdots, \\ \pi = \pi^{(0)} + \epsilon\pi^{(1)} + \epsilon^2\pi^{(2)} + \cdots. \end{cases} \tag{3.14}$$

Assume that the velocity is of order ϵ,

$$u^{(0)} = \lambda_\alpha^{(0)} = 0, \qquad \alpha = 1,\dots,d. \tag{3.15}$$

and denote $\lambda_0^{(0)} = r$. Using the expansions (3.14) and assuming that the zeroth order terms of π are constants, we have

$$\begin{aligned} \epsilon\partial_t\rho^{(1)} + \sum_{j=1}^d \partial_j \pi_{0,j}^{(1)} + \epsilon\sum_{j=1}^d \partial_j\pi_{0,j}^{(2)} &= \epsilon\gamma\Delta\rho^{(1)} + O(\epsilon^2), \\ \epsilon\partial_t u_\alpha^{(1)} + \sum_{j=1}^d \partial_j \pi_{\alpha,j}^{(1)} + \epsilon\sum_{j=1}^d \partial_j\pi_{\alpha,j}^{(2)} &= \epsilon\gamma\Delta u_\alpha^{(1)} + O(\epsilon^2), \qquad \alpha = 1,\dots,d. \end{aligned} \tag{3.16}$$

The zeroth order of the second equation in (3.16) gives

$$\sum_{j=1}^{d} \partial_j \pi_{\alpha j}^{(1)} = 0, \qquad \alpha = 1, \ldots, d.. \tag{3.17}$$

The first order of the second equation in (3.16) gives

$$\partial_t u_\alpha^{(1)} + \sum_{j=1}^{d} \partial_j \pi_{\alpha j}^{(2)} = \gamma \Delta u_\alpha^{(1)}, \qquad \alpha = 1, \ldots, d. \tag{3.18}$$

Finally, the zeroth order of the first equation gives

$$\operatorname{div} \pi_0^{(1)} = 0. \tag{3.19}$$

We will prove that (3.17) is maintained by system. The latter two equations, (3.18) and (3.19), are the INS equations.

We now compute the expansions. Recall the defintions (2.26) and (2.25) of θ and $\theta^{(i)}$. If

$$\zeta(r, a, b) = \frac{e^{r+\epsilon a+\epsilon^2 b}}{1 + e^{r+\epsilon a+\epsilon^2 b}},$$

then

$$\begin{aligned} \zeta &= \theta + \epsilon\theta^{(1)} a + \epsilon^2 \left[\theta^{(1)} b + \frac{\theta^{(2)}}{2} a^2\right] + O(\epsilon^3), \\ \zeta - \zeta^2 &= \theta^{(1)} + \epsilon\theta^{(2)} a + \epsilon^2 \left[\theta^{(2)} b + \theta^{(3)} a^2\right] + O(\epsilon^3). \end{aligned} \tag{3.20}$$

Let $n = |\mathcal{V}|$ denote the number of velocities. From (3.6), (3.7) and (3.13), and using the invariance of $\mathcal{V}$ under reflections with respect to the coordinate planes,

$$\begin{aligned} \rho =& n\theta + \epsilon n\theta^{(1)} \lambda_0^{(1)} + \epsilon^2 \left\{ n\theta^{(1)} \lambda_0^{(2)} + \frac{\theta^{(2)}}{2} \left((\lambda_0^{(1)})^2 + v^{(2)} \sum_{i=1}^{d} (\lambda_i^{(1)})^2 \right) \right\} + O(\epsilon^3), \\ u_\alpha =& \epsilon\theta^{(1)} v^{(2)} \lambda_\alpha^{(1)} + \epsilon^2 \left\{ \theta^{(2)} v^{(2)} \lambda_0^{(1)} \lambda_\alpha^{(1)} + \theta^{(1)} \Omega_\alpha^4(\lambda) \right\} + O(\epsilon^3), \\ \pi_{0,j} =& -\epsilon\theta^{(2)} v^{(2)} \lambda_j^{(1)} - \epsilon^2 \left\{ 2\theta^{(3)} v^{(2)} \lambda_0^{(1)} \lambda_j^{(1)} + \theta^{(2)} v^{(2)} \lambda_j^{(2)} \right\} + O(\epsilon^3), \end{aligned} \tag{3.21}$$

and, for $\alpha = 1, \ldots, d$,

$$\begin{aligned} \pi_{\alpha j} =& -\delta_{\alpha,j} \theta^{(1)} v^{(2)} - \epsilon\delta_{\alpha,j} \theta^{(2)} v^{(2)} \lambda_0^{(1)} \\ & - \epsilon^2 \left\{ \delta_{\alpha,j} \theta^{(2)} v^{(2)} \lambda_0^{(2)} + \delta_{\alpha,j} \theta^{(3)} v^{(2)} (\lambda_0^{(1)})^2 + \theta^{(3)} \Omega_{\alpha,j}(\lambda) \right\} + O(\epsilon^3), \end{aligned} \tag{3.22}$$

where $\Omega_{\alpha,j}(\lambda) = \sum_{\beta,\gamma} v_{\alpha,j,\beta,\gamma}^{(4)} \lambda_\beta^{(1)} \lambda_\gamma^{(1)}$. By the reflection symmetry, $v_{\alpha,j,\beta,\gamma}^{(4)}$ vanishes unless the parameters come in pairs (cf.(2.12)) so that,

$$\Omega_{\alpha,j}(\lambda) = \delta_{\alpha,j} v_{12}^{(4)} \sum_{i=1}^{d} \left(\lambda_i^{(1)}\right)^2 + \delta_{\alpha,j} \left[v_{11}^{(4)} - 3v_{12}^{(4)}\right] (\lambda_\alpha^{(1)})^2 + 2v_{12}^{(4)} \lambda_\alpha^{(1)} \lambda_j^{(1)}. \tag{3.23}$$

The second term vanishes by assumption (see (2.10)). In order to satisfy (3.17), we need $\lambda_0^{(1)} = 0$. Assuming this, we have

$$\begin{aligned} \rho &= n\theta + O(\epsilon^2), \\ u_\alpha &= \epsilon\theta^{(1)} v^{(2)} \lambda_\alpha^{(1)} + O(\epsilon^2) \\ \pi_{0,j} &= \epsilon(\theta^{(2)}/\theta^{(1)}) u_j^{(1)} + O(\epsilon^2) \end{aligned} \tag{3.24}$$

and, for $\alpha = 1, \dots, d$,

$$\pi_{\alpha j} = -\delta_{\alpha,j}\theta^{(1)} v^{(2)} - \epsilon^2 \left\{ \delta_{\alpha,j} p + \theta^{(3)} v_{12}^{(4)} \left(\theta^{(1)} v^{(2)}\right)^{-2} u_\alpha^{(1)} u_j^{(1)} \right\} + O(\epsilon^3) \tag{3.25}$$

where the pressure p is given by

$$p = \theta^{(2)} v^{(2)} \lambda_0^{(2)} + v_{12}^{(4)} \left(\theta^{(1)} v^{(2)}\right)^{-2} \sum_{i=1}^{d} \left(u_i^{(1)}\right)^2. \tag{3.26}$$

Equations (3.18) and (3.19) become the INS equations,

$$\begin{cases} \partial_t u^{(1)} + c\left(u^{(1)} \cdot \nabla u^{(1)}\right) + \nabla p = \gamma \Delta u^{(1)}, \\ \operatorname{div} u^{(1)} = 0. \end{cases} \tag{3.27}$$

Our task is to carry out this procedure microscopically. In fact the above derivation is not correct for the microscopic model considered in this paper. Fluctuations contribute to the diffusion coefficient so that it is in fact strictly larger than γ.

4. A priori tightness estimates.

In this section we start to prove the two main compactness estimates on the set P_ϵ of distributions of the empirical momentum densities given by (2.23). The first estimate shows that the P_ϵ are tight as a sequence of measures on $L^2(0,T;H_{-2})$ with the weak topology.

Theorem 4.1. *Let $\mathcal{C}_\ell = \left\{\nu \in L^2(0,T;H_{-2}) : \int_0^T \|\nu\|_{-2}^2 dt \le \ell\right\}$. The sets $\mathcal{C}_\ell$ are compact in the weak topology, their increasing union is the whole space and their complements $\mathcal{C}_\ell^C$ are superexponentially small in the sense that*

$$\limsup_{\ell\to\infty} \limsup_{\epsilon\to 0} \epsilon^{d-2} \log P_\epsilon(\mathcal{C}_\ell^C) = -\infty. \tag{4.1}$$

Recall the spaces H and V of divergence free L^2 and H_1 functions from (2.39). When one derives weak solutions of the INS from some regularization of those equations, the key point is that the approximants u_ϵ are shown to satisfy uniform estimates

$$\mathcal{I}_0(u_\epsilon) =: \|u_\epsilon\|^2_{L^\infty(0,T;H)} + \|u_\epsilon\|^2_{L^2(0,T;V)} + \|\partial_t u_\epsilon\|_{L^{\frac{4}{d}}(0,T,V')} \le \ell. \tag{4.2}$$

The empirical momentum density fields on the other hand are rapidly fluctuating and therefore cannot be expected to satisfy such estimates before convolution. Let

$$u^a = u * \omega^a \tag{4.3}$$

where $\omega_a(x)$ is a smooth approximate identity on the torus and $u = \mathbf{P}\nu$. For the sake of concreteness let us take $\omega(x)$ to be a smooth positive function of mass 1 on $\boldsymbol{R}^d$, symmetric about the axes and supported in a box of side length 1. Then $\omega^a(x) = a^{-d}\omega(a^{-1}x)$ makes sense as a function on the torus for $a < 1$, and gives a nice approximate identity. For each $\ell > 0$ let

$$\mathcal{K}_{\ell,a} = \left\{u \in L^2(0,T;H_{-2}) : \mathcal{I}_0(u^a) \le \ell\right\}. \tag{4.4}$$

The second compactness result which will be proved in this section and section 8 is

Theorem 4.2.

$$\lim_{a\to 0} \limsup_{\epsilon\to 0} \epsilon^{d-2} \log P_\epsilon\left(\mathcal{K}_{\ell,a}^C\right) = -\infty. \tag{4.5}$$

Theorem 4.1 is a corollary of Lemma 4.8. Theorem 4.2 will not be proved completely in this section; to obtain the estimate on the time derivative requires an analysis of the dynamics which we put off untill Section 8. Lemmas 4.4 and 4.7 constitute the part of Theorem 4.2 which will be obtained in this section. The analysis in this secton is based on the following well known corollary of the local limit theorem. A proof can be found, for example, in [P] or [Y].

Lemma 4.3. *Let $X_1, X_2, \dots$ be independent mean 0 random variables with $|X_i| \le B$ for some $B < \infty$, and $c_1, c_2, \dots$ a uniformly bounded sequence. There exists a constant $C < \infty$, depending only on B, such that for sufficiently small β, for all n,*

$$E\left[\exp \beta \left(\frac{c_1X_1 + \cdots + c_nX_n}{\sqrt{n}}\right)^2 \;\middle|\; X_1 + \cdots + X_n = 0\right] \le C. \tag{4.6}$$

Note that the only nontrivial aspect of the lemma is the conditioning. The next lemma is a probabilistic version of the classical L^2 bound on the gradient of the velocity field for the Navier-Stokes equations.

Lemma 4.4. *For sufficiently small β, there exists a constant $C < \infty$ independent of a such that for all $a > 0$,*

$$\limsup_{\epsilon \to 0} \epsilon^{d-2} \log E\left[\exp \beta \epsilon^{2-d} \left\| \nabla \nu_\epsilon^a \right\|^2_{L^2([0,T]\times \mathbf{T}^d)}\right] \leq C. \tag{4.7}$$

Proof. *Step 1.* First we prove the estimate in equilibrium, ie. with initial distribution μ_r, and with $C = 0$ in (4.7). Fix $\alpha \in \{0, \dots, d\}$ and $i \in \{1, \dots, d\}$ and let

$$v = \epsilon \partial_i \nu_\alpha^a(0) = \epsilon^d \sum_x I_\alpha(\eta_x) \partial_i \omega(a^{-1}\epsilon x) a^{-(d+1)}. \tag{4.8}$$

Here ∂_i is the standard partial derivative in the ith direction. By the Feynman-Kac formula (see Lemma 5.4), the exponential estimate can be reduced to an eigenvalue problem,

$$\epsilon^{d-2} \log E\left[\exp \beta \int_0^T \sum_x \tau_x v^2 \, dt\right] \leq T\epsilon^{d-2} \sup_{\substack{f \geq 0 \\ \int f \, d\mu_r = 1}} \left\{\beta \int \sum_x \tau_x v^2 f \, d\mu_r - \epsilon^{-2}\mathcal{D}(\sqrt{f})\right\}. \tag{4.9}$$

Note that v depends only on variables in a box Λ of side length $M = a\epsilon^{-1}$. Let z denote the empirical densities of particles of each velocity on Λ and $\mu_{\Lambda,z}^{c,ex}$ denote the canonical measure on Λ with densities given by z. Let $\mathcal{D}_{\Lambda,z}$ be the corresponding Dirichlet form on Λ. If we project the density in (4.9) onto this set then by convexity the above eigenvalue problem is dominated by another,

$$T\epsilon^{-2} \sup_z \sup_{\substack{f \geq 0 \\ \int f d\mu_{\Lambda,z}^{c,ex} = 1}} \left\{\beta \int v^2 f d\mu_{\Lambda,z}^{c,ex} - \frac{1}{2}\epsilon^{-2} M^{-d} \mathcal{D}_{\Lambda,z}(\sqrt{f})\right\}. \tag{4.10}$$

Since the Dirichlet form $\mathcal{D}^{ex}(f) = \langle f, -\mathcal{L}^{ex} f\rangle$ of the exclusions is less than or equal to the full Dirichlet form, for each fixed z we can use the logarithmic Sobolev inequality for symmetric simple exclusions [Y],

$$\sup_{\substack{f \geq 0 \\ \int f d\mu_{\Lambda,z}^{c,ex} = 1}} M^{-2} S(f/\mu_{\Lambda,z}^{c,ex}) / \mathcal{D}_{\Lambda,z}^{ex}(\sqrt{f}) \leq C. \tag{4.11}$$

to control the above by

$$\sup_f \left\{\epsilon^{-2}\beta \int v^2 f \, d\mu_{\Lambda,z}^{c,ex} - C^{-1}\epsilon^{-4} M^{-(d+2)} S(f/\mu_{\Lambda,z}^{c,ex})\right\}. \tag{4.12}$$

There is a standard convexity inequality, called the entropy inequality, which says that for two probability measures P and Q and a function F,

$$\int F dQ \leq \log \int \exp\{F\} dP + S(Q/P). \tag{4.13}$$

We can use the entropy inequality to bound (4.12) by

$$C^{-1}\epsilon^{-4} M^{-(d+2)} \log \int \exp\left\{C\epsilon^2 M^{d+2} \beta v^2\right\} d\mu_{\Lambda,z}^{c,ex}. \tag{4.14}$$

Because of the antisymmetry of $\partial_i \omega$, v is just an (unnormalized) average of M^d mean zero random variables,

$$\left\{\epsilon^2 M^{d+2}\right\}^{1/2} v = M^{-d/2} \sum_{x\in\Lambda} (I_\alpha(\eta_x) - I_\alpha(\eta_{x'}))\partial_i \omega(a^{-1}\epsilon x) \tag{4.15}$$

Here x' is the reflection of x across $x_i = 0$. Since $I_\alpha(\eta_x) - z$ are independent, mean 0 variables under the grand canonical measure, $\mu^{gc,ex}_{\Lambda,z}$, by the previous lemma, for sufficiently small β,

$$\limsup_{\epsilon\to 0} \epsilon^{-4} M^{-(d+2)} \log \int \exp\left\{C\epsilon^2 M^{d+2}\beta v^2\right\} d\mu^{c,ex}_{\Lambda,z} = 0. \tag{4.16}$$

This proves (4.7) for the equilibrium measure with a constant $C = 0$ on the right hand side. The proof extends to nonequilibrium with the aid of the following lemma which will be used throughout the article.

Lemma 4.5. *Let P^{eq}_ϵ and P^{non}_ϵ denote, respectively, the equilibrium and nonequilibrium processes, with initial distributions μ_r and μ^ϵ_λ given by (2.21). Suppose one has a sequence of functionals F_ϵ and an equilibrium statement of either of the following forms: (i) There exists a constant $C < \infty$ such that for sufficiently small $\beta > 0$,*

$$\limsup_{\epsilon\to 0} \epsilon^{d-2} \log E^{P^{eq}_\epsilon} \left[\exp \epsilon^{2-d}\beta F_\epsilon\right] \le C. \tag{4.17}$$

or (ii) For all $\beta > 0$,

$$\limsup_{\epsilon\to 0} \epsilon^{d-2} \log E^{P^{eq}_\epsilon} \left[\exp \epsilon^{2-d}\beta F_\epsilon\right] \le 0. \tag{4.18}$$

Then an identical statement holds for the nonequilibrium measures P^{non}_ϵ (in case (i) with suitably modified C and β).

Proof. Note that $dP^{non}_\epsilon / dP^{eq}_\epsilon = d\mu^\epsilon_\lambda / d\mu_r$. For any $1 \le p < \infty$ we can therefore compute

$$\limsup_{\epsilon\to 0} \epsilon^{d-2} \log E^{P^{eq}_\epsilon} \left[(dP^{non}_\epsilon / dP^{eq}_\epsilon)^p\right] < \infty. \tag{4.19}$$

The lemma follows by Hölder's inequality. □

The next two lemmas give the probabilistic analogue of the uniform bound in time on the L^2 norm of the velocity field in the Navier-Stokes equations.

Lemma 4.6. *Let ω be a smooth positive function of mass 1 and supported in a box of side length 1 and for each $\alpha = 1, \dots, d$ let*

$$\hat{I}_{\alpha,x} = \epsilon^d \sum_x I_\alpha(\eta_x)\omega(a^{-1}\epsilon x)a^{-d}. \tag{4.20}$$

Then for sufficiently small β,

$$\lim_{\epsilon\to 0} \epsilon^{d-2} \log E^{\mu_r} \left[\exp \beta \sum_x \left(\hat{I}_{\alpha,x}\right)^2\right] = 0. \tag{4.21}$$

Proof. Let $M = \epsilon^{-1}a$ and divide $\Lambda_{\epsilon^{-1}}$ into disjoint boxes of side length M. Label the boxes by σ and rewrite the sum $\sum_x \hat{I}^2_{\alpha,x}$ as $\sum_\sigma M^d Av_{x\in\sigma}\left(\hat{I}_{\alpha,x}\right)^2$ .. By Jensen's inequality on the one hand, and the product nature of μ_r on the other, we have

$$\epsilon^{d-2}\log E^{\mu_r}\left[\exp\beta\sum_\sigma M^d Av_{x\in\sigma}\left(\hat{I}_{\alpha,x}\right)^2\right] \le \epsilon^{d-2}\sum_\sigma Av_{x\in\sigma}\log E^{\mu_r}\left[\exp\beta M^d\left(\hat{I}_{\alpha,x}\right)^2\right]. \tag{4.22}$$

By the local limit theorem, Lemma 4.3, the right hand side is bounded by $\epsilon^{d-2}a^{-d}C$ which vanishes in the limit $\epsilon \to 0$. □

Lemma 4.7. *For sufficiently small β, there exists a constant $C < \infty$ independent of a such that for all $a > 0$,*

$$\limsup_{\epsilon\to 0}\epsilon^{d-2}\log E\left[\exp\beta\epsilon^{2-d}\left\|\nu^a_\epsilon\right\|^2_{L^\infty(0,T;H)}\right] \le C. \tag{4.23}$$

Proof. Fix $\alpha = 1,\dots,d$, let $\hat{I}_{\alpha,x}$ be as in the previous lemma and denote $X_t = \epsilon^d\sum_x\left(\epsilon^{-1}\hat{I}_{\alpha,x}(t)\right)^2$. By the previous lemma, for sufficiently small β,

$$\lim_{\epsilon\to 0}\epsilon^{d-2}\log E^{\mu_r}\left[\exp\beta\epsilon^{2-d}X_0\right] = 0. \tag{4.24}$$

Now X_t is a Poisson process driven by η_t; in other words, with jump rate $\gamma(\eta_t)$ and jump size $\Delta(\eta_t)$. One can check from the definition that for some $C < \infty$,

$$|\Delta| \le Ca^{-d-2}\epsilon^{2d} \tag{4.25}$$

and

$$\gamma \le C\epsilon^{-(d+2)}. \tag{4.26}$$

In (4.25) we used the smoothness of ω. Let τ be small time interval; small enough so that $\tau = o(\epsilon^4)$ but also large enough that $\tau^{-1} = o(\exp\{\epsilon^{2-d}\})$. Then

$$\log E\left[\exp\beta\epsilon^{2-d}\sup_{0\le t\le\tau}X_t - X_0\right] \le \gamma\tau(\exp\{\beta\epsilon^{2-d}\Delta\} - 1). \tag{4.27}$$

We conclude from (4.24), (4.25),(4.26), (4.27) that for sufficiently small β,

$$\lim_{\epsilon\to 0}\epsilon^{d-2}\log E\left[\exp\beta\epsilon^{2-d}\sup_{0\le t\le\tau}X_t\right] \le 0. \tag{4.28}$$

But then using the stationarity of the equilibrium measure,

$$\begin{aligned}\lim_{\epsilon\to 0}\epsilon^{d-2}\log E\left[\exp\beta\epsilon^{2-d}\sup_{0\le t\le T}X_t\right] &\le \lim_{\epsilon\to 0}\epsilon^{d-2}\log\sum_{i=0}^{\lfloor T/\tau\rfloor}E\left[\exp\beta\epsilon^{2-d}\sup_{i\tau\le t\le(i+1)\tau}X_t\right]\\ &= \lim_{\epsilon\to 0}\epsilon^{d-2}\log(T/\tau)E\left[\exp\beta\epsilon^{2-d}\sup_{0\le t\le\tau}X_t\right]\end{aligned} \tag{4.29}$$

which vanishes by the choice of τ. This proves (4.7) in equilibrium with a constant $C + 0$ on the right hand side. The general statement follows from Lemma 4.5. □

The main tightness estimate Theorem 4.1 is an immediate corollary of the following lemma and the exponential Chebyshev inequality.

Lemma 4.8. *For sufficiently small β, there exists a constant $C < \infty$ independent of β such that*

$$\limsup_{\epsilon\to 0} \epsilon^{d-2} \log E\left[\exp \beta\epsilon^{2-d}\|\nu_\epsilon\|^2_{L^2(0,T;H_{-2})}\right] \leq C. \tag{4.30}$$

Proof. Let λ_n, $\phi_n(x)$ be the eigenvalues and eigenfunctions of $-\Delta$ on the three dimensional unit torus. Note that

$$\epsilon^{2-d}\|\nu_{\alpha,\epsilon}\|^2_{L^2(0,T;H_{-2})} = \int_0^T \epsilon^d \sum_{x,y}\sum_n (1+\lambda_n)^{-2}\phi_n(x)\phi_n(y)I_\alpha(\eta_x)I_\alpha(\eta_y)dt. \tag{4.31}$$

In three dimensions $\sum_n (1+\lambda_n)^{-2} = B < \infty$, so by Jensen's inequality using $(BT)^{-1}\int_0^T dt(1+\lambda_n)^2$ as a probability measure, the left hand side of (4.30) is bounded by

$$B^{-1}\limsup_{\epsilon\to 0}\epsilon^{d-2}\sum_n (1+\lambda_n)^{-2}\log E\left[\exp\beta BT\left(\epsilon^{d/2}\sum_x \phi_n(x)I_\alpha(\eta_x)\right)^2\right]. \tag{4.32}$$

The eigenvalues are uniformly bounded, so as in the proofs of the previous two lemmas, by the local limit theorem we have in equilibrium for sufficiently small β, for each n,

$$\limsup_{\epsilon\to 0}\epsilon^{d-2}\log E\left[\exp\beta BT\left(\epsilon^{d/2}\sum_x \phi_n(x)I_\alpha(\eta_x)\right)^2\right] = 0. \tag{4.33}$$

Furthermore the rate of convergence depends only on the L_∞ norm of the test function and therefore is uniform in n. This proves (4.30) in equilibrium with 0 on the right hand side. The estimate is transfered to nonequilibrium by Lemma 4.5. □

We conclude this section with a technical a priori estimate which will be needed in the proof of the upper bound.

Lemma 4.9. *For any local function F with $E^\mu[F] = 0$ any test function $J(x)$ and any $\delta > 0$ we have*

$$\lim_{\epsilon\to 0}\epsilon^{d-2}\log E^\mu\left[\exp\left\{\epsilon^{1+\delta}\sum_x J(\epsilon x)\tau_x F(\eta)\right\}\right] = 0 \tag{4.34}$$

Proof. Suppose that F has range K. Divide $\Lambda_{\epsilon^{-1}}$ into disjoint boxes σ of side length $2K$ and rewrite the sum as

$$\sum_x J(\epsilon x)\tau_x F = \sum_\sigma (2K)^d Av_{x\in\sigma}J(\epsilon x)\tau_x F. \tag{4.35}$$

Using Jensen's inequality and the independence

$$\begin{aligned}&\epsilon^{d-2}\log E^\mu\left[\exp\left\{\epsilon^{1+\delta}\sum_\sigma (2K)^d Av_{x\in\sigma}J(\epsilon x)\tau_x F\right\}\right]\\ &= \sum_\sigma Av_{x\in\sigma}\epsilon^{d-2}\log E^\mu\left[\exp\left\{\epsilon^{1+\delta}(2K)^d J(\epsilon x)\tau_x F\right\}\right].\end{aligned} \tag{4.36}$$

Expanding the exponent we obtain (4.34). Note that we need the condition $E^\mu[F] = 0$ to have the first order term in the expansion vanish. □

5. Multiscale analysis.

In this section we will prove the main eigenvalue estimate on which the law of large numbers and large deviation results are based. We start with some notation. Consider first the process without collisions on a box Λ_ℓ of side length ℓ. Particles of each velocity perform asymmetric simple exclusions and do not interact with those of other velocities. The conserved quantities are $\ell^{-1}\sum_{x\in\Lambda_\ell}\eta_x(v),\ v\in\mathcal{V}$, whereas the conserved quantities of the full dynamics are

$$\bar{I}_\alpha=\ell^{-1}\sum_{x\in\Lambda_\ell}I_\alpha(\eta_x),\ \ \alpha=0,\dots,d. \tag{5.1}$$

It is convenient to complete the latter to a set

$$\bar{I}_{-n_0},\dots,\bar{I}_d \ = \ \text{conserved quantities of collisionless process} \tag{5.2}$$

where $n_0=n-d$. We will use I to denote $I_\alpha,\ \alpha=-n_0,\dots,d$ and I^+ to denote I_α, $\alpha=0,\dots,d$. The grand-canonical invariant measure of the exclusion part of the dynamics is the product measure with marginals

$$\mu^{ex}_\lambda=Z^{-1}\exp\left\{\sum_{\alpha=-n_0}^{d}\lambda_\alpha I_\alpha(\eta_x)\right\} \tag{5.3}$$

The canonical invariant measure of the exclusion part $\mu^{c,ex}_{\ell,z}$ is the uniform measure given the averages $\bar{I}_\ell=z$. Let h be a local function depending on variables inside the box Λ_ℓ. We will need four different averages of h:

$$\begin{aligned}
&\widetilde{h}(z)=E^{\mu^{ex}_\lambda}[h] \quad \text{where} \quad E^{\mu^{ex}_\lambda}[I_\alpha]=z_\alpha,\ \ \alpha=-n_0,\dots,d,\\
&\bar{h}_\ell(z)=E\left[h \ \mid\ \bar{I}_\ell=z\right]=:E^{\mu^{c,ex}_{\ell,z}}[h],\\
&\widetilde{h}^+(z)=E^{\mu_\lambda}[h] \quad \text{where} \quad E^{\tilde{\mu}_\lambda}[I_\alpha]=z_\alpha,\ \ \alpha=0,\dots,d,\\
&\bar{h}^+_\ell(z)=E\left[h \ \mid\ \bar{I}^+_\ell=z\right]=:E^{\mu^{c}_{\ell,z}}[h].
\end{aligned} \tag{5.4}$$

We will also use $\bar{h}_\ell$ and $\bar{h}^+_\ell$ to denote $\bar{h}_\ell(\bar{I}_\ell)$ and $\bar{h}^+_\ell(\bar{I}^+_\ell)$. We need the following spaces of local functions orthogonal to the conserved quantities:

$$\begin{aligned}
&\tilde{\mathcal{G}}^{ex}=\left\{h: h\ \text{local},\ E^{\mu_r}[h]=0,\ E^{\mu_r}[h;\bar{I}]=0\right\};\\
&\mathcal{G}=\left\{h: h\ \text{local},\ E^{\mu_r}[h\mid \bar{I}^+_k]=0 \text{ for some } k>0 \text{ and for all values of } \bar{I}_k,\ \alpha\ge 0\right\};\\
&\tilde{\mathcal{G}}=\left\{h: h\ \text{local},\ E^{\mu_r}[h]=0, E^{\mu_r}[h;\bar{I}^+]=0\right\}.
\end{aligned} \tag{5.5}$$

Here $E^{\mu_r}[f;h]=E^{\mu_r}[fh]-E^{\mu_r}[f]E^{\mu_r}[h]$. Note $\mathcal{G}\subset\tilde{\mathcal{G}}$.

For such a local function we can define the variance $\mathbf{V}(h,r)$ as follows. Let

$$\mathbf{V}_\ell(h,z)=(2\ell'+1)^d\left\langle\left(Av_{|x|\le\ell'}\tau_x h-\bar{h}^+_\ell\right),(-\mathcal{L}_{s,\ell})^{-1}\left(Av_{|x|\le\ell'}\tau_x h-\bar{h}^+_\ell\right)\right\rangle_{\mu^c_{\ell,z}}. \tag{5.6}$$

Here $\ell'=\ell-\ell_0$ where ℓ_0 is the range of h and $\mathcal{L}_{s,\ell}$ is the symmetric part of the generator restricted to Λ_ℓ, ie. with reflecting boundary conditions. The expectation is with respect to the canonical invariant measure for the full process. This definition makes sense because of the finite volume ergodicity of the generator $\mathcal{L}_{s,\ell}$. The following theorem is proved in [LOY]and [EMY2]

Theorem 5.1. *For each $h \in \tilde{\mathcal{G}}$ the variance*

$$\mathbf{V}(h,r) =: \limsup_{\ell\to\infty} E^{\mu_r}\left[\mathbf{V}_\ell(h, \bar{I}^+_\ell)\right]. \tag{5.7}$$

is finite. For any local function h, $h - \bar{h}^+_R \in \mathcal{G}$,

$$h^r = h - E^{\mu_r}[h] - \sum_{\alpha\geq 0} \frac{\partial\tilde{h}(m)}{\partial m_\alpha}\Big|_{m=r} (I_\alpha(\eta(0)) - r_\alpha) \in \tilde{\mathcal{G}}, \tag{5.8}$$

and

$$\mathbf{V}(h,r) =: \mathbf{V}(h^r, r) = \lim_{R\to\infty} \mathbf{V}(h - \bar{h}^+_R, r). \tag{5.9}$$

Furthermore (5.9) coincides with (5.7) if $h \in \tilde{\mathcal{G}}$.

The main result of the this section is

Theorem 5.2. *For any smooth ϕ and any local function h,*

$$\lim_{a\to 0}\lim_{\epsilon\to 0}\epsilon^{d-2}\log E\left[\exp\left\{\int_0^T \sum_x \phi(\epsilon x)\tau_x\left[Av_{|y|\leq a\epsilon^{-1}}\tau_y h - \bar{h}^+_{a\epsilon^{-1}}\right]dt\right\}\right] \leq \frac{1}{2}\|\phi\|_2^2\,\mathbf{V}(h,r). \tag{5.10}$$

The expectation here, and in the rest of this section, is with respect to the equilibrium process, with initial distribution μ_r.

Proof. *Step 1.* Let $M = a\epsilon^{-1}$. For each $0 < R < M$ we decompose

$$Av_{|y|\leq M}\tau_y h - \bar{h}^+_M = h_1 + h_2 + h_3. \tag{5.11}$$

where

$$\begin{aligned} h_1 &= Av_{|y|\leq M}\tau_y\left(h - \bar{h}^+_R\right), \\ h_2 &= Av_{|y|\leq M}\tau_y\bar{h}^+_R - \widetilde{\bar{h}^+_R}(\bar{I}_M), \\ h_3 &= \widetilde{\bar{h}^+_R}(\bar{I}_M) - \bar{h}^+(\bar{I}^+_M). \end{aligned} \tag{5.12}$$

For each $i = 1, 2, 3$ let

$$\Omega_i = \epsilon^{d-2}\log E\left[\exp\left\{\int_0^T\sum_x \phi(\epsilon x)\left[h_i\right]dt\right\}\right], i = 1, 2, 3.$$

We will bound each $\Omega_i, i = 1, 2, 3$ seperately, and then use Hölder's inequality to obtain Theorem 5.2 at the end. Roughly what we are going to do is use perturbation theory to bound Ω_1 by $\frac{1}{2}\|\phi\|_2^2\,\mathbf{V}(h,r)$ in the limit; the local limit theorem and equivalence of ensembles to show that Ω_2 vanishes in the limit; and various orthogonality relations to show that Ω_3 vanishes in the limit.

Step 2: Bound on Ω_1. We first quote the following lemma summarizing the standard perturbation theory. Proofs can be found in [V,Q, QY, EMY2]

Lemma 5.3. *For any local function h, any smooth test function ϕ, any ℓ, and any fixed $R \ll \ell$,*

$$\lim_{\epsilon\to 0} \sup_f \left\{ \epsilon^{-2} \, Av_x \phi(\epsilon x) \int f \, Av_{|y-x|\le \ell} \tau_y \left(h - \bar{h}_R^+ \right) \, d\mu_r \right.$$
$$\left. - Av_x \phi^2(\epsilon x) \int f \, \mathbf{V}_\ell(h - \bar{h}_R^+, \bar{I}_{x,\ell}) \, d\mu_r - \epsilon^{d-4} D(\sqrt{f}) \right\} \le 0 \tag{5.13}$$

The next lemma is a basic tool which allows us to transform equilibrium exponential estimates into either eigenvalue problems, or entropy estimates.

Lemma 5.4. *Let L be the generator of a Markov process $x(t)$ with invariant measure μ. Let*

$$u(x,t) = E_x \left[\exp \left\{ \int_0^T V(x(s)) ds \right\} \right], \tag{5.14}$$

where $E_x[\cdot]$ denotes the expectation with respect to the process starting at x. Then

$$(i) \quad (2T)^{-1} \log \int u(x,T) \, d\mu(x) \le \sup_{\int f d\mu = 1, \; f\ge 0} \left\{ \int V f d\mu - \mathcal{D}(\sqrt{f}) \right\}$$
$$(ii) \quad T^{-1} \log \int u(x,T) \, d\mu(x) \le \sup_{\int f d\mu = 1, \; f\ge 0} \left\{ \int V f \, d\mu - T^{-1} S(f) \right\}. \tag{5.15}$$

Here $\mathcal{D}(f) = -\int f L f d\mu$ is the Dirichlet form and $S(f) = \int f \log f d\mu$ is the entropy.

Proof. By Jensen's inequality, $\log \int u \, d\mu \le \frac{1}{2} \log \int u^2 \, d\mu$. By the Feynman-Kac formula, u solves the equation $\frac{\partial u}{\partial t} = Lu + Vu$. Multiplying by u and integrating by parts we obtain

$$(2T)^{-1} \log \int u^2 \, d\mu \Big|_0^T = \left(\int u^2 \, d\mu \right)^{-1} T^{-1} \int_0^T \left\{ \int V u^2 \, d\mu - \mathcal{D}(u) \right\} dt \tag{5.16}$$

Taking $f = u^2 / \int u^2 d\mu$ and noting $u(0) \equiv 1$ we obtain (i). By Jensen's inequality again,

$$\log E^\mu \left[\exp \left\{ \int_0^T V(x(s)) \, ds \right\} \right] \le \log E^\mu \left[\exp TV(x) \right]. \tag{5.17}$$

This equals the right hand side of (ii) by the variational formulation of the entropy, which proves (ii). □

Lemma 5.5. *For any local function h and any smooth test function ϕ,*

$$\lim_{R\to\infty} \lim_{\ell\to\infty} \lim_{\epsilon\to 0} \epsilon^{d-2} \log E \left[\exp \int_0^T \sum_x \phi(\epsilon x) Av_{|y-x|\le \ell} \tau_y \left(h - \bar{h}_R^+ \right) dt \right] \le \frac{1}{2} \|\phi\|_2^2 \mathbf{V}(h,r) \tag{5.18}$$

Proof. Note that $h-\bar{h}_R^+ \in \mathcal{G}$. By Hölder's inequality with $1/p+1/q=1$ we can bound the left hand side of (5.18) by

$$p^{-1}\epsilon^{d-2}\log E\left[\exp\left\{p\int_0^T (A-B)(\eta_t)\right]dt\right\}+q^{-1}\epsilon^{d-2}\log E\left[\exp\left\{q\int_0^T B(\eta_t)dt\right\}\right] \tag{5.19}$$

where

$$\begin{aligned} A &= \sum_x \phi(\epsilon x) Av_{|y-x|\le \ell}\tau_y\left(h-\bar{h}_R^+\right), \\ B &= \frac{\epsilon^2 p}{2}\sum_x \phi^2(\epsilon x)\mathbf{V}_\ell(h-\bar{h}_R^+, \bar{I}_{x,\ell}). \end{aligned} \tag{5.20}$$

We can use Lemma 5.4 to bound the first term in (5.19) by the eigenvalue problem: For each R, p and ϕ, by the standard perturbation theory Lemma 5.3,

$$\lim_{\ell\to\infty}\lim_{\epsilon\to 0} p^{-1}\sup_f\left\{\epsilon^{d-2}p\int f\ (A-B)\ d\mu_r - \epsilon^{d-4}\mathcal{D}(\sqrt{f})\right\}\le 0. \tag{5.21}$$

We can rewrite the last term of (5.19) as the sum of $\frac{p}{2}\epsilon^d\sum_x \phi^2(\epsilon x)\mathbf{V}(h,r)$ plus an error term given by

$$q^{-1}\epsilon^{d-2}\log E\left[\exp\left\{qT\frac{\epsilon^2 p}{2}\sum_x \phi^2(\epsilon x)\left[\mathbf{V}_\ell(h-\bar{h}_R^+,\bar{I}_{x,\ell})-\mathbf{V}_\ell(h,r)\right]\right\}\right] \tag{5.22}$$

By Theorem 5.1 the error term vanishes as $\epsilon\to 0$. Letting $p\downarrow 1$ gives the desired result.$\square$

We can now bound Ω_1. Note that the difference between Ω_1 and the left side of (5.18) is that the summation in the average of y in (5.18) should be replaced by $|y-x|\le a\epsilon^{-1}$ for Ω_1. We can rewrite Ω_1 in the form of the left side of (5.18) by summing by parts, and replacing ϕ by a new version ϕ^a averaged over blocks of size $a\epsilon^{-1}$. Since $\lim_{a\to 0}\lim_{\epsilon\to 0}\epsilon^d\sum_x(\phi^a(\epsilon x))^2=\|\phi\|_2^2$ we have now proven that

$$\lim_{a\to 0}\lim_{R\to\infty}\lim_{\epsilon\to 0}\Omega_1\le \frac{1}{2}\|\phi\|_2^2\mathbf{V}(h,r). \tag{5.23}$$

Our goal is to prove such an estimate with R replaced by $a\epsilon^{-1}$ for small a. We bootstrap our way up scale by scale.

Step 3. Bound on Ω_2. Let $M=a\epsilon^{-1}$. For each x, divide the cube $\Lambda_{x,M}$ into disjoint subcubes both of size $K=\ell\epsilon^{-2/d}$ and of size R. We label cubes of size K by ω and size R by σ. The decomposition is compatible in the sense that each cube of size R is contained in a cube of size K. Let $\bar{I}_\omega$ be the density on ω and $\bar{I}_\alpha$ be the density on α. The following lemma summarizes *Case 1* in the proof of Theorem 4.6 in [EMY].

Lemma 5.6. *For any local function h any $\gamma>0$ and smooth test function ϕ,*

$$\begin{aligned} \lim_{R\to\infty}\lim_{\epsilon\to 0}\sup_f\Bigg\{\epsilon^{d-2}\int f\sum_x \phi(\epsilon x)Av_\omega\{Av_{\sigma\in\omega}\bar{h}_R^+(\bar{I}_\sigma)-\widetilde{\left(\bar{h}_R^+\right)}_K(\bar{I}_\omega)\}d\mu_r \\ -\gamma\epsilon^{d-4}\mathcal{D}^{ex}(\sqrt{f})\Bigg\}\le 0. \end{aligned} \tag{5.24}$$

Note the two meanings of $\bar{h}_R^+$ in (5.24). On the one hand $\bar{h}_R^+$ is simply a polynomial in the density. On the other hand $\bar{h}_R^+=\bar{h}_R^+(\bar{I}_R^+)$ is a local function of the particle configuration. Our goal in this step is to extend (5.24) from $K=\ell\epsilon^{-2/d}$ to the scale $a\epsilon^{-1}$. We state it as the following

Lemma 5.7. *Let $M = a\epsilon^{-1}$ and $K = \ell\epsilon^{-2/d}$. For any local function h, for sufficiently large β,*

$$\lim_{a\to 0}\lim_{\ell\to\infty}\lim_{\epsilon\to 0}\sup_f \left\{ M^2 \int f\{Av_\omega \widetilde{(\widetilde{h_R^+})}_K(\bar{I}_\omega) - \widetilde{h_R^+}(\bar{I}_M))\} d\mu_{M,z}^{c,ex} - \beta M^{4-d}\mathcal{D}_{\Lambda_M}^{ex}(\sqrt{f}) \right\} \le 0. \tag{5.25}$$

The supremum is over densities f with respect to $\mu_{M,z}^{c,ex}$ and (5.25) holds uniformly in the density z.

Combining Lemma 5.7 and Lemma 5.4 we have

$$\lim_{a\to 0}\lim_{R\to\infty}\lim_{\epsilon\to 0}\Omega_2 \le 0$$

The rest of this step is devoted to proving Lemma 6.7. We need a few results from previous work which we state as Lemmas 6.8 to 6.10. The following lemma is proved in [LOY].

Lemma 5.8. *Suppose $\ell^d \le K^d/2$. There are universal constants C_1 and C_2 such that*

$$\mu_{K,z}^{c,ex}\left(\left|\bar{I}_\ell - z\right| \ \ge\ a\right) \ \le\ C_1 \exp\left\{-C_2\ell^d a^2\right\}\ . \tag{5.26}$$

As a corollary one has the following version of the equivalence of ensembles.

Lemma 5.9. *Let g be a C^2 function on* Range$(\bar{I})$*. Then for $\ell^d \le K^d/2$ there is a universal constant C_3 such that,*

$$\left| E^{\mu_{K,z}^{c,ex}}\left[g(\bar{I}_\ell)\right] - g(z)\right| \ \le\ C_3\|D^2 g\|_\infty \ell^{-d}.$$

Proof. Let $q(y) = g(y) - g(z) - \nabla g(z)\cdot(y-z)$. Note that $|q(y)| \le \|D^2 g\|_\infty |y-z|^2$ and

$$E^{\mu_{K,z}^{c,ex}}\left[q(\bar{I}_\ell)\right] = E^{\mu_{K,z}^{c,ex}}\left[g(\bar{I}_\ell)\right] - g(z). \tag{5.27}$$

By the previous lemma,

$$\left|E^{\mu_{K,z}^{c,ex}}\left[q(\bar{I}_\ell)\right]\right| \le C_1C_2\ell^d\|D^2g\|_\infty \int_0^\infty a^3 \exp\left\{-C_2\ell^d a^2\right\} da = C_3\left\|D^2 g\right\|_\infty \ell^{-d}. \tag{5.28}$$

This proves Lemma 5.9. □

The next tool we need is the following large deviation lemma [EMY]. We repeat the proof here because it contains some basic idea used in many places in this paper.

Lemma 5.10. *Let G be a smooth function satisfying*

$$G(\theta) = 0, \quad \left.\frac{\partial G}{\partial z_i}\right|_{z=\theta} = 0, \quad i = -n_0, \dots, d. \tag{5.29}$$

Then for any ϕ with sufficiently small $\|\phi\|_\infty$, and any scale $\ell = \ell(\epsilon)$ such that $\lim_{\epsilon\to 0}\epsilon^{-2}\ell^{-d} = 0$,

$$\lim_{\epsilon\to 0}\epsilon^{d-2}\log E^{\mu_r}\left[\exp\left\{\sum_x \phi(\epsilon x)G(\bar{I}_{x,\ell})\right\}\right] = 0. \tag{5.30}$$

Proof. It follows from the assumptions that $|G(u)| \leq \text{const.}|u-\theta|^2$, so it will suffice to prove for each $\alpha = 1, \ldots, d$, for sufficiently small β, that

$$\lim_{a\to 0}\lim_{\epsilon\to 0} \epsilon^{d-2} \log E^{\mu_r} \left[\exp\left\{\beta\left(\bar{I}_{\alpha,x,\ell} - \theta_\alpha\right)\right\}\right] = 0. \tag{5.31}$$

Note that $\bar{I}_{\alpha,x,\ell} - \theta_\alpha$ is the average on the box $\Lambda_{\ell,x}$ of side length ℓ centered at x, of $\xi_x = I_\alpha(\eta_x) - \theta_\alpha$ which are independent and identically distributed under μ_r. Divide $\Lambda_{\epsilon^{-1}}$ into disjoint cubes $\Lambda_{\ell,q}$. We can write the sum in the exponent as the sum first over q and then over $\Lambda_{\ell,q}$, with an appropriate normalizing factor. By exploiting the independence of the summands on different cubes $\Lambda_{\ell,q}$ and applying Jensen's inequality, (5.31) reduces to

$$\lim_{a\to 0}\lim_{\epsilon\to 0} \epsilon^{-2}\ell^{-d} \log E^{\mu_r} \left[\exp\left\{\beta\left(\ell^{-d/2}\sum_{x\in\Lambda_\ell}\xi_x\right)^2\right\}\right] = 0. \tag{5.32}$$

Since the variables ξ_x are uniformly bounded, one can verify that the expectation is uniformly bounded in ℓ, either by the local limit theorem, or by a straighforward expansion of the exponential. This proves (5.32) and the lemma follows. □

Proof of Lemma 5.6. By the equivalence of ensemble, we can replace $\widetilde{(\bar{h}_R^+)}_K(\bar{I}_\omega)$ by $\widetilde{(\bar{h}_R^+)}\,(\bar{I}_\omega)$ with an error of order K^{-d} negligible in the limit since $K^{-d}M^2 \to 0$. We can also subtract a linear term $\sum_{\alpha=-n_0}^{d} c_\alpha(\bar{I}_M)\left(\bar{I}_{\alpha,\omega} - \bar{I}_{\alpha,M}\right)$ where $c_\alpha(z) = \frac{\partial \widetilde{\bar{h}_R^+}}{\partial z_\alpha}\Big|_{z=\bar{I}_M}$ since such a term vanishes after the summation over ω. From the logarithmic Sobolev inequality for symmetric simple exclusions [Y],

$$S(f/d\mu_{M,z}^{c,ex}) \leq \text{const.} M^2 \mathcal{D}_{\Lambda_M}(\sqrt{f}), \tag{5.33}$$

and the entropy inequality the left side of (5.25) is bounded by

$$\text{const.}\beta M^{-d+2} \log \int \exp\left\{\beta^{-1} M^d \text{Av}_\omega \xi_\omega\right\} d\mu_{M,z}^{c,ex} \tag{5.34}$$

where

$$\xi_\omega = \widetilde{(\bar{h}_R^+)}(\bar{I}_\omega) - \widetilde{(\bar{h}_R^+)}(\bar{I}_M) - \sum_{\alpha=-n_0}^{d} c_\alpha(\bar{I}_M)\left(\bar{I}_{\alpha,\omega} - \bar{I}_{\alpha,M}\right) \tag{5.35}$$

Note that R is fixed and $\widetilde{\bar{h}_R^+}$ is just a polynomial. By Lemma 5.10, (5.34) vanishes in the limit as $\epsilon \to 0$, then $\ell \to \infty$ and $a \to 0$. □

Step 4: Bound on Ω_3. Let h be a local function and let

$$u_R(z) = \widetilde{(\bar{h}_R^+)}(z) - \bar{h}^+(z) \tag{5.36}$$

Note that $\bar{h}^+(z) = \bar{h}^+(z^+)$. We first prove a few properties of u.

Lemma 5.11.

$$\begin{aligned} &(i) \quad \lim_{R\to\infty} \sup_z |u_R| = 0, \\ &(ii) \quad \sup_R \sup_{|\theta - z|\leq 1/4} \left|\frac{\partial^3 u_R}{\partial z_\alpha \partial z_\beta \partial z_\gamma}\right| \leq \textit{const.}, \qquad \alpha,\, \beta,\, \gamma = 0, \ldots, d. \end{aligned} \tag{5.37}$$

Proof. To prove (i), first from the definition of $\bar{h}^+(z)$, we have $\bar{h}^+(z) = \lim_{R\to\infty} \bar{h}^+_R(z)$. From the equivalence of ensembles,

$$\lim_{R\to\infty} \sup_z |\bar{h}^+_R(z) - \bar{h}^+(z)| = 0. \tag{5.38}$$

so it suffices to prove that

$$\lim_{R\to\infty} \left| E^{\mu_z^{gc,ex}} [\bar{h}^+(\bar{I}^+_R)] - \bar{h}^+(z) \right| = 0, \tag{5.39}$$

As $R \to \infty$, by the law of large numbers, $\bar{I}_{\alpha,R} \to z_\alpha$ with respect to $\mu_z^{gc,ex}$. This proves (i). For (ii), the derivatives can be computed explicitly and (ii) checked. The restriction on the range of z is needed because some correlation functions appearing in the denominator of the explicit expressions for the third derivatives blow up as the z approaches the boundary. □

Lemma 5.12. *Let*

$$a_{\alpha,R} = \frac{\partial u_R}{\partial a_\alpha}\Big|_{z=\theta}, \qquad \alpha = -n_0, \dots, d. \tag{5.40}$$

Suppose that $h \in \tilde{\mathcal{G}}$. *Then*

$$\begin{aligned} &(i) \quad a_{\alpha,R} = 0, \qquad \alpha = 0, \dots, d, \\ &(ii) \quad \lim_{R\to\infty} a_{\alpha,R} = 0, \quad \alpha = -n_0, \dots, -1. \end{aligned} \tag{5.41}$$

Proof. To prove (i) consider the identity

$$\frac{\partial u}{\partial z_\alpha}\Big|_{z=\theta} = \sum_{\beta=-n_0}^{d} \sum_x \left(E^{\mu_z^{gc,ex}} [\bar{h}^+_R; \tau_x I_\beta] \frac{\partial \lambda_\beta}{\partial z_\alpha} - E^{\mu^{gc}_{z^+}} [h; \tau_x I_\beta] \frac{\partial n_\beta}{\partial z_\alpha} \right) \Big|_{z=\theta}. \tag{5.42}$$

Here we have used n to denote the chemical potential in the grand-canonical invariant measure for the full dynamics (exclusion + collision). $\mu_z^{gc,ex}$ and $\mu^{gc}_{z^+}$ are the grand-canonical invariant measures of the exclusion part, and the full dynamics respectively. By definition $n_\beta = 0$ for $\beta < 0$ and thus $\frac{\partial n_\beta}{\partial z_\alpha}|_{z=\theta} = 0$, $\alpha < 0$. By the orthogonality relation $E^{\mu_r}[I_\alpha; I_\beta] = 0$ for $\alpha \neq \beta$, one has $\frac{\partial \lambda_\beta}{\partial z_\alpha}|_{z=\theta} = 0$ if $\alpha \neq \beta$. Therefore only the $\alpha = \beta$ terms of the right side of (5.42) survive. But if $\beta \geq 0$, since $h \in \tilde{\mathcal{G}}$, we have $\sum_x E^{\mu_r}[\bar{h}^+_R; \tau_x I_\beta\ ] = E^{\mu_r}[h; \sum_x \tau_x I_\beta\ ] = 0$. This proves (i). To prove (ii): By Schwarz's inequality,

$$\begin{aligned} \sum_x E^{\mu_r} \left[\bar{h}^+_R; I_\alpha(\eta_x) \right] &= E^{\mu_r} \left[\bar{h}^+_R; \sum_{|x|\leq R} I_\alpha(\eta_x) \right] \\ &\leq E^{\mu_r} \left[\bar{h}^+_R; \bar{h}^+_R \right]^{1/2} E^{\mu_r} \left[\sum_{|x|\leq R} I_\alpha(\eta_x); \sum_{|x|\leq R} I_\alpha(\eta_x) \right]^{1/2} \\ &\leq R^{d/2} E^{\mu_r} \left[\bar{h}^+_R; \bar{h}^+_R \right]^{1/2}. \end{aligned} \tag{5.43}$$

It was shown in [EMY2] that if $h \in \tilde{\mathcal{G}}$, the last term vanishes in the limit of large R. We sketch the idea: If one just uses the central limit theorem the most one can hope for is that the last term is bounded. But since $h \in \tilde{\mathcal{G}}$, we know that $\bar{h}^+_R(z^+)$ is approximately a quadratic funtion of z^+. Hence the variance is of order R^{-2d} instead of R^{-d} and the last term in (5.43) vanishes as $R \to \infty$. □

We need the following lemma from [EMY] (Lemma 6.1 in [EMY]).

Lemma 5.13. *Let* $Qf = \sum_{q\in\mathcal{Q}} r(0,q)Q_0^q f$. *Then there exist local functions* v_α *and functions* $\xi_\alpha \in \tilde{\mathcal{G}}^{ex}$ *such that*

$$I_\alpha - \theta_\alpha = Qv_\alpha + \xi_\alpha, \qquad \alpha < 0\,; \tag{5.44}$$

We now state our estimate

$$\lim_{R\to\infty}\lim_{M\to\infty} \Omega_3 = 0. \tag{5.45}$$

as the following Lemma.

Lemma 5.14. *For any* $h \in \tilde{\mathcal{G}}$, *any smooth* ϕ, *and any* $\gamma > 0$

$$\lim_{R\to\infty}\lim_{M\to\infty} \epsilon^{d-2}\log E\left[\exp\left\{\beta^{-1}\int_0^T \sum_x \phi(\epsilon x)\left[\widetilde{(\bar{h}_R^+)}(\bar{I}_M) - \bar{h}^+(\bar{I}_M^+)\right] dt\right\}\right] \le 0. \tag{5.46}$$

Proof. Recall $u_R(z) = \widetilde{(\bar{h}_R^+)}(z) - \bar{h}^+(z)$. Expanding u_R about $z = \theta$,

$$u_R(z) = \sum_{\alpha=-n_0}^{d} a_{\alpha,R}\,(z_\alpha - \theta_\alpha) + \sum_{\alpha,\beta=-n_0}^{d} b_{R,\alpha,\beta}\,(z_\alpha - \theta_\alpha)(z_\beta - \theta_b) + O((z-\theta)^3). \tag{5.47}$$

From Lemmas 5.11 and 5.12 we conclude that $\lim_{R\to\infty} b_{R,\alpha,\beta} = 0$ for $\alpha, \beta = -n_0, \ldots, d$. Define $\tilde{u}_R(z)$ by

$$\tilde{u}_R(z) = u_R(z) - \sum_{\alpha<0} a_{R,\alpha}\,(z_\alpha - \theta_\alpha)\,. \tag{5.48}$$

Then

$$|u_R(z)| \le \sigma_R\,|z-\theta|^2\,. \tag{5.49}$$

for some σ_R satisfying $\lim_{R\to\infty}\sigma_R = 0$. By Jensen's inequality,

$$\begin{aligned}
&\epsilon^{d-2}\log E\left[\exp\left\{2\beta^{-1}\int_0^T \sum_x \phi(\epsilon x)\tau_x\left[\tilde{u}(\bar{I}_M)\right] dt\right\}\right]\\
&\le \epsilon^{d-2}\log E\left[\exp\left\{2\beta^{-1}T\sum_x \phi(\epsilon x)\tau_x\left[\tilde{u}(\bar{I}_M)\right]\right\}\right]
\end{aligned} \tag{5.50}$$

By Lemma 5.10, this term vanishes in the limit of small ϵ. By Schwarz's inequality, it therefore suffices to prove that for any $\beta > 0$,

$$\lim_{R\to\infty}\lim_{M\to\infty} \epsilon^{d-2}\log E\left[\exp\left\{2\beta^{-1}\int_0^T \sum_x \phi(\epsilon x)\tau_x\left[\sum_{\alpha<0} a_{R,\alpha}(\bar{I}_{\alpha,M} - \theta_\alpha)\right] dt\right\}\right] \le 0. \tag{5.51}$$

By Lemma 5.13 and the Schwarz inequality, it suffices to prove (5.51) with $(\bar{I}_{\alpha,M} - \theta_\alpha)$ replaced by Qv_α or ξ_α. For the term with $(\bar{I}_{\alpha,M} - \theta_\alpha)$ replaced by ξ_α with $\xi_\alpha \in \tilde{\mathcal{G}}^{ex}$, (5.51) follows from the large deviation Lemma 5.10. For the other term, using Lemma 5.4 we can bound it by

$$\begin{aligned}
&\sup_{\int f^2 d\mu^r = 1}\left\{2\beta^{-1}\epsilon^{-2}\int f^2 Av_x\left[\phi(\epsilon x)\tau_x\sum_{\alpha<0} a_{R,\alpha}Qv_\alpha\right] - \epsilon^{-4+d}\mathcal{D}(f)\right\}\\
&\le Av_x\left\{\sup_{\int f^2 d\mu^r = 1}\left[\phi(\epsilon x)2\beta^{-1}\epsilon^{-2}\int f^2\left[\sum_{\alpha<0} a_{R,\alpha}Qv_\alpha\right] - \epsilon^{-4}\int f(-Q)f\right]\right\}\;;
\end{aligned} \tag{5.52}$$

Now the variational problem is finite dimensional. Since Qv_α is orthogonal to the kernel of Q, the last term is bounded by

$$\beta^2 \sum_\alpha a_{R,\alpha}^2 Av_x \phi^2(\epsilon x). \tag{5.53}$$

From (ii) of Lemma 5.12, this term is again negligible. This concludes the proof of Lemma 6.13 and hence we have proved Theorem 6.2. $\square$

6. Upper bound I.

We first recall the currents and their expansions (3.14). For our purpose in this section, it is convenient to consider these currents as functions of I^+ instead of the empirical field u. Note that I^+ and u differ by only a scaling factor ϵ. Define

$$\overline{w_i^{\alpha}}^+(z) = \pi_{\alpha,i}(z) \tag{6.1}$$

where the left hand side is defined in (3.3). We can expand the currents using the Taylor theorem as

$$\pi = \pi^{(0)} + \pi^{(1)} + \pi^{(2)} + \cdots \tag{6.2}$$

where $\pi^{(1)} = \partial\pi/\partial z|_{z=(r,0,0,0)}(z-(r,0,0,0))$. Note that compared with (3.14) we have lost factors of ϵ in the expansion because we use the variable z instead of the rescaled variables u used in (3.14). This expansion was computed explicitly in section 3 and we quote the following results from (3.21) and (3.22):

$$\begin{aligned} \pi_{0,j}^{(1)} &= (\theta^{(2)}/\theta^{(1)})z_j, \\ \pi_{\alpha,j}^{(1)} &= -\theta^{(2)}v^{(2)}\delta_{\alpha,j}z_0, \qquad \alpha = 1,\dots,d, \\ \pi_{\alpha,j}^{(2)} &= \delta_{\alpha,j}p + cz_\alpha z_j, \qquad \alpha = 1,\dots,d, \end{aligned} \tag{6.3}$$

where $c = \theta^{(3)}v_{12}^{(4)}(\theta^{(1)}v^{(2)})^{-2}$. The reader should keep in mind that $z_0 \sim \theta$ and $z_\alpha = \epsilon u_\alpha$, $\alpha = 1,\dots,d$. A basic result concerning the currents is the following corollary of Lemma 5.10, which, together with Lemma 4.5, implies that $\pi^{(2)}$ is of order ϵ^2.

Corollary 6.1. *For each $\alpha = 0,1,\dots,d$ and $j = 1,\dots,d$, for sufficiently small $\|J\|_\infty$,*

$$\limsup_{a\to 0}\limsup_{\epsilon\to 0} \epsilon^{d-2}\log E^{\mu_r}\left[\exp\sum_x J(\epsilon x)\,\tau_x\left\{\pi_{\alpha,j}(\bar I^+_{a\epsilon^{-1}}) - \pi^{(1)}_{\alpha,j}(\bar I^+_{a\epsilon^{-1}})\right\}\right] \le 0. \tag{6.4}$$

At this point we change our point of view from that of obtaining estimates to a precise study of the microscopic evolution. The evolution of the microscopic system can be described either by a set of stochastic differential equations as in (3.5), or by specifying a class of martingales. For our purposes we need the following specific class of exponential martingales. Let $F = F_{\alpha,i},\ \alpha = 0,\dots,d,\ i = 1,\dots,d$ be local mean 0 functions of η, and $J(x,t) = (J_0(x,t),\dots,J_d(x,t))$ smooth test functions and define

$$V(t,\eta) = \sum_x \left\{\epsilon J^a(\epsilon x,t)I(\eta_x) - \epsilon^2\nabla J^a(\epsilon x,t)\tau_x F\right\}. \tag{6.5}$$

We use the notation $JI = \sum_{\alpha=0}^{d} J_\alpha I_\alpha$ and $\nabla J\, F = \sum_{i,\alpha=1}^{d} (\partial_i J_\alpha) F_{\alpha,i}$ for the inner products. As usual ∂_i is the partial derivative in the ith direction acting on smooth functions on the torus. The result V is therefore a scalar. The key to both Theorem 2.2 and 2.3 is that for each J, F, and G,

$$\Psi_T(J, F, G) = \exp\left\{\Omega_\epsilon(J, F) + \Omega_\epsilon^0(G)\right\} \tag{6.6}$$

is a martingale under Q_ϵ, where

$$\Omega_\epsilon(J, F) = V(t, \eta_t)\Big|_{t=0}^{t=T} - \int_0^T \frac{\partial V}{\partial t}(t, \eta_t)\, dt + \int_0^T \epsilon^{-2} e^{-V} \mathcal{L} e^V\, dt \tag{6.7}$$

and

$$\Omega_\epsilon^0(G) = \sum_x \left\{\epsilon G(\epsilon x) I(\eta_x) - \log\left[Z(\epsilon^2 \lambda_0^{(2)}(\epsilon x), \epsilon(\lambda^{(1)} + G)(\epsilon x)) / Z(\epsilon^2 \lambda_0^{(2)}(\epsilon x), \epsilon \lambda^{(1)}(\epsilon x))\right]\right\}. \tag{6.8}$$

where Z is the moment generating function given by

$$Z(\lambda_0, \lambda) = E^{\mu_r}\left[\exp\left\{\lambda_0 I_0(\eta) + \sum_\alpha \lambda_\alpha I_\alpha(\eta)\right\}\right], \tag{6.9}$$

Since Ψ_T is a martingale under Q_ϵ we have, in particular, $E^{Q_\epsilon}[\Psi_T] = E^{\mu_\lambda^\epsilon}\left[\exp\left\{\Omega_\epsilon^0(G)\right\}\right] = 1$, and this is in fact the only property of martingales that we use. Let B denote

$$B(J, \nu) = \int J(x, t)\nu(x, t)\, dx\Big|_{t=0}^{t=T} - \int_0^T \int \partial_t J(x, t)\nu(x, t)\, dx dt. \tag{6.10}$$

From the definitions of V and ν (2.23) the first two terms containing V on the right hand side of (6.7) can be written as

$$V(t, \eta_t)\Big|_{t=0}^{t=T} - \int_0^T \frac{\partial V}{\partial t}(t, \eta_t)\, dt = \epsilon^{2-d} B(J, \nu^a) + \Gamma_\epsilon^1 \tag{6.11}$$

where Γ_ϵ^1 corresponds to those term in V containing F. Lemma 4.4 shows that Γ_ϵ^1 is negligible in sense that for each $q > 0$,

$$\lim_{\epsilon \to 0} \epsilon^{d-2} \log E\left[\exp q\Gamma_\epsilon^1\right] = 0 \tag{6.12}$$

The last term in (6.7) is the key term. From the definition of $\mathcal{L}$, we have

$$\epsilon^{-2} e^{-V} \mathcal{L} e^V = \epsilon^{-2} \sum_{\substack{y,\ |e|=1 \\ v \in \mathcal{V}}} p(y, e, v)\left(\exp\{\nabla_{y,e}^v V\} - 1\right) + \epsilon^{-2} \sum_{\substack{y \\ q \in \mathcal{Q}}} r(y, q)\left(\exp\{Q_y^q V\} - 1\right) \tag{6.13}$$

We expand the exponential in (6.13) to have

$$\epsilon^{-2} e^{-V} \mathcal{L} e^V = A_1 + A_2 + \Gamma^2. \tag{6.14}$$

Here $A_1 = \epsilon^{-2} \mathcal{L} V$ is the linear term and A_2 is the quadratic term. The last term in (6.14), $\Gamma_\epsilon^2(J, F)$, is the error consisting of higher order terms. Since J is smooth, we have $|\Gamma_\epsilon^2(J, F)| \le C\epsilon^{3-d}$. The constant depends on J and F but is independent of ϵ and η. In particular we have the following lemma.

Lemma 6.2. *For each fixed smooth J and local F, $\Gamma_\epsilon^2(J,F)$ is negligible in the sense that for any $q>0$,*

$$\limsup_{\epsilon\to 0} \epsilon^{d-2} \log E^{Q_\epsilon}\left[\exp\left\{q\int_0^T \Gamma_\epsilon^2(J,F)\,dt\right\}\right] = 0. \tag{6.15}$$

Note that (6.15) is the natural sense of negligibility in the context of large deviations. By taking various cases of J and F in V we will obtain the upper bounds in Theorems 2.2 and 2.3. First we need the following apriori estimate.

Lemma 6.3. *There exists a constant $C<\infty$ such that for $J=(J_0,J_1,\dots,J_d)$ with $\|\nabla J\|_\infty$ sufficiently small,*

$$\begin{aligned}&\lim_{a\to 0}\lim_{\epsilon\to 0} \epsilon^{d-2}\log E^{Q_\epsilon}\left[\exp\left\{\int_0^T \sum_x\sum_{\alpha=0}^d\sum_{j=1}^d \epsilon^{-1}\nabla_j^+ J_\alpha(\epsilon x)\tau_x\pi^{(1)}_{\alpha,j}(\bar I_{a\epsilon^{-1}})dt\right\}\right]\\ &\le C\left(\|J\|_2^2+\|\partial_t J\|_2^2+\|\nabla J\|_2^2\right).\end{aligned} \tag{6.16}$$

Proof. Let $F=0$ and $J=(J_0,J_1,\dots,J_d)$ in the definition of V. The strategy is to break up $\Omega_\epsilon+O_\epsilon^0$ into a finite sum $\sum_i A_i$ and estimate the pieces term by term. Now $A_1=\epsilon^{-2}\mathcal{L}V$ is given by

$$-\sum_x\sum_{\alpha=0}^d\sum_{j=1}^d \epsilon^{-1}\nabla_j^+ J_\alpha^a(\epsilon x) w^\alpha_{x,j}. \tag{6.17}$$

We rewrite the integrand in (6.16) as $-A_1+A_3+A_4$ where

$$\begin{aligned} A_3 &= -\sum_x\sum_{\alpha=0}^d\sum_{j=1}^d \epsilon^{-1}\nabla_j^+ J_\alpha(\epsilon x)\,\tau_x\left[\pi_{\alpha,j}(\bar I^+_{a\epsilon^{-1}})-\pi^{(1)}_{\alpha,j}(\bar I^+_{a\epsilon^{-1}})\right],\\ A_4 &= -\sum_x\sum_{\alpha=0}^d\sum_{j=1}^d \epsilon^{-1}\nabla_j^+ J_\alpha(\epsilon x)\,\tau_x\left(Av_{|y|\le a\epsilon^{-1}}\tau_y w_j^\alpha-\pi_{\alpha,j}(\bar I^+_{a\epsilon^{-1}})\right).\end{aligned} \tag{6.18}$$

One can obtain an estimate of the form (6.16) for A_3 from Corollary 6.1 using Jensens's inequality on the time integration. One can also obtain such an estimate for A_4 from the multiscale estimate, Theorem 5.2, since the corresponding variances are finite. The term A_2cna be computed explicitly. Since this will be done shortly in a context where the explicit form is important we will not write it out here. One easily shows that $\lim_{\epsilon\to 0}\epsilon^2 A_2\le C\|\nabla J\|_2^2$ from which an estimate of the form (6.16) follows. In addition one can estimate $B(J,\nu)$ in terms of the right hand side of (6.16) by Schwarz's inequality. Finally the lemma is obtained from these estimates by Hölder's inequality and the fact that $E^{Q_\epsilon}[\exp\Omega_\epsilon]=1$. □

In particular we have now obtained the following lemma, from which the upper bound for the non-divergence free part of the field, Theorem 2.2, follows immediately by the exponential Chebyshev inequality.

Lemma 6.4. *For any smooth test function J,*

$$\lim_{\epsilon\to 0}\epsilon^{d-2}\log E^{Q_\epsilon}\left[\exp\left\{\epsilon\int_0^T\int \nabla J\cdot\nu dx\right\}\right]<\infty. \tag{6.19}$$

Proof. From the expansion (6.3), $\pi_0^{(1)} = (\theta^{(2)}/\theta^{(1)})z$, so if one replaces the differential operator acting on J by the lattice derivative this is just a special case of the previous lemma. To control the error made by replacing the differential operator by the difference operator one can use Schwarz's inequality and then apply Lemma 4.7. □

We now consider the divergence free part of the field and prove some apriori estimates needed for the proofs of Theorems 2.1 and 2.3. To prove the upper bound for the divergence free part of the field we will specialise to $J_0 = 0$ and $J = (J_1, \dots, J_d)$ satisfying $\nabla \cdot J = 0$. We will also need F to be chosen very carefully. Now we need to write out the quadratic term explicitly.

$$A_2 = \frac{\epsilon^2}{2} \sum_x \tau_x \Theta(J, F, J, F; \tau_x \eta) + O(\epsilon^{3-d}), \tag{6.20}$$

where $\Theta = \Theta(J, F, \phi, H; \eta)$ is given by

$$\begin{aligned}\Theta = &\sum_{e,v} p(0,e,v)\left(\partial_e J\, v \nabla_e \eta(0,v) - \nabla^v_{0,e} \sum_y \nabla J\, \tau_y F\right)\left(\partial_e \phi\, v \nabla_e \eta(0,v) - \nabla^v_{0,e} \sum_y \nabla J\, \tau_y H\right)\\ &+ \sum_{q\in\mathcal{Q}} r(0,q)\left(Q_0^q \sum_y \nabla J\, \tau_y F\right)\left(Q_0^q \sum_y \nabla \phi\, \tau_y H\right)\end{aligned} \tag{6.21}$$

The $O(\epsilon^{3-d})$ error comes from replacement of the lattice gradients by differential operators and is negligible by Lemma 4.9. If we let

$$\Gamma^2_\epsilon(J, F) = \epsilon^2 \sum_x \tau_x \left(\Theta(J, F, J, F) - E^{\mu_r}\left[\Theta(J, F, J, F)\right]\right) \tag{6.22}$$

then by (4.34), $\Gamma^2_\epsilon(J, F)$ is negligible in the same sense (6.15). Now recall the currents w_i^α given in (3.3), the diffusion coefficient $D^{\alpha,b}_{i,j}$, given in (2.31). Let I_β, $\beta = 1, \dots, d$ be the momenta, $\mathcal{L}$ the generator and define

$$h^F_{\alpha,i} := w_i^\alpha - \sum_{\beta=1}^d \sum_{j=1}^d D^{\alpha,\beta}_{i,j} \nabla_j I_\beta(\eta_0) - \mathcal{L}\, F_{\alpha,i}, \qquad \alpha = 1, \dots, d. \tag{6.23}$$

Theorem 6.5. *Let $\mathbf{V}(h^F_{\alpha,i}, r)$ be the corresponding variances (see (5.6), (5.7)). Then*

$$\inf_F \mathbf{V}(h^F_{\alpha,i}, r) = 0. \tag{6.24}$$

Furthermore, for any minimizing sequence F_n,

$$\lim_{m,n\to\infty} E^{\mu_r}\left[\Theta(J, F_n, J, F_m)\right] = 2\chi(J, DJ) \tag{6.25}$$

The first statement is the main result of [EMY]. The second statement will be proved in Section 10. Let

$$\Gamma^3_\epsilon(J, F) = A_2 - \epsilon^2 \chi \sum_x \left(\nabla J(\epsilon x), D \nabla J(\epsilon x)\right). \tag{6.26}$$

By Theorem 6.5, we thus have the following lemma

Lemma 6.6. *$\Gamma^3_\epsilon(J,F)$ is negligible in the sense that for any $q>0$, if F_n is a minimizing sequence from (6.24),*

$$\lim_{n\to\infty}\limsup_{\epsilon\to 0}\epsilon^{d-2}\log E^{Q_\epsilon}\left[\exp\left\{q\int_0^T \Gamma^3_\epsilon(J,F_n)\,dt\right\}\right]=0. \tag{6.27}$$

For each smooth, divergence free $J(x,t)$, each $\delta>0$ and each $\nu\in L^2(0,T;H_1)$ let

$$b(J,\nu(\cdot,\cdot))=-\int c\sum_{\alpha,i=1}^d \partial_i J_\alpha(x)\nu_\alpha\nu_i(x)\,dx+\int(\nabla J(x),D\nabla\nu(x))\;dx \tag{6.28}$$

As usual, $c=\theta^{(3)}v^{(4)}_{12}/\left(\theta^{(1)}v^{(2)}\right)^2$ is the model specific constant in the limiting Navier-Stokes equations. Recall that $A_1=\epsilon^{-2}\mathcal{L}V$ is given by

$$A_1=-\sum_x\sum_{\alpha,j=1}^d\nabla^+_j J^a_\alpha(\epsilon x)\,\tau_x\left[w^\alpha_j+\mathcal{L}F_{\alpha,j}\right] \tag{6.29}$$

and let $\Gamma^4_{\epsilon,a}(J,F)$ denote the following difference

$$\Gamma^4_{\epsilon,a}(J,F)=A_1-\epsilon^{2-d}b(J,\nu^a) \tag{6.30}$$

Lemma 6.7. *Γ^4 is negligible in the sense that for each divergence free $J\in L^\infty(0,T;W^{2,\infty})$ and $q>0$,*

$$\lim_{n\to\infty}\lim_{\ell\to\infty}\limsup_{a\to 0}\limsup_{\epsilon\to 0}\epsilon^{d-2}\log\int_{\{\mathcal{I}_0(u^a)\le\ell\}}\exp\left\{q\int_0^T\Gamma^4_{\epsilon,a}(J,F_n)\,dt\right\}dQ_\epsilon=0. \tag{6.31}$$

where F_n is taken over the minimizing sequence for (6.24). Moreover the convergence is uniform over J with two bounded space derivatives.

Proof. *Step 1.* The strategy is to decompose Γ^4 into a sum of terms, for each of which we can prove an estimate of the form (6.31). The lemma will then follow by Hölder's inequality. Recall the definition (6.23) of $h^F_{\alpha,i}$. We can rewrite $A_1=A_3+A_4+A_5$ where

$$\begin{aligned}
A_3&=-\sum_x\sum_{\alpha,j=1}^d\nabla^+_j J_\alpha(\epsilon x)\tau_x\left(Av_{|y|\le a\epsilon^{-1}}\tau_y h^F_{\alpha,j}-\pi_{\alpha,j}\left(\bar I^+_{a\epsilon^{-1}}\right)\right)\\
A_4&=-\sum_x\sum_{\alpha,j=1}^d\nabla^+_j J_\alpha(\epsilon x)\tau_x\pi_{\alpha,j}\left(\bar I^+_{a\epsilon^{-1}}\right)\\
A_5&=-\sum_x\sum_{\alpha,\beta,i,j=1}^d\nabla^+_j J_\alpha(\epsilon x)D^{\alpha,\beta}_{i,j}\tau_x\nabla^+_j\bar I_{\beta,a\epsilon^{-1}}
\end{aligned} \tag{6.32}$$

Replacing the difference operator in A_5 by the differential operator we obtain

$$\epsilon^{2-d}\int(\nabla J(x),D\nabla\nu(x))\,dx \tag{6.33}$$

plus an error which can be shown to be negligible using Lemma 4.7 and Schwarz's inequality.

Step 2. We now estimate A_3. Since for any α and i

$$E^{\mu_\lambda}\left[\sum_{\beta,j} D_{i,j}^{\alpha,\beta}\nabla_j I_\beta(\eta_0) + \mathcal{L}\, F_{\alpha,i}\right] = 0 \tag{6.34}$$

for all λ where μ_λ is the product measure defined in (2.14), we have

$$\overline{h^F}^{+}_{\alpha,i}(z) = \overline{w_i^\alpha}^{+}(z) = \pi_{\alpha,i}(z). \tag{6.35}$$

Applying Theorem 5.2 to bound A_3, we have

$$\lim_{a\to 0}\lim_{\epsilon\to 0}\epsilon^{d-2}\log E^{Q_\epsilon}\left[\exp\left\{\int_0^T A_3\, dt\right\}\right] \le \frac{1}{2}\|\nabla J\|_2^2\sum_{\alpha,j=1}^{d}\mathbf{V}(h_{\alpha,j}^F, r) \tag{6.36}$$

By Theorem 5.3 the infimum over F of the variance vanishes. Therefore A_3 is negligible in the sense of (6.31).

Step 3. We can rewrite $A_4 = A_6 + A_7$ where

$$\begin{aligned} A_6 &= -\sum_x\sum_{j=1}^{d}\epsilon^{-1}\nabla_j^+ J_j(\epsilon x)\,\tau_x\pi_{j,j}^{(1)}\left(\bar I^+_{a\epsilon^{-1}}\right) \\ A_7 &= -\sum_x\sum_{\alpha,j=1}^{d}\epsilon^{-1}\nabla_j^+ J_\alpha(\epsilon x)\,\tau_x\left[\pi_{\alpha,j}\left(\bar I^+_{a\epsilon^{-1}}\right) - \pi_{\alpha,j}^{(1)}\left(\bar I^+_{a\epsilon^{-1}}\right)\right] \end{aligned} \tag{6.37}$$

Recall that $\pi_{j,j}^{(1)} = -c_2 z_0$. Since $\nabla\cdot J = 0$, if we could replace the difference operator in the definition of A_6, the resulting term vanishes. Using Lemma 4.7 one can show that the error is bounded, but not that it goes to zero. Now by Taylor's theorem, since $\nabla\cdot J = 0$,

$$\sum_{j=1}^{d}\epsilon^{-1}\left(J_j(\epsilon(x+e_j)) - J_j(\epsilon x)\right) = \frac{1}{2}\sum_{j=1}^{d}\left(\partial_j J_j(\epsilon(x+e_j)) - \partial_j J_j(\epsilon x)\right) + O(\epsilon^2). \tag{6.38}$$

Lemma 4.9 gives us the superexponential estimate we need for the $O(\epsilon^2)$ term. On the other hand if J has two bounded derivatives, by Lemma 6.3 we get an estimate for the right hand side of (6.38) with 0 on the right hand side of (6.16) since we got an extra factor of ϵ for free. This proves that

$$\limsup_{\epsilon\to 0}\epsilon^{d-2}\log E^{Q_\epsilon}\left[\exp\left\{\int_0^T A_6\, dt\right\}\right] \le 0. \tag{6.39}$$

Let A_8 be defined by

$$A_8 = -c\sum_{\alpha,j=1}^{d}\partial_j J_\alpha(\epsilon x)\,\tau_x\left(\bar I^+_{\alpha,a\epsilon^{-1}}\right)\left(\bar I^+_{j,a\epsilon^{-1}}\right) = -\sum_{\alpha,j=1}^{d}\partial_j J_\alpha(\epsilon x)\,\tau_x\pi^{(2)}\left(\bar I^+_{\alpha,a\epsilon^{-1}}\right). \tag{6.40}$$

Note that the pressure term p in $\pi^{(2)}$ disappears after multiplying $\partial_j J_\alpha$ since $\nabla \cdot J = 0$. It remains to prove that

$$\lim_{\ell\to\infty} \lim_{a\to 0} \limsup_{\epsilon\to 0} \epsilon^{d-2} \log \int_{\{\mathcal{I}_0(u^a)\le \ell\}} \exp\left\{ \int_0^T (A_7 - A_8)\, dt \right\} dQ_\epsilon \le 0 \tag{6.41}$$

Now

$$A_7 - A_8 = -\sum_x \sum_{\alpha,j=1}^d \partial_j J_\alpha(\epsilon x)\, \tau_x \left[\pi_{\alpha,j}(\bar I^+_{a\epsilon^{-1}}) - \pi^{(1)}_{\alpha,j}(\bar I^+_{a\epsilon^{-1}}) - \pi^{(2)}_{\alpha,j}(\bar I^+_{a\epsilon^{-1}}) \right] \tag{6.42}$$

plus an error term coming from the replacement of the difference operator by the differential operator in A_7. This error is negligible by (6.4). From the definition, $\pi_{\alpha,j}(z)$ is analytic in the interior of the domain of definition and is continuous up to the boundary. Hence

$$|\pi_{\alpha,j}(z) - \pi^{(1)}_{\alpha,j}(z) - \pi^{(2)}_{\alpha,j}(z)| \le C z^3. \tag{6.43}$$

Therefore (6.41) follows from the following Lemma 6.8. Recall that $\bar I_\alpha$, $\alpha = 1, \dots, d$, are the average momenta on a box Λ_ℓ of side length ℓ, centered at the origin in the lattice: $\bar I_{\alpha,\ell} = \ell^{-1} \sum_{x\in\Lambda_\ell} I_\alpha(\eta_x)$.

Lemma 6.8. *Let $\bar I_\ell = \bar I_{\alpha,\ell}$ for any $\alpha = 1, \dots, d$. For any $\beta > 0$, for any $a > 0$,*

$$\lim_{\ell\to\infty} \lim_{a\to 0} \lim_{\epsilon\to 0} \epsilon^{d-2} \log \int_{\{\mathcal{I}_0(u^a)\le \ell\}} \exp\left\{ \beta \sup_{0\le t\le T} \sum_x \tau_x \left(\bar I_{a\epsilon^{-1}}\right)^3 \right\} dQ_\epsilon = 0. \tag{6.44}$$

Proof. We claim that the following inequality holds:

$$\sup_x |\tau_x \bar I_{a\epsilon^{-1}}| \le a^{\frac{-d}{d+2}} \epsilon^{\frac{2}{d+2}} \|u^a\|_H^{\frac{2}{d+2}} \tag{6.45}$$

Suppose that for some x

$$|\tau_x \bar I_{a\epsilon^{-1}}| \ge \gamma a^{\frac{-d}{d+2}} \epsilon^{\frac{2}{d+2}}. \tag{6.46}$$

Since

$$|\tau_x \bar I_{a\epsilon^{-1}} - \tau_y I_{a\epsilon^{-1}}| \le C a^{-1} \epsilon |x-y|, \tag{6.47}$$

we have

$$|\tau_y \bar I_{a\epsilon^{-1}}| \ge \gamma a^{\frac{-d}{d+2}} \epsilon^{\frac{2}{d+2}} \tag{6.48}$$

if $|y - x| \le C\gamma a^{\frac{2}{d+2}} \epsilon^{\frac{-d}{d+2}}$. Hence

$$\|u^a\|_H^2 \ge C\epsilon^{d-2} \left[\gamma a^{\frac{2}{d+2}} \epsilon^{\frac{-d}{d+2}}\right]^d \left[\gamma a^{\frac{-d}{d+2}} \epsilon^{\frac{2}{d+2}}\right]^2 = C\gamma^{d+2}. \tag{6.49}$$

This proves (6.45). Therefore, as long as we are in the the set $\{\mathcal{I}_0(u^a) \le \ell\}$, we have, as $\epsilon \to 0$,

$$|\tau_x \left(\bar I_{a\epsilon^{-1}}\right)^3| \le \delta \tau_x \left(\bar I_{a\epsilon^{-1}}\right)^2 \tag{6.50}$$

for δ arbitrary small. Therefore, Lemma 6.8 follows from Lemma 4.7. □

7. The rate function.

Recall the functionals $\mathcal{I}_0$, $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}$ defined in section 1. In this section we derive several properties of these functionals which will be important in our proof of the law of large numbers and large deviations in the next two sections. All of these results are well known, but we need them in an unusual setting. Recall the definitions (2.39) of the spaces H and V of divergence free functions.

Lemma 7.1. *For any divergence free u,*

$$\|\mathbf{P}(u\cdot\nabla u)\|_{L^{4/d}(0,T;V')} \leq C\|u\|_{L^\infty(0,T;H)}^{\frac{4-d}{2}}\|u\|_{L^2(0,T;V)}^{\frac{d}{2}}. \tag{7.1}$$

Proof. Integrating by parts, for any divergence free J and u,

$$\int J\mathbf{P}(u\cdot\nabla u)dx = -\int \nabla J\,(u\otimes u)\,dx. \tag{7.2}$$

By Schwarz's inequality

$$\int |\nabla J||u\otimes u|dx \leq \|\nabla J\|_{L^2}\,\|u\|_{L^4}^2 \leq \|\nabla J\|_{L^2}\,\|u\|_{L^2}^{\frac{4-d}{2}}\,\|u\|_{L^{\frac{2d}{d-2}}}^{\frac{d}{2}}. \tag{7.3}$$

By Sobolev's inequality,

$$\|u\|_{L^{\frac{2d}{d-2}}} \leq C\,\|u\|_V\,. \tag{7.4}$$

Another application of Hölder's inequality in the time integration gives,

$$\left|\int_0^T\int J\mathbf{P}(u\cdot\nabla u)dxdt\right| \leq C\|u\|_{L^\infty(0,T;H)}^{\frac{4-d}{2}}\|u\|_{L^2(0,T;V)}^{\frac{d}{2}}\|J\|_{L^{\frac{4}{4-d}}(0,T;V)}. \tag{7.5}$$

(7.1) follows by duality. $\square$

Recall the space $\mathcal{S}$ of nice profiles u satisfying $\mathcal{I}_0(u)<\infty$, $\frac{\partial u}{\partial t}\in L^2(0,T;V')$, and $\mathbf{P}(u\cdot\nabla u)\in L^2(0,T;V')$. The class $\mathcal{S}$ consists of profiles u for which the dynamic part of the rate function $\mathcal{I}_1(u)=\frac{1}{4\chi}\left\|\frac{\partial u}{\partial t}+c\mathbf{P}\left(u\cdot\nabla u\right)-\nabla D\nabla u-f\right\|_{L^2(0,T;V')}^2$ can be defined term by term.

Lemma 7.2. *Suppose that $\mathcal{I}_0(u)<\infty$ and $\mathcal{I}_1(u)<\infty$. Then $u^a\in\mathcal{S}$.*

Proof. By the previous lemma, the assumptions imply that $\left\|\frac{\partial u}{\partial t}\right\|_{L^{4/d}(0,T;V')}<\infty$. Therefore there exists a matrix $F\in L^{4/d}(0,T;L^2)$ with $\frac{\partial u}{\partial t}=\nabla\cdot F$ in the sense of distributions and $\left\|\frac{\partial u}{\partial t}*\omega^a\right\|_{V'}=\|F*\omega^a\|_{L^2}$. By Young's inequality,

$$\|F*\omega^a\|_{L^2}\leq\|F\|_{L^{4/d}}\,\|\omega^a\|_{L^{4/d}}\,. \tag{7.6}$$

Since ω^a is smooth and $L^{4/d}(0,T;L^2)\subset L^{4/d}(0,T;L^{4/d})$ the lemma is proved by integrating this inequality over time. $\square$

The next lemma is standard in deriving weak solutions of the INS equations. The proof can be found in [CF].

Lemma 7.3. *Let u_n be a sequence of functions satisfying*

$$\left\{ \begin{array}{l} \|u_m\|_{L^2(0,T;V)} \le C, \\ \|\frac{\partial u_m}{\partial t}\|_{L^p(0,T;V')} \le C, \end{array} \right. \tag{7.7}$$

for some $C < \infty$ and $p > 1$. Then there exists a subsequence u_{m_j} converging strongly in $L^2(0,T;H)$ to some $u \in L^2(0,T;H)$.

The dynamic rate function $\mathcal{I}_1$ is not a nice rate function in the sense that it does not have compact level sets. To remedy the situation it is often convenient to replace $\mathcal{I}_1$ by

$$\mathcal{I}_1^\gamma(u) = (1-\gamma)\left\{\mathcal{I}_{\text{stat}}(u^0) + \mathcal{I}_1(u)\right\} + \gamma\left(\beta\mathcal{I}_0(u) - C\right). \tag{7.8}$$

where $0 < \gamma \le 1$ and $\beta > 0$ and $C < \infty$ are given constants to be described in the sequel. We have the following lemma:

Lemma 7.4. *On sets of bounded $\mathcal{I}_0$, the functionals $\mathcal{I}_1$ and $\mathcal{I}_2$ are lower semicontinuous with respect to the weak topology on $L^2(0,T;H_{-2})$, $\mathcal{I}_1 \le \mathcal{I}_2$ and for any closed set $\mathcal{U}$, any $0 < \gamma \le 1$ and $\beta > 0$, $C < \infty$,*

$$\limsup_{a\to 0} \inf_{u\in\mathcal{U}} \mathcal{I}_1^\gamma(u^a) \ge \inf_{u\in\mathcal{U}} \left\{(1-\gamma)\mathcal{I}(u) + \gamma\left(\beta\mathcal{I}_0(u) - C\right)\right\}. \tag{7.9}$$

Proof. From the definition $\mathcal{I}_2(u) = \liminf_{\substack{u_\epsilon \to u \\ u_\epsilon \in \mathcal{S}}} \mathcal{I}_1(u)$ it is clear that $\mathcal{I}_2 \ge \mathcal{I}_1$ and the lower semicontinuity of $\mathcal{I}_2$ follow from the lower semicontinuity of $\mathcal{I}_1$. To show that $\mathcal{I}_1$ is lower semicontinous we note that since $\mathcal{I}_1(u) = \sup_{J\in C^\infty} \Lambda(J,u)$ it suffices to show that $\Lambda(J,u)$ is continuous. Suppose that $u_n \rightharpoonup u$ weakly in $L^2(0,T;H_{-2})$ and $\mathcal{I}_0(u_n)$, $\mathcal{I}_0(u) \le C$. Let J be a smooth test function and suppose for some subsequence u_{m_j}

$$\alpha = \lim_{j\to\infty} \int_0^T \int \nabla J(u_{m_j} \otimes u_{m_j}) \tag{7.10}$$

By the previous lemma there is a further subsequence u_{m_k} converging strongly in $L^2(0,T;H)$ to some $v \in L^2(0,T;H)$. Clearly $v = u$ and therefore

$$\alpha = \int_0^T \int \nabla J(u \otimes u). \tag{7.11}$$

Therefore $\Lambda(J,u)$ is continuous.

We now prove (7.9). We can assume $\mathcal{I}_0(u) < \infty$ for some $u \in \mathcal{U}$ for otherwise the statement holds trivially. Fix $\gamma > 0$ and let u_a be a sequence in $\mathcal{U}$ with

$$\lim_{a\to 0} \mathcal{I}_1^\gamma(u_a) = \limsup_{a\to 0} \inf_{u\in\mathcal{U}} \mathcal{I}_1^\gamma(u_a). \tag{7.12}$$

For some $M < \infty$ we have $\mathcal{I}_{\text{stat}}((u_a^0)) + \mathcal{I}_0(u_a) \le M$ so there exists a subsequence converging strongly in $L^2(0,T;H)$, to $u \in \mathcal{U}$ say and a further subsequence with u_a^0 converging weakly in L^2, to v say. Since $\mathcal{I}_0(u) < \infty$ we can identify $v = u^0$. We continue to call the subsequence u_a. By Lemma 7.2, $(u_a)^a \in \mathcal{S}$ and therefore,

$$\lim_{a\to 0} \mathcal{I}_1((u_a)^a) \ge \liminf_{\substack{u_\epsilon \to u \\ u \in \mathcal{S}}} \mathcal{I}_1(u_\epsilon). \tag{7.13}$$

Therefore by lower semicontinuity of $\mathcal{I}_{\text{stat}}$ and $\mathcal{I}_0$ we have

$$\lim_{a\to 0} \mathcal{I}_1^\gamma((u_a)^a) \ge (1-\gamma)\mathcal{I}(u) + \gamma\left(\beta\mathcal{I}_0(u) - C\right). \tag{7.14}$$

Since $u \in \mathcal{U}$, (7.9) follows. $\square$

Lemma 7.5. *Suppose that $u \in \mathcal{S}$. Then $\mathcal{I}_1(u) = \mathcal{I}_2(u)$ and furthermore u^a are smooth functions converging strongly to u with $\mathcal{I}_1(u^a) \to \mathcal{I}_1(u)$.*

Proof. By the lower semicontinuity of $\mathcal{I}_1$ on bounded sets of $\mathcal{I}_0$, the first statement follows from the second since by convexity, $\mathcal{I}_0(u^a) \leq \mathcal{I}_0(u) < \infty$. Now clearly we have $\frac{\partial u^a}{\partial t} \to \frac{\partial u}{\partial t}$ and $\nabla D \nabla u^a \to \nabla D \nabla u$ strongly in $L^2(0,T;V')$. For the nonlinear term one can check that the $L^2(0,T;V')$ norm of $\mathbf{P}(u \cdot \nabla u)$ is equivalent to the $L^4([0,T] \times \mathbf{T}^d)$ norm. Then, since $u \in L^4([0,T] \times \mathbf{T}^d)$, $u^a \to u$ strongly in $L^4([0,T] \times \mathbf{T}^d)$. Therefore $\mathcal{I}_1(u^a) \to \mathcal{I}_1(u)$. □

Lemma 7.6. *Suppose that $\mathcal{I}_1(u) < \infty$. Then $u \in C_{\mathrm{w}}(0,T;H)$. In particular, the zeros of $\mathcal{I}_1(u)$ are exactly the weak solutions of the INS equations.*

Proof. Since $u \in L^\infty(0,T;H)$ and V is dense in H it is enough to check it for $J \in V$. Let $J \in V$ be independent of t. By Lemma 7.1, $\partial_t u \in L^{\frac{4}{d}}(0,T;V')$ and so it makes sense to write

$$\begin{aligned} \int J(x)u(t,x)dx - \int J(x)u(s,x)dx &= \int_s^t \int J\left(\partial_\tau u + c\mathbf{P}\left(u \cdot \nabla u\right) - \nabla D \nabla u\right) dx d\tau \\ &\quad - c\int_s^t \int J\,\mathbf{P}(u \cdot \nabla u)\, dx d\tau \\ &\quad - \int_s^t \int (\nabla J, D\nabla u)\, dx d\tau \end{aligned} \tag{7.15}$$

The weak continuity follows by Schwarz's inequality and (7.5). □

8. Upper bound II.

Recall the exponential martingales defined in 6.2. For the purposes of Theorem 2.3, we will specialize to the case where $J_0 = 0$ and $J = (J_1, \ldots, J_d)$ is divergence free. But now F will have to be chosen very carefully. Let

$$\Lambda^0(G, \nu^0) = \int dx \left\{ G(x)(\nu^0(x) - u^0(x)) - \frac{\theta^{(1)} v^{(2)}}{2} |G|^2(x) \right\}. \tag{8.1}$$

Recall the definitions (2.47) of $\mathcal{I}_{\text{stat}}$ and (6.8) of $\Omega^0_\epsilon(G)$ and note that

$$\Omega^0_\epsilon(G) = \epsilon^{2-d} \Lambda^0(G, \nu^0) + O(1), \tag{8.2}$$

and

$$I_{\text{stat}}(\nu^0) = \sup_G \Lambda^0(G, \nu^0). \tag{8.3}$$

Recall $\mathcal{I}_1(u) = \mathcal{I}_1(u, f = 0)$ from (2.43). Let

$$\begin{aligned} \Lambda(J,\nu) = \int J\nu\, dx \Big|_{t=0}^{t=T} - \int_0^T \int \partial_t J \nu\, dx dt - \int_0^T \int c \sum_{\alpha,i=1}^d \partial_i J_\alpha \nu_\alpha \nu_i\, dx dt \\ + \int_0^T \int \left(\nabla J, D\nabla \nu\right)\, dx dt - \chi \int_0^T \int (\nabla J, D \nabla J)\, dx dt \end{aligned} \tag{8.4}$$

and note that

$$\mathcal{I}_1(\nu) = \sup_J \Lambda(J,\nu). \tag{8.5}$$

From Lemmas 6.2, 6.6 and 6.7 we have the following immmediate corollary.

Corollary 8.1. *Let $\Omega_\epsilon(J,F)$ be as in (6.7) with $J_0 = 0$ and J be smooth and divergence free. Then*

$$\Omega_\epsilon(J,F) = \epsilon^{2-d}\Lambda(J,\nu^a) + \Gamma_{\epsilon,a}(J,F) \tag{8.6}$$

where $\Gamma_{\epsilon,a}$ is negligible in the sense that for each divergence free J with bounded second spacial derivatives, and each $q > 0$,

$$\lim_{n\to\infty}\lim_{\ell\to\infty}\limsup_{a\to0}\limsup_{\epsilon\to0}\epsilon^{d-2}\log\int_{\{\mathcal{I}_0(u^a)\le\ell\}}\exp\left\{q\int_0^T\Gamma_{\epsilon,a}(J,F_n)\,dt\right\}dQ_\epsilon = 0. \tag{8.7}$$

where F_n is taken over the minimizing sequence for (6.24). Moreover the convergence is uniform over J with two bounded space derivatives.

Note that the cutoff $\mathcal{I}_0(u^a) \le \ell$ is needed to prove this main superexponential estimate. Since we know that the complement has superexponentially small probability this will not affect the large deviations. We need the following preliminary upper bound.

Lemma 8.2. *Let $\mathcal{I}_1^\gamma(u)$ be as in (7.8). There exist $\beta > 0$ and $C < \infty$ such that for each $0 < \gamma \le 1$, for any closed set $\mathcal{U}$,*

$$\limsup_{\epsilon\to0}\epsilon^{d-2}\log P_\epsilon(\mathcal{U}) \le -\limsup_{a\to0}\inf_{u\in\mathcal{U}}\mathcal{I}_1^\gamma(u^a). \tag{8.8}$$

Proof. By definition

$$\sup_{J,G}\left\{\Lambda^0(G,u^0) + \Lambda(J,u)\right\} = \mathcal{I}_{\text{stat}}(u^0) + \mathcal{I}_1(u). \tag{8.9}$$

By Hölder's inequality for each J, for any $p > 1$,

$$\limsup_{a\to0}\limsup_{\epsilon\to0}\epsilon^{d-2}\log E^{P_\epsilon}\left[\exp p^{-1}\epsilon^{2-d}\left\{\Lambda^0(G,(u^0)^a) + \Lambda(J,u^a)\right\}\right] \le 0. \tag{8.10}$$

For each J and a, $\Lambda^0(G,(u^0)^a) + \Lambda(J,u^a)$ is a continuous function of u and therefore $\sup_{J,G}\left\{\Lambda^0(G,(u^0)^a) + \Lambda(J,u^a)\right\}$ can be approximated uniformly by the supremum over a finite subset on the set $\mathcal{C}_\ell = \left\{\int_0^T \|u\|_{-2}^2 dt \le \ell\right\}$ which is compact in our topology of weak convergence on $L^2(0,T,H_{-2})$. In an estimate of the form (8.10) one can automatically put a supremum over a finite set into the exponent. Therefore we have proved that for each $\ell < \infty$, $a > 0$ and $p > 1$,

$$\limsup_{\epsilon\to0}\epsilon^{d-2}\log\int_{\mathcal{C}_\ell}\exp\left[p^{-1}\epsilon^{2-d}\left\{\mathcal{I}_{\text{stat}}((u^0)^a) + \mathcal{I}_1(u^a)\right\}\right]dP_\epsilon \le 0. \tag{8.11}$$

By (7.5) there exist constants K_1 and K_2 such that

$$\|\partial_t u\|_{L^{4/d}(0,T,V')} \le K_1\left[\mathcal{I}_1(u) + \|u\|^2_{L^\infty(0,T;H)} + \|u\|^2_{L^2(0,T;V)}\right] + K_2. \tag{8.12}$$

By (8.11) and Lemmas 4.4 and 4.7 we therefore can find $\beta > 0$ and $C < \infty$ such that

$$\limsup_{\epsilon\to0}\epsilon^{d-2}\log\int_{\mathcal{C}_\ell}\exp\left[p^{-1}\epsilon^{2-d}\left\{\beta\mathcal{I}_0(u^a) - C\right\}\right]dP_\epsilon \le 0. \tag{8.13}$$

The lemma follows from Jensen's inequality, the exponential Chebyshev inequality, (8.11), (8.13), and (4.1). □

From the Lemmas 8.2 and 7.4 we have the following immediate corollary:

Corollary 8.3. *There exist $\beta > 0$ and $C < \infty$ such that for each $0 \le \gamma \le 1$, for each closed set $\mathcal{U}$,*

$$\limsup_{\epsilon \to 0} \epsilon^{d-2} \log P_\epsilon(\mathcal{U}) \le - \inf_{u \in \mathcal{U}} \{(1-\gamma)\mathcal{I}(u) + \gamma(\beta \mathcal{I}_0(u) - C)\}. \tag{8.14}$$

Taking $\gamma = 0$ in particular gives the upper bound (2.50).

Note that if $\gamma > 0$ then $(1-\gamma)\mathcal{I}(u) + \gamma(\beta\mathcal{I}_0(u) - C)$ has compact level sets. This makes Corollary 8.3 often more useful in practice than the best upper bound (2.50). In particular we can now obtain the law of large numbers.

Lemma 8.4. *Suppose that P_n is a sequence of probability measures on a Polish space X and $P_n \Rightarrow P$. Suppose that there exists a lower semicontinuous function $H : X \to [-\infty, \infty]$ with $H \le \ell$ compact for each $\ell \in (-\infty, \infty)$ and for each closed set $\mathcal{U} \subset X$,*

$$\limsup_{n \to \infty} \frac{1}{n} \log P_n(\mathcal{U}) \le - \inf_{x \in \mathcal{U}} H(x). \tag{8.15}$$

Then P is supported on $H \le 0$.

Proof. Let $\mathcal{U}$ be any closed set with $\mathcal{U} \cap \{H \le 0\} = \emptyset$. Suppose that there exists a sequence $x_n \in \mathcal{U}$ with $H(x_n) \to 0$. Since $H \le \ell$ is compact, there exists a subsequence $x_{n_j} \to x \in \mathcal{U}$. Since H is lower semicontinuous, $H(x) \le 0$, contradicting $\mathcal{U} \cap \{H \le 0\} = \emptyset$. Therefore $\inf_{x \in \mathcal{U}} H(x) > 0$, so by (8.15), $P_n(\mathcal{U}) \to 0$. $\square$

Corollary 8.5. *Any weak limit P of the set P_ϵ is supported entirely on weak solutions of the INS satisfying $\mathcal{I}_2(u) = 0$.*

Proof. Let $\beta > 0$ and $C < \infty$ be as in Corollary 8.3. By Lemmas 7.1, 7.3 and 7.4, for each $\gamma > 0$, $(1-\gamma)\mathcal{I}(u) + \gamma(\beta\mathcal{I}_0 - C)$ is lower semicontinuous with compact level sets. By the previous lemma and Corollary 8.3, for each $\gamma > 0$, P is supported on $\mathcal{U}_\gamma = \{(1-\gamma)\mathcal{I}(u) + \gamma(\beta\mathcal{I}_0(u) - C) \le 0\}$. Therefore P is supported on $\bigcap_{\gamma > 0} \mathcal{U}_\gamma = \{\mathcal{I}(u) = 0\}$. $\square$

9. Lower bound.

For each given smooth function ϕ, local function H, and initial smooth $\tilde{\lambda}$, let $Q_\epsilon^{\phi,H}$ be the process defined by

$$\frac{dQ_\epsilon^{\phi,H}}{dQ_\epsilon}\Big|_{\mathcal{F}_T} = \Psi_T(\phi, H)\frac{d\mu_{\tilde{\lambda}}^\epsilon}{d\mu_\lambda^\epsilon}. \tag{9.1}$$

where Ψ_T is defined in (6.6) with J replaced by ϕ and F replaced by H. The process $Q_\epsilon^{\phi,H}$ has initial distribution $\mu_{\tilde{\lambda}}^\epsilon$ corresponding to initial data $\tilde{u}^0$. A straightforward calculation shows that $Q_\epsilon^{\phi,H}$ has time dependent generator

$$\mathcal{L}_\epsilon^{\phi,H} = \sum_{y,e,v} p(y,e,v)\exp\left\{\nabla_{y,e}^v V\right\}\nabla_{y,e}^v + \sum_{q\in\mathcal{Q},y} r(y,q)\exp\left\{Q_y^q V\right\}Q_y^q. \tag{9.2}$$

We can rewrite the generator as $\mathcal{L}_\epsilon^{\phi,H} = \mathcal{L}_\epsilon + \mathcal{L}_{1,\epsilon}^{\phi,H} + O(\epsilon^2)$ where

$$\mathcal{L}_{1,\epsilon}^{\phi,H} = \sum_{y,e,v} p(y,e,v)\left\{\nabla_{y,e}^v V\right\}\nabla_{y,e}^v + \sum_{q\in\mathcal{Q},y} r(y,q)\left\{Q_y^q V\right\}Q_y^q \tag{9.3}$$

Certainly, we do not specify the meaning of the order of the generator and the above decomposition is merely heuristic. But it helps to keep this expansion in mind in the proof of Theorem 9.1 below.

Let F_n be a minimising sequence of (6.24). It is not hard to show that $E^{\mu_r}\left[\Theta(J,F_n,\phi,H)\right]$ also has a limit in n, and since this limit is automatically a quadratic form for each H, we can write it as

$$\lim_{n\to\infty} E^{\mu_r}\left[\Theta(J,F_n,\phi,H)\right] =: 2\chi\left(\nabla J, D_H\nabla\phi\right) \tag{9.4}$$

for some diffusion matrix D_H. In general $D_H \neq D$ and D_H may not even be a symmetric matrix. From Theorem 6.3, however,

$$\lim_{n\to\infty}(D_{F_n})_s = D \tag{9.5}$$

Let $P_\epsilon^{\phi,H}$ be the corresponding distributions of the evolving empirical momentum density fields $\nu(\cdot,\cdot)$ and $\tilde{P}_\epsilon^{\phi,H}$ the distribution of $u = \mathbf{P}\nu$. We have the following law of large numbers for the driven dynamics.

Theorem 9.1. *The set of measures $P_\epsilon^{\phi,H}$, $\epsilon > 0$ on $L^2(0,T;H_{-2})$ with the weak topology is tight, and any limit point is concentrated on weak solutions of the INS equations with initial velocity $\tilde{u}^0$ and with force*

$$f = 2\chi\nabla(D_H)_s\nabla\phi \tag{9.6}$$

These weak solutions satisfy in addition $\mathcal{I}_2(u,f) = 0$. Note that only the symmetric part of D_H enters the equation because $\partial_i\partial_j\phi = \partial_j\partial_i\phi$.

Proof. Let $\Psi_T^{\phi,H}(J,F)$ be the same as $\Psi_T(J,F)$ defined in (6.6) except that the generator $\mathcal{L}$ in (6.7) is replaced by $\mathcal{L}^{\phi,H}$. Then $\Psi_T^{\phi,H}(J,F)$ is a martingale with respect to $Q_\epsilon^{\phi,H}$. Using the decomposition of the generator $\mathcal{L}^{\phi,H}$ we can rewrite $\Psi_T^{\phi,H}(J,F)$ as

$$\Psi_T^{\phi,H}(J,F) = \Psi_T(J,F)\exp\left\{\int_0^T\left[\epsilon^{-2}\mathcal{L}_{1,\epsilon}^{\phi,H}V + \Gamma^5\right]dt\right\} \tag{9.7}$$

where Γ^5 is negligible. Recall $\Theta(J, F, \phi, H; \eta)$ from (6.21). By definition of $\mathcal{L}^{\phi,H}_{1,\epsilon}$, we have

$$\epsilon^{-2}\mathcal{L}^{\phi,H}_{1,\epsilon}V = \epsilon^{-2}\sum_y \tau_y\Theta(J, F, \phi, H). \tag{9.8}$$

Now a straightforward computation shows that for $1 < p < \infty$,

$$E^{Q_\epsilon}\left[\left(dQ^{\phi,H}_\epsilon/dQ_\epsilon\right)^p\right] \le \exp\left\{C\epsilon^{2-d}\right\} \tag{9.9}$$

with a constant independent of ϵ. Therefore all the error estimates obtained for Q_ϵ in the proof of the upper bound extend, by Hölder's inequality, to $Q^{\phi,H}_\epsilon$. Therefore for any $p > 1$,

$$\lim_{\ell\to\infty}\lim_{a\to 0}\lim_{\epsilon\to 0}\epsilon^{d-2}\log E^{Q^{\phi,H}_\epsilon}\left[\exp\left\{p^{-1}\epsilon^{2-d}\Lambda^{\phi,H}(J, G, \nu^a)\right\}\right] \le 0 \tag{9.10}$$

where

$$\Lambda^{\phi,H}(J,\nu) = \Lambda(J,\nu) - 2\chi\int_0^T\int(\nabla J, D_H\nabla\phi)\,dxdt + \Lambda^0(G,\nu^0) \tag{9.11}$$

Optimizing J and G as before, we obtain the upper bounds for $P^{\phi,H}_\epsilon$ and $\tilde{P}^{\phi,H}_\epsilon$ with an appropriately modified rate function. In particular, we have the Theorem. □

We now use the theorem to prove the lower bound. We need some standard estimates from the elliptic theory.

Lemma 9.2. *Let $D^{\alpha,b}_{i,j}$ satisfy*

$$\sum_{\alpha,i,\beta,j=1}^d D^{\alpha,\beta}_{i,j}a^\alpha_i a^\beta_j \ge \gamma\sum_{i,a}(a^\alpha_i)^2. \tag{9.12}$$

for some $\gamma > 0$ for all a^α_i. Let $f_\alpha \in C^\infty([0,T]\times T^d)$, $\alpha = 1,\ldots,d$, and

$$\nabla D\nabla u = f. \tag{9.13}$$

Then for any $\delta > 0$,

$$\|u\|_\infty \le C(\delta)\gamma^{-1}\|f\|_{H_{\frac{d}{2}-2+\delta}}. \tag{9.14}$$

Proof. Let $\hat{f}(k)$ denote the Fourier coefficients. We have

$$\sum_{i,j,\beta} D^{\alpha,\beta}_{i,j}k_ik_j\hat{u}_\beta(k) = \hat{f}_\alpha(k). \tag{9.15}$$

Let $B_{\alpha,\beta}(k) = \sum_{i,j}D^{\alpha,\beta}_{i,j}k_ik_j$. Then

$$B\hat{u} = \hat{f}. \tag{9.16}$$

Note that $\langle x, B(k)x\rangle \ge \gamma|k|^2|x|^2$ for all x_α. So $0 \le \left\langle x, B^{-1}(k)x\right\rangle \le \gamma^{-1}|k|^{-2}|x|^2$ for all x_α. In particular $\|B^{-1}(k)\|_{\mathbf{R}^d\otimes\mathbf{R}^d} \le g^{-1}|k|^{-2}$ and therefore since

$$u(x) = \sum_k B^{-1}(k)\hat{f}(k)e^{ik\cdot x} \tag{9.17}$$

we can estimate

$$|u| \le \sum_k |B^{-1}(k)\hat{f}(k)| \le \gamma^{-1}\sum_k |k|^{-2}|\hat{f}(k)|. \tag{9.18}$$

By Schwarz's inequality,

$$\sum_k |k|^{-2}|\hat{f}(k)| \le \sqrt{\sum_k |\hat{f}(k)|^2|k|^{-4+d+2\delta}}\sqrt{\sum_k |k|^{-(d+2\delta)}}. \tag{9.19}$$

Note that $\sum_k |k|^{-(d+2\delta)}$ is convergent for $\delta > 0$. This completes the proof of the lemma.□

Lemma 9.3. *The viscosity matrix D is positive definite in the sense that*

$$\sum_{\alpha,i,\beta,j=1}^{d} D_{i,j}^{\alpha,\beta} a_i^\alpha a_j^\beta \geq 2\gamma \sum_{i,a} (a_i^\alpha)^2. \tag{9.20}$$

Furthermore, D_{F_n} satisfies the analogous estimate with γ replaced by $\gamma/2$ for sufficiently large n.

Proof. Letting $F = 0$ in the variational formula (2.37) one obtains $(a, D^{-1}a) \leq (2\gamma)^{-1} \sum_{i,a} (a_i^\alpha)^2$. (9.20) follows and the second statement follows since $\lim_{n\to\infty} D_{F_n} = D$. □

Proof of the lower bound of theorem 2.3. In view of Theorem 4.1 we can assume without loss of generality that $\mathcal{O} \subset \{\mathcal{I}_0(u^a) < \ell\}$ for some ℓ and a. For any $\delta > 0$ we can find a smooth $v \in \mathcal{O}$ such that

$$\mathcal{I}(v) \leq \inf_{u\in\mathcal{O}} \mathcal{I}(u) + \delta \tag{9.21}$$

Let

$$f = \frac{\partial v}{\partial t} + \mathbf{P}(v \cdot \nabla v) - \nabla D \nabla v. \tag{9.22}$$

By Lemma 9.2, for sufficiently large F_n we can solve the Poisson equation for ϕ_n,

$$2\chi \nabla D_{F_n} \nabla \phi_n = f. \tag{9.23}$$

By definition,

$$P_\epsilon^{\phi_n, F_n}(\mathcal{O}) = \int_{\mathcal{O}} \exp\left\{\Omega_\epsilon(\phi_n, F_n) + \Omega_\epsilon^0(G)\right\} dP_\epsilon \tag{9.24}$$

where Ω_ϵ^0 is defined in (8.1) and $G = \tilde{\lambda}^{(1)} - \lambda^{(1)}$. By (8.2) and Corollary 8.1 we can write

$$\Omega_\epsilon(\phi_n, F_n) + \Omega_\epsilon^0(G) = \epsilon^{2-d}\left[\Lambda(\phi, u^a) + \Lambda^0(G, u^0)\right] + \Gamma_{\epsilon,a}(\phi_n, F_n). \tag{9.25}$$

By Chebyshev's inequality, for each $p > 1$,

$$\liminf_{\epsilon\to 0} \epsilon^{d-2} \log P_\epsilon(B) \geq -\lim_{n\to\infty} \limsup_{a\to 0} \inf_{\substack{B\subset\{\mathcal{I}_0(u^a)\leq\ell\} \\ v\in B}} \sup_{u\in B} \left\{\Lambda(\phi_n, u) + \Lambda^0(G, u^0)\right\} - \Xi^p + \Delta(B) \tag{9.26}$$

where

$$\Xi^p = \frac{p}{q} \lim_{n\to\infty} \limsup_{\epsilon\to 0} \epsilon^{d-2} \log \int_{\{\mathcal{I}_0(u^a)\leq\ell\}} \exp\left\{\frac{p}{p-1}\epsilon^{2-d}\Gamma_{\epsilon,a}(\phi_n, F_n)\right\} dP_\epsilon \tag{9.27}$$

and

$$\Delta(B) = p \liminf_{n\to\infty} \liminf_{\epsilon\to 0} \epsilon^{d-2} \log P_\epsilon^{\phi_n, F_n}(B). \tag{9.28}$$

From Corollary 8.1 we conclude that for each $p > 1$, $\Xi^p = 0$. Since v is smooth, it is the unique solution of (9.23). Therefore for each n, $P_\epsilon^{\phi_n, F_n} \Rightarrow \delta_v$. In particular $\Delta(B) = 0$. Since for any smooth ϕ, $\Lambda(\phi, u)$ is continuous on bounded sets of $\mathcal{I}_0$,

$$\limsup_{a\to 0} \inf_{\substack{B\subset\{\mathcal{I}_0(u^a)\leq\ell\} \\ v\in B}} \sup_{u\in B} \left\{\Lambda(\phi, u^a) + \Lambda^0(G, u^0)\right\} = \Lambda(\phi, v) + \Lambda^0(G, v^0). \tag{9.29}$$

Furthermore

$$\Lambda(\phi_n, v) = \chi \int_0^T \int (\nabla\phi_n, [2D_{F_n} - D]\,\nabla\phi_n)\,dxdt. \tag{9.30}$$

Sending $n \to \infty$ then $p \downarrow 1$ in (9.26) we obtain

$$\liminf_{\epsilon\to 0} \epsilon^{d-2} \log P_\epsilon(\mathcal{O}) \geq -\mathcal{I}(v) \geq -\inf_{u\in\mathcal{O}} \mathcal{I}(u) - \delta. \tag{9.31}$$

Since $\delta > 0$ is arbitrary we are done. □

Remark. In fact what we have done shows that the driven system $P_\epsilon^{\phi,H}$ has the same large deviations property as P_ϵ with rate function

$$\mathcal{I}^{\phi,H}(u) = \mathcal{I}_{\mathrm{stat}}(u^0) + \mathcal{I}_2(u, 2\chi\nabla D_H \nabla\phi). \tag{9.32}$$

10. Appendix: Variance computations.

Recall the expressions (3.3) of the currents w_j^β (we omit the position label). Note that $w_j^\beta \notin \tilde{\mathcal{G}}$. In order to obtain functions in $\mathcal{G}$, we have to substract linear terms from w as in (2.33). We call the resulting function after this subtraction $\sigma_j^\beta \in \tilde{\mathcal{G}}$:

$$\sigma_j^\beta = w_j^\beta - \sum_{\alpha=0}^{d} c_{j,\alpha}^\beta I_\alpha(\eta_0) \tag{10.1}$$

with a suitable choice of $c_{j,\alpha}^\beta$. The explicit expression for $c_{j,\alpha}^\beta$ is not important for our purposes here.

For any two local functions f and g, define an inner product

$$\langle f, g\rangle_0 = E\left[f \sum_{x\in\mathbb{Z}^d} \tau_x g\right] = E\left[g \sum_{x\in\mathbb{Z}^d} \tau_x f\right]. \tag{10.2}$$

Recall the inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ defined by polarizing the variance $\mathbf{V}$ from Theorem 5.1. From the definition of the variance,

$$\langle\!\langle L_s f, g\rangle\!\rangle = \langle f, g\rangle_0 \tag{10.3}$$

Hence

$$\langle\!\langle L_s f, \nabla_\beta I_\alpha\rangle\!\rangle = 0 \tag{10.4}$$

Furthermore,

$$\langle\!\langle \mathcal{L}_s f, \mathcal{L} f\rangle\!\rangle = -\langle f, \mathcal{L} f\rangle_0 = -\langle f, \mathcal{L}_s f\rangle_0 \tag{10.5}$$

Note that, for $g \in \tilde{\mathcal{G}}$, $\langle g, w_j^\beta\rangle_0 = \langle g, \sigma_j^\beta\rangle_0$. We also need the currents $w_i^{*\,\alpha}$ of the reversed process. They can be defined as in (3.2) with the generator $\mathcal{L}$ replaced by its adjoint $\mathcal{L}^*$. Clearly, we can define the associated $\sigma_j^{*\,\beta}$ and we have the analogous identity $\langle g, w_j^{*\,\beta}\rangle_0 = \langle g, \sigma_j^{*\,\beta}\rangle_0$. From the rotational symmetry, we have that $\mathbf{V}(\nabla_i I_\alpha)$ is independent of α or i. We need to know the variance explicitly for some special functions. The results can be summarized by the following lemma.

Lemma 10.1. *Suppose $g \in \mathcal{G}$. Then for $i, j = 1, \dots, d$ and $\alpha, \beta = 1, \dots, d$*

$$
\begin{aligned}
&\ll \nabla_i I_\alpha, \mathcal{L} g \gg = -\gamma^{-1} \langle g, w_i^{*\ \alpha} \rangle_0 = -\gamma^{-1} \langle g, \sigma_i^{*\ \alpha} \rangle_0, \\
&\ll \sigma_j^\beta, \nabla_i I_\alpha(\eta) \gg = 2\chi \delta_{i,j} \delta_{\alpha,\beta} = \ll \nabla_j I_\beta, \nabla_i I_\alpha(\eta) \gg
\end{aligned}
\tag{10.6}
$$

where $\chi = E\left[I_\alpha; I_\alpha\right]$.

Proof. From the definition of the variance and of $\mathcal{L}^*$ we have ,

$$
\ll \nabla_i I_\alpha, \mathcal{L} g \gg = \gamma^{-1} \lim_{\ell \to \infty} E^\mu[\mathcal{L} g, \sum_{|x| \le \ell} (x \cdot e_i) I_\alpha(\eta(x))] = \gamma^{-1} E^\mu[g, \mathcal{L}^* \sum_{|x| \le \ell} (x \cdot e_i) I_\alpha(\eta(x))].
\tag{10.7}
$$

Also

$$
\mathcal{L}^* \sum_x (x \cdot e_i) I_\alpha = -w_i^{*\alpha} \quad .
\tag{10.8}
$$

Some boundary terms arise but one can check they vanish in the limit $\ell \to \infty$. This proves the first identity in (10.6). The second identity on the first line follows from the definition of the inner product $\langle , \rangle_0$. To prove the identity in the second line of (10.6) , by definition,

$$
\langle\langle \nabla_i I_\alpha, \sigma_j^\beta \rangle\rangle = \gamma^{-1} \lim_{\ell \to \infty} E^\mu[\sigma_j^\beta \sum_{|x| \le \ell} (x \cdot e_i) I_\alpha(\eta(x))].
\tag{10.9}
$$

If $i \neq j$ or $\alpha \neq \beta$ then $\langle\langle \nabla_i I_\alpha, \sigma_j^\beta \rangle\rangle = 0$ by the reflection symmetry. Hence we can assume $i = j$ and $\alpha = \beta$. Since μ is a product measure, $E^\mu[\sigma_j^\beta I_\alpha(\eta_x)] = 0$ unless $x = 0$ or $x = e_j$. Hence

$$
\langle\langle \nabla_i I_\alpha, \sigma_i^\alpha \rangle\rangle = \gamma^{-1} E^\mu[\sigma_i^\alpha I_\alpha(\eta_{e_i})] = E^\mu[(\nabla_i I_\alpha)^2] = 2\chi.
\tag{10.10}
$$

We can also compute $\langle\langle \nabla_j I_\beta, \nabla_i I_\alpha(\eta) \rangle\rangle$ in the same way. $\square$

Proof of Theorem 6.5. *Step 1.* Recall the definition (6.21) of $\Theta(J, F, J, F; \eta)$. We will write it as the sum $\Theta^{ex}(J, F)$ and $\Theta^c(J, F)$ corresponding to the exclusion and collision parts of the dynamics. We first assume $n = m$. When we expand the square in $E^{\mu_r}[\Theta_2^{ex}(J, F)]$ the cross term is a linear combination of terms of the form

$$
E^{\mu_r}[\nabla_e \eta(0, v) \nabla_{0,e}^v \sum_y \tau_y F];
\tag{10.11}
$$

The infinite sum $\sum_y \tau_y F$ is purely formal, but $\nabla_{0,e}^v \sum_y \tau_y F$ makes sense for any local function F. Summing by parts one obtains

$$
E^{\mu_r}[(\eta(0, v) - \eta(e, v)) \sum_y \tau_y F].
\tag{10.12}
$$

This expression is also well defined because all but a finite number of terms vanish. But by translation invariance of the measure μ_r the entire expression vanishes for any local function F. We conclude that the cross term vanishes. Thus $E^{\mu_r}[\Theta^{ex}(J, F; \eta)] = \Omega_1(J, F) + \Omega_2(J, F)$ where

$$
\begin{aligned}
\Omega_1(J, F) &= E^{\mu_r}\left[\sum_{|e|=1, v \in \mathcal{V}} p(0, e, v; \eta) \left(\sum_{\alpha=1}^d (v \cdot e_\alpha) \partial_e J_\alpha(0) \nabla_e \eta(0, v)\right)^2\right] \\
\Omega_2(J, F) &= E^{\mu_r}\left[\sum_{|e|=1, v \in \mathcal{V}} p(0, e, v; \eta) \left(\nabla_{0,e}^v \sum_y \sum_{\alpha, i=1}^d \partial_e J_\alpha(0) \tau_y F_{\alpha, i}\right)^2\right]
\end{aligned}
\tag{10.13}
$$

The first expression can be computed explicitly. Since $E[(\nabla_j \eta(0,v))(\nabla_j \eta(0,v'))] = 0$ if $v \neq v'$ and $p(e,v) + p(-e,v) = \gamma$, we have

$$\Omega_1(J,F) = \gamma E\left[\sum_{j=1}^{d}\left(\sum_{v\in\mathcal{V}}\sum_{\alpha=1}^{d}(v\cdot e_\alpha)\partial_j J_\alpha(0)\nabla_j\eta(0,v)\right)^2\right]. \tag{10.14}$$

Since $\sum_{v\in\mathcal{V}}(v\cdot e_\alpha)\eta(0,v) = I_\alpha(0)$ we can use the last identity from Lemma 10.1 to obtain

$$\Omega_1(J,F) = 2\gamma\chi(\nabla J, \nabla J). \tag{10.15}$$

Since $p(e,v) + p(-e,v) = \gamma$, and using the translation invariance of $\sum_{y,\alpha,i} \partial_i J_\alpha(0)\tau_y F_{\alpha,i}$ we can collect the terms with $\pm e$ to obtain,

$$\Omega_2(J,F) = 2\sum_{\substack{e>0\\ v\in\mathcal{V}}} E^{\mu_r}\left[p(0,e,v;\eta)\left(\nabla^v_{0,e}\sum_{y,\alpha,i}\partial_i J_\alpha(0)\tau_y F_{\alpha,i}\right)^2\right]. \tag{10.16}$$

Recall $p(0,e,v;\eta) = p(e,v)\eta^v_0(1-\eta^v_e)$, $p(e,0,v;\eta) = p(-e,v)\eta^v_e(1-\eta^v_0)$ and $p(e,v) + p(-e,v) = \gamma$, hence

$$\Omega_2(J,F;\eta) = 2\gamma\sum_{e>0}\sum_{v\in\mathcal{V}} E\left[\left(\nabla^v_{0,e}\sum_y(\nabla J, \tau_y F)\right)^2\right] \tag{10.17}$$

Together with the collision term, we have the following expression for $E^{\mu_r}[\Theta(J,F)]$, which also appears in [LOY2]:

$$2\gamma\chi|\nabla J|^2 + 2\gamma\sum_{\substack{e>0\\ v\in\mathcal{V}}} E\left[\left(\nabla^v_{0,e}\sum_y(\nabla J,\tau_y F)\right)^2\right] + \sum_{q\in\mathcal{Q}} E\left[r(0,q)\left(Q^q_0\sum_y(\nabla J,\tau_y F)\right)^2\right] \tag{10.18}$$

Step 2. Suppose we choose F such that

$$\mathbf{V}(w^\alpha_i - \sum_{\beta,j=1}^{d} D^{\alpha,\beta}_{i,j}\nabla_j I_\beta - \mathcal{L}F^\alpha_i) \le \delta \tag{10.19}$$

We compute the inner product of $w^\alpha_i - \sum_{\beta,j=1}^{d} D^{\alpha,\beta}_{i,j}\nabla_j I_\beta - \mathcal{L}F^\alpha_i$ with $\nabla_\ell I_\gamma$ using Lemma 10.1. We obtain in this way, applying Schwarz inequality, that

$$\left|2\chi\delta_{i,\ell}\delta_{\alpha,\gamma} \;-\; 2\chi D^{\alpha,\gamma}_{i,\ell} \;+\; \gamma^{-1}\left\langle w^{*\gamma}_\ell, F^\alpha_i\right\rangle_0\right| = o(1)\;. \tag{10.20}$$

in the sense $o(1) \to 0$ as $\delta \to 0$. Repeating the same procedure with $\mathcal{L}_s F^\gamma_\ell$ instead of $\nabla_\ell I_\gamma$, we obtain, together with (10.3) that

$$\left| -\left\langle w^\alpha_i, F^\gamma_\ell\right\rangle_0 \;+\; \left\langle \mathcal{L}F^\alpha_i, F^\gamma_\ell\right\rangle_0\right| \;\le\; C_\delta\Big\{\mathbf{V}(\mathcal{L}_s F^\alpha_i)\Big\}^{1/2} \tag{10.21}$$

A simple computation shows that $\mathbf{V}(\mathcal{L}_s F_\ell^\alpha) \le \mathbf{V}(\mathcal{L} F_\ell^\alpha)$. By Schwarz inequality we have that

$$\mathbf{V}(\mathcal{L}F_i^\alpha) \;\le\; o(1) \;+\; 2\mathbf{V}\Big(w_i^\alpha - \sum_{\beta,j=1}^{d} D_{i,j}^{\alpha,\beta}\nabla_j I_\beta\Big)\,. \tag{10.22}$$

This shows that that the right hand side of (10.21) is of order $o(1)$ in δ. Since $\Big\langle w_i^\alpha + w_i^{*,\alpha}, F_\ell^\gamma \Big\rangle_0 = 0$, from the two previous equalities we get that

$$2\gamma\chi\delta_{i,l}\delta_{\alpha,\gamma} \;-\; 2\gamma\chi D_{i,\ell}^{\alpha,\gamma} \;-\; \langle \mathcal{L}F_i^\alpha, F_\ell^\gamma\rangle_0 \;= o(1) \tag{10.23}$$

Thus,

$$2\gamma\chi|\nabla J|^2 - 2\gamma\chi(\nabla J, D\nabla J) \;=\; \langle \mathcal{L}(\nabla J, F), (\nabla J, F)\rangle_0 \;+\; o(1)\,. \tag{10.24}$$

Since $\langle \mathcal{L}f, f\rangle_0 = \langle \mathcal{L}_s f, f\rangle_0$, the first term on the right hand side is equal to

$$-\frac{\gamma}{2}\sum_{e>0}\sum_{v\in\mathcal{V}} E^{\mu_r}\left[\left(\nabla_{0,e}^v \sum_y (\nabla J, \tau_y F)\right)^2\right] - \frac{1}{2}\sum_{q\in\mathcal{Q}} E^{\mu_r}\left[r(0,q)\left(Q_0^q \sum_y (\nabla J, \tau_y F)\right)^2\right]. \tag{10.25}$$

We have thus concluded the proof for $n = m$ and $j = \phi$ and we have also proved the following formula for $(\nabla J, D\nabla J)$

$$\begin{aligned}(\nabla J, D\nabla J) &= |\nabla J|^2 \\ &+ \frac{1}{2\chi}\lim_{n\to\infty} E^{\mu_r}\left[\gamma \sum_{\substack{e>0\\ v\in\mathcal{V}}}\left(\nabla_{0,e}^v \sum_y \nabla J \tau_y F_n\right)^2 + \sum_{q\in\mathcal{Q}} r(0,q)\left(Q_0^q \sum_y (\nabla J, \tau_y F_n)\right)^2\right].\end{aligned} \tag{10.26}$$

where F_n is any minimising sequence in (6.24). This formula is very similar to the variational formula for the diffusion coefficient for reversible systems. But, unlike the reversible system where the correct choice of F is simply the minimizer of this formula, the choice of F is now through a much more sophisticated mechanism.

Step 3. We now prove the general case $n \ne m$. From the definition of Θ, the decomposition (4.7) and the Schwarz inequality

$$\begin{aligned}&E^{\mu_r}\left[\Theta(J, F_n, J, (F_m - F_n))\right] \\ &\le E^{\mu_r}\left[\Theta(J, F_n, J, F_n)\right]^{1/2}\left\{E^{\mu_r}\left[\Theta^c(J, (F_m - F_n), J, (F_m - F_n))\right] + \Omega_2(J, F_n - F_m)\right\}^{1/2}\end{aligned} \tag{10.27}$$

Clearly, the first term on the right side is uniformly bounded in n. Hence it suffices to prove the last term vanishes as $n, m \to 0$. By definition this is given by

$$-\langle \mathcal{L}_s(\nabla J, F_n - F_m), (\nabla J, F_n - F_m)\rangle_0 = \mathbf{V}(\mathcal{L}_s(\nabla J, F_n - F_m)) \tag{10.28}$$

Recall that $\mathbf{V}(\mathcal{L}_s f) \le \mathbf{V}(\mathcal{L} f)$. By Schwarz's inequality,

$$\mathbf{V}(\mathcal{L}(\partial J, F_n - F_m)) \le 2\mathbf{V}(w_i^\alpha - \sum_{\beta,j=1}^{d} D_{i,j}^{\alpha,\beta}\nabla_j I_\beta - \mathcal{L}(F_n)_i^\alpha) + 2\mathbf{V}(w_i^\alpha - \sum_{\beta,j=1}^{d} D_{i,j}^{\alpha,\beta}\nabla_j I_\beta - \mathcal{L}(F_m)_i^\alpha). \tag{10.29}$$

Since F_n is a minimizing sequence, the right hand side vanishes as $n, m \to \infty$. $\square$.

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