Diffusion Semigroups and Diffusion Processes Corresponding to Degenerate Divergence Form Operators

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1. Introduction.

In this article we study existence and uniqueness problems for semigroups and processes determined by time dependent second order linear partial differential operators of the form

$$L_t u = \nabla \cdot a\nabla u + \sigma c \cdot \nabla u \tag{1.1}$$

For simplicity we will take the space to be the d-dimensional torus $\mathbf{T}^d$. The matrix $a : [0,T] \times \mathbf{T}^d \to \mathbf{R}^d \otimes \mathbf{R}^d$ is assumed to be symmetric and non-negative definite. We define σ to be the symmetric non-negative definite square root of a, and $c : [0,T] \times \mathbf{T}^d \to \mathbf{R}^d$ is a function about which assumptions will be made later. The focus of our study is on cases where the coefficients a and c have minimal regularity, and a is allowed to degenerate.

From the point of view of partial differential equations one considers parabolic equations of the form

$$\frac{\partial u}{\partial t} + L_t u = 0 \tag{1.2}$$

on $[0,T]\times\mathbf{T}^d$, with a (final) boundary condition $u(T,x) = f(x)$, and tries to prove existence and uniqueness in some suitable class. The class of potential solutions is to be chosen carefully. Formally one can try to obtain an apriori energy estimate by multiplying the solution by u and integrating. From the maximum principle one expects the solution to be in $L^\infty([0,T] \times \mathbf{T}^d)$ if we start with an f in $L^\infty(\mathbf{T}^d)$. It is therefore not unnatural to look for solutions in the class $H_{1,a}$ consisting of u such that

$$\int_0^T dt \int_{\mathbf{T}^d} dx\, \nabla u \cdot a\nabla u < \infty \tag{1.3}$$

provided we impose some control on c. But a serious question that arises is the meaning of a condition like (1.3) especially when a is degenerate. If $a = \sigma^2$ with a symmetric σ, what we have is

$$\int_0^T dt \int_{\mathbf{T}^d} dx\, |\sigma\nabla u|^2 < \infty$$

and we can try to define $\sigma\nabla u$ as a distribution. It is therefore natural to impose on σ the condition

$$\int_0^T dt \int dx \sum_{i,j=1}^{d} |\nabla\sigma_{ij}|^2 < \infty.$$

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Another issue that has to be faced is the control of large values of a or equivalently σ. In any case we have assumed no regularity of a in t and some control on a is needed. Our motivation comes from an effort to understand the motion of a tagged particle in an interacting particle system away from equilibrium. Such systems in their hydrodynamic scaling limits can be described approximately by their particle density $\rho(t,x)$ which is a function of time and space. A tagged particle in equilibrium at a constant density ρ will diffuse like a Brownian Motion with a diffusion matrix $a(\rho)$. One expects that, to within first order corrections that are due to density gradients, the leading term in the generator for the tagged particle in nonequilibrium will given by

$$\frac{1}{2}\nabla a(\rho(t,x))\nabla u.$$

In other words one expects the generator to be

$$L_t u = \frac{1}{2}\nabla a(\rho(t,x))\nabla u + \sigma c \nabla u$$

for some decent choice of c. Self consistency dictates that ρ itself should be a solution of

$$\frac{\partial \rho}{\partial t} = L_t^* \rho$$

We cannot expect $a(t,x) = a(\rho(t,x))$ to have any more regularity than ρ even when $a(\rho)$ is a fairly regular function of ρ. While $a(\rho)$ is nondegenerate or elliptic, the ellipticity could degenerate at the fringes, especially at high density. For what are known as simple exclusion models, there is a natural upper bound of 1 for the density and $a(\rho) \to 0$ as $\rho \to 1$. If the regularity of ρ is not strong enough to force continuity of ρ we do not even have local ellipticity. In the context of large deviation theory the only regularity one has on ρ is the finiteness of some energy-like rate function and so, for wide applicability, it is important to deal with a satisfying minimal regularity. It will be useful to investigate the case with minimal control on the size of a, namely $a \in L^1([0,T] \times \mathbf{T}^d)$. We will therefore assume that

$$\int_0^T dt \int dx \left\{ \sum_{i,j=1}^d |\nabla \sigma_{ij}|^2 + \sum_{i,j=1}^d |\sigma_{ij}|^2 \right\} < \infty. \tag{\textbf{A1}}$$

If one can establish that $a(\rho)$ is Lipshitz in ρ then any minimal regularity that one has on $\rho(t,x)$ can be transferred to $a(t,x)$.

From the probabilistic point of view one is intersted in proving a uniqueness theorem for nonnegative weak solutions of the adjoint equation

$$\frac{\partial p}{\partial t} = L_t^* p \tag{1.4}$$

with the initial condition $p(0,x) = p_0(x)$. Section 2 contains some basic approximation tools that are needed and in section 3 we establish the existence and uniqueness theorems for parabolic equations under suitable conditons on a and c. The applications that we have in mind have a peculiar twist to the usual martingale problem corresponding to a given generator. We are given L_t as well as a nonnegative weak solution p of the equation (1.4). We want to show the existence and uniqueness of a measure P on the space of continuous functions that has p for its marginals and is a solution to the martingale problem for L_t. The vector field c can have singularities and in fact need not even be defined where $p = 0$, which can be a set of positive Lebesgue measure. In our application c will involve some sort of logarithmic gradient of p. In a much simpler context this is like taking an arbitrary nonnegative function $p(x)$ and writing it as a solution of

$$0 = \frac{\partial p}{\partial t} = \frac{1}{2}\Delta p - \nabla . \left(\frac{\nabla p}{2p}\right) p$$

and asking if there is a unique process with generator

$$L_t = L = \frac{1}{2}\Delta \quad + \frac{\nabla p}{2p} . \nabla$$

that has p for an invariant measure. The first order term $\frac{\nabla p}{2p}$ can be quite bad. This is the motivation for sections 4 and 5. For some of the results in these sections we need to strengthen the hypothesis (**A**1) by replacing it with (**A**2).

$$\int_0^T dt \int dx \, Trace\left(\dot{a} a^{-1} \dot{a}\right) < \infty. \tag{$\mathbf{A2}$}$$

Here $\dot{a}$ denotes $\frac{\partial a}{\partial x_i}$ for any $i = 1, \ldots, d$ and

$$Trace\left(\dot{a} a^{-1} \dot{a}\right) = \sum_{i,j,k,\ell=1}^{d} \frac{\partial a_{jk}}{\partial x_i} (a^{-1})_{k\ell} \frac{\partial a_{\ell j}}{\partial x_i}.$$

The condition (**A**2) is technical; we need it in order to prove the uniqueness of very weak solutions, which is the type of uniqueness that comes up when one studies the martingale problem for L_t. It implies both (**A1**) and the classical finite energy condition $\int_0^T dt \int dx (\nabla \cdot a) a^{-1} (\nabla \cdot a) < \infty$. In one dimension it is equivalent to either, while in higher dimensions it is considerably stronger in general. We do not claim that it is optimal. However when $a(t, x) = a(\rho(t, x))$ for a nice function $a(\rho)$ the two conditions (**A1**) and (**A2**) are not too far apart and both demand the finiteness of some weighted quadratic norms on $\nabla \rho$.

2. Some approximation results.

We start with the bilinear form

$$\langle \phi, \psi \rangle_{1,a} = \int_0^T dt \int dx \, \left(\nabla \phi \cdot a \nabla \psi + \phi \psi \right) \tag{2.1}$$

that defines an inner product on $C^\infty \left([0,T] \times \mathbf{T}^d \right)$. The Hilbert space $H_{1,a}$ is defined as the completion of $C^\infty \left([0,T] \times \mathbf{T}^d \right)$ under the norm

$$\|\phi\|_{H_{1,a}} = \langle \phi, \phi \rangle_{1,a}^{1/2}.$$

If ϕ_n is a Cauchy sequence of smooth functions in the space $H_{1,a}$, that converges to $\phi \in L_2([0,T] \times \mathbf{T}^d)$ the corresponding sequence $f_n = \sigma \nabla \phi_n$ has a limit f in $L_2([0,T] \times \mathbf{T}^d)$ and it is natural to call $f = \sigma \nabla \phi$. On the other hand for $\phi \in L_2([0,T], \mathbf{T}^d)$ we can try to define the norm by duality as given by

$$N^2(\phi) = \sup \left\{ 2 \int_0^T dt \int dx \phi \left(\nabla \cdot \sigma g + f \right) - \int_0^T dt \int dx (|g|^2 + |f|^2) \right\} \tag{2.2}$$

with the supremum being taken over $f, g_1, \dots g_d \in C^\infty([0,T] \times \mathbf{T}^d)$. As long as $(\mathbf{A1})$ holds, the above definition makes sense and for $\phi \in H_{1,a}$

$$N(\phi) = \|\phi\|_{H_{1,a}}$$

A key point is that if $(\mathbf{A1})$ holds, the converse is also true. If $N(\phi) < \infty$ then $\phi \in H_{1,a}$ and

$$\|\phi\|_{H_{1,a}} = N(\phi).$$

This requires a certain amount of work, because the the way in which the usual mollification process of convolution by a smooth function works on $H_{1,a}$ is far from clear. It will be proved below.

We first prove the easy fact that condition $(\mathbf{A2})$ implies condition $(\mathbf{A1})$.

Lemma 2.1. *Let $a : [0,T] \times \mathbf{T}^d \to \mathbf{R}^d \otimes \mathbf{R}^d$ and let σ denote the non-negative definite square root of a. Then*

$$\sum_{i,j} |\dot{\sigma}_{ij}|^2 \le Trace \left(\dot{a} a^{-1} \dot{a} \right).$$

Proof. Suppose that a is diagonal at some point (t,x). Then

$$\dot{\sigma}_{ij} = \frac{\dot{a}_{ij}}{\sqrt{a_{ii}} + \sqrt{a_{jj}}}.$$

Therefore at (t,x),

$$\sum_{i,j} |\dot{\sigma}_{ij}|^2 = \sum_{i,j} \left(\frac{\dot{a}_{ij}}{\sqrt{a_{ii}} + \sqrt{a_{jj}}} \right)^2 \le \sum_{i,j} \frac{\dot{a}_{ij} \dot{a}_{ji}}{a_{jj}} = Trace \left(\dot{a} a^{-1} \dot{a} \right).$$

But $Trace(MM^*) = \sum_{ij} |M_{ij}|^2$ and $Trace(M)$ are invariant under orthogonal change of basis, and a is diagonalizable at every point. The lemma follows.

Let $\mathcal{G}_\epsilon$ be the fundamental solution of the heat equation at time ϵ^2, and let $\phi_\epsilon := \phi * \mathcal{G}_\epsilon$ be the spatial convolution. The following lemma provides a crucial estimate that will be used repeatedly.

Lemma 2.2. *Let $f \in W^{1,2}$. Then the family of functions*

$$g_\epsilon(x) = \int dy \, |f(x+y) - f(x)|^2 \, \frac{|\nabla \mathcal{G}_\epsilon|^2}{\mathcal{G}_\epsilon}(y).$$

for $\epsilon > 0$ is uniformly integrable.

Proof. Recall that a family of functions $g_\gamma, \ \gamma \in \Gamma$ is uniformly integrable if and only if there exists a function $\Psi : \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$\begin{aligned} &(a) \qquad \lim_{x\to\infty} \frac{\Psi(x)}{x} = \infty \\ &(b) \qquad \sup_{\gamma\in\Gamma} \int dx \, \Psi\left(|g_\gamma(x)|\right) < \infty. \end{aligned}$$

Furthermore if g is any function in L^1 then there exists an increasing, convex function Ψ satisfying (a) such that $\Psi(|g|) \in L^1$. Choose such a Ψ for $|\nabla f|^2$. For points $x \in \mathbf{T}^d$ and $y \in \mathbf{R}^d$ the point $x + y$ is of course well defined on $\mathbf{T}^d$ and the convolution can be performed by integrating with respect to y on $\mathbf{R}^d$. For $x \in \mathbf{T}^d$ and $y \in \mathbf{R}^d$, let us define

$$h_y(x) = \frac{|f(x+y) - f(x)|^2}{\|y\|^2}$$

By Jensen's inequality,

$$\begin{aligned} \int dx \, \Psi(h_y(x)) &= \int dx \, \Psi\left(\frac{|f(x+y)-f(x)|^2}{\|y\|^2}\right) \\ &= \int dx \, \Psi\left(|\int_0^1 dt \, \frac{\langle y \cdot \nabla f(x+ty)\rangle}{\|y\|}|^2\right) \\ &\leq \int dx \int_0^1 dt \, \Psi\left(|\nabla f(x+ty)|^2\right) \\ &= \int dx \, \Psi\left(|\nabla f|^2\right) \end{aligned}$$

The last term is finite, and therefore the family $h_y, \ y \in \mathbf{R}^d$ is uniformly integrable. Let

$$c_\epsilon = \int_{\mathbf{R}^d} dy \|y\|^2 \frac{|\nabla \mathcal{G}_\epsilon|^2}{\mathcal{G}_\epsilon}(y)$$

and note that $c_\epsilon = c < \infty$, is independent of ϵ. Therefore

$$\begin{aligned} \int dx \, \Psi\left(c^{-1} g_\epsilon(x)\right) &= \int dx \, \Psi\left(\int dy \ |f(x+y)-f(x)|^2 \, c^{-1} \frac{|\nabla \mathcal{G}_\epsilon|^2}{\mathcal{G}_\epsilon}(y)\right) \\ &\leq \int dx \int dy \, \Psi\left(h_y(x)\right) c^{-1} \|y\|^2 \frac{|\nabla \mathcal{G}_\epsilon|^2}{\mathcal{G}_\epsilon}(y) \\ &\leq \sup_{y\in\mathbf{R}^d} \int dx \, \Psi\left(h_y(x)\right). \end{aligned}$$

We have just shown that the last term is bounded. Therefore the family $\{c^{-1}g_\epsilon,\ \epsilon > 0\}$ is uniformly integrable and so is the family $\{g_\epsilon,\ \epsilon > 0\}$.

The basic fact regarding convolutions in $H_{1,a}$ is contained in the next lemma. Although the lemma is an easy consequence of the earlier lemma, its conclusion is far from obvious. When σ is not invertible, control over $|\sigma\nabla u|$ can hide a lot of cancellations. Convolving u while leaving σ alone could royally mess up these cancellations. The assumption $(\mathbf{A}1)$ prevents this.

Lemma 2.3. *Suppose that* $(\mathbf{A1})$ *holds.*

$$(i)\quad \text{If } \phi \in L^2\left([0,T]\times \mathbf{T}^d\right) \quad \text{then} \quad \|\sigma\nabla\phi_\epsilon - (\sigma\nabla\phi)_\epsilon\|_{L^1([0,T]\times\mathbf{T}^d)} \to 0$$

$$(ii)\quad \text{If } \phi \in L^\infty\left([0,T]\times \mathbf{T}^d\right) \quad \text{then} \quad \|\sigma\nabla\phi_\epsilon - (\sigma\nabla\phi)_\epsilon\|_{L^2([0,T]\times\mathbf{T}^d)} \to 0$$

In particular for $\phi \in H_{1,a}$ *with* $\sigma\nabla\phi = f$, $\|\sigma\nabla\phi_\epsilon - f\|_{L^1([0,T]\times\mathbf{T}^d)} \to 0$; *if we assume in addition that* $\phi \in L^\infty\left([0,T]\times\mathbf{T}^d\right)$, *then* $\|\sigma\nabla\phi_\epsilon - f\|_{L^2([0,T]\times\mathbf{T}^d)} \to 0$

Proof. Let us prove (i) first. After an integration by parts we can write

$$\begin{aligned}
&|\sigma\nabla\phi_\epsilon - (\sigma\nabla\phi)_\epsilon|(x) \\
&\quad = \left|\int dy\ (\sigma(x+y)-\sigma(x))\,\nabla\phi(x+y)\mathcal{G}_\epsilon(y)\right| \\
&\quad = \left|\int dy\ \left[\nabla\cdot\sigma(x+y) + (\sigma(x+y)-\sigma(x))\,\frac{\nabla\mathcal{G}_\epsilon}{\mathcal{G}_\epsilon}(y)\right]\phi(x+y)\mathcal{G}_\epsilon(y)\right| \\
&\quad \le \frac{1}{2}\int dy\ \left|\nabla\cdot\sigma(x+y) + (\sigma(x+y)-\sigma(x))\,\frac{\nabla\mathcal{G}_\epsilon}{\mathcal{G}_\epsilon}(y)\right|^2 \mathcal{G}_\epsilon(y) \\
&\qquad\qquad + \frac{1}{2}\int dy\,\phi^2(x+y)\mathcal{G}_\epsilon(y)
\end{aligned} \tag{2.1}$$

Integrating both side with respect to x,

$$\sup_{\epsilon>0}\|\sigma\nabla\phi_\epsilon - (\sigma\nabla\phi)_\epsilon\|_{L^1([0,T]\times\mathbf{T}^d)} \le \frac{1}{2}\left[C^2 + \|\phi\|^2_{L^2([0,T]\times\mathbf{T}^d)}\right]$$

which in turn implies

$$\sup_{\epsilon>0}\|\sigma\nabla\phi_\epsilon - (\sigma\nabla\phi)_\epsilon\|_{L^1([0,T]\times\mathbf{T}^d)} \le C\|\phi\|_{L^2([0,T]\times\mathbf{T}^d)} \tag{2.2}$$

We note that for smooth ϕ the claim is obvious and given the estimate (2.2), the result follows for ϕ in $L^2([0,T]\times\mathbf{T}^d)$.

We now turn to (ii). Since we already have convergence in $L^1([0,T]\times\mathbf{T}^d)$, we only need to establish the uniform integrability of the functions $|\sigma\nabla\phi_\epsilon - (\sigma\nabla\phi)_\epsilon|^2$. They are dominated by

$$\|\phi\|^2_{L^\infty([0,T]\times\mathbf{T}^d)}\int dy\ \left|\nabla\cdot\sigma(x+y) + (\sigma(x+y)-\sigma(x))\,\frac{\nabla\mathcal{G}_\epsilon}{\mathcal{G}_\epsilon}(y)\right|^2 \mathcal{G}_\epsilon(y). \tag{2.3}$$

Since we have assumed in $(\mathbf{A}1)$ that $|\nabla\cdot\sigma|^2 \in L^1([0,T]\times\mathbf{T}^d)$ the family $\{(|\nabla\cdot\sigma|^2)_\epsilon : \epsilon > 0\}$ is uniformly integrable. The remaining piece in (2.3) is uniformly integrable as a consequence of Lemma 2.2.

The following is a useful variant of Lemmas 2.2 and 2.3.

Lemma 2.4. *Let $\mathcal{H}_\epsilon$ be any sequence of functions such that $\frac{\|y\|^2}{\epsilon^2}\mathcal{H}_\epsilon(y)$ is (a bounded multiple of) an approximate identity on $\mathbf{R}^d$.*
(i) For any $f \in W^{1,2}$, the family of functions

$$\tilde{g}_\epsilon(x) = \epsilon^{-2}\int dy\, |f(x+y) - f(x)|^2\, \mathcal{H}_\epsilon(y)$$

for $\epsilon > 0$ is uniformly integrable.
(ii) Suppose that $(\mathbf{A1})$ holds and $\phi \in L^\infty\left([0,T]\times\mathbf{T}^d\right)$. Then

$$\int dx \int dz\, |\sigma(x+z)\nabla\phi_\epsilon(x) - (\sigma\nabla\phi)_\epsilon(x)|^2 \mathcal{H}_\epsilon(z) \to 0.$$

Proof. To prove *(i)*, recalling the notation of the proof of Lemma 2.2, if

$$\tilde{g}_\epsilon(x) = \int dy\, h_y(x)\|y/\epsilon\|^2\mathcal{H}_\epsilon(y).$$

then

$$\begin{aligned}\sup_{\epsilon>0}\int dx \Psi\left(\tilde{g}_\epsilon(x)\right) &\le \sup_{\epsilon>0}\int dx \int dy\, \Psi\left(h_y(x)\right)\|y/\epsilon\|^2\mathcal{H}_\epsilon(y)\\ &\le \sup_{y\in\mathbf{R}^d}\int dx\, \Psi(h_y(x)).\end{aligned}$$

Since $\{h_y(x),\ y\in\mathbf{R}^d\}$ is uniformly integrable, so is $\{\tilde{g}_\epsilon(x),\ \epsilon>0\}$.

To prove *(ii)* from *(i)* we use exactly the same argument as in the proof of Lemma 2.3 to reduce it to the uniform integrability of

$$\int dz \int dy\, |f(x+y) - f(x+z)|^2\, \frac{|\nabla\mathcal{G}_\epsilon|^2}{\mathcal{G}_\epsilon}(y)\mathcal{H}_\epsilon(z)$$

for any $f\in W^{1,2}$. Now expand $|f(x+y)-f(x+z)| \le |f(x+y)-f(x)| + |f(x+z)-f(x)|$. For the first term appeal to the previous lemma. For the second term use *(i)* and the fact that

$$\int dy\, \frac{|\nabla\mathcal{G}_\epsilon|^2}{\mathcal{G}_\epsilon}(y) \le C\epsilon^{-2}.$$

Lemma 2.5. *Suppose that $(\mathbf{A1})$ holds and $N(\phi)<\infty$ for some $\phi\in L_2([0,T]\times\mathbf{T}^d)$. Then there exists $f=(f_1,\cdots,f_d)\in L^2([0,T]\times\mathbf{T}^d)$ such that for all $\psi=(\psi_1,\cdots,\psi_d)\in C^\infty$,*

$$\int_0^T dt \int dx\, \phi\nabla\cdot(\sigma\psi) = -\int_0^T dt \int dx\, f\psi,$$

and $N^2(\phi) = \|f\|^2_{L^2([0,T]\times\mathbf{T}^d)} + \|\phi\|^2_{L^2([0,T]\times\mathbf{T}^d)}$. Furthermore we have the following chain rule: let $F:\mathbf{R}\to\mathbf{R}$ satisfy $|F(x)|\le C|x|$ and $|F'(x)|\le C$ for all $x\in\mathbf{R}$ and some $C<\infty$. Then for all $\psi\in C^\infty$,

$$\int_0^T dt \int dx\, F(\phi)\nabla\cdot(\sigma\psi) = -\int_0^T dt \int dx\, F'(\phi)f\psi. \tag{2.4}$$

Proof. The first statement is obvious. To prove (2.4), note first that it holds for $\phi \in C^\infty$. In particular

$$\int_0^T dt \int dx\, F(\phi_\epsilon)\nabla \cdot (\sigma\psi) = -\int_0^T dt \int dx\, F'(\phi_\epsilon)\sigma\nabla\phi_\epsilon\psi.$$

Since $\phi \in L^2$, $\phi_\epsilon \to \phi$ strongly in L^2. Since $|F(x)| \le C|x|$ and $\nabla \cdot (\sigma\psi) \in L^2$ we conclude

$$\int_0^T dt \int dx\, F(\phi_\epsilon)\nabla \cdot (\sigma\psi) \to \int_0^T dt \int dx\, F(\phi)\nabla \cdot (\sigma\psi).$$

By lemma 2.3, we have $\sigma\nabla\phi_\epsilon \to f$ strongly in L^1. Since F' is bounded we conclude that

$$\int_0^T dt \int dx\, F'(\phi_\epsilon)\sigma\nabla\phi_\epsilon\psi \to \int_0^T dt \int dx\, F'(\phi) f\psi.$$

This completes the proof.

Theorem 2.6. *Suppose that* (**A1**) *holds, and* $\phi \in L_2([0,T]\times \mathbf{T}^d)$ *with* $N(\phi) < \infty$. *Then* $\phi \in H_{1,a}$.

Proof. First let us suppose that ϕ is in $L^\infty([0,T]\times\mathbf{T}^d)$. Let, as before, ϕ_ϵ denote the convolution of ϕ with the heat kernel $\mathcal{G}_\epsilon$. For $\epsilon_1, \epsilon_2 > 0$,

$$\begin{aligned}\|\phi_{\epsilon_1} - \phi_{\epsilon_2}\|_{H_{1,a}} \le{}& \|\sigma\nabla\phi_{\epsilon_1} - (\sigma\nabla\phi)_{\epsilon_1}\|_{L^2([0,T]\times\mathbf{T}^d)} + \|\sigma\nabla\phi_{\epsilon_2} - (\sigma\nabla\phi)_{\epsilon_2}\|_{L^2([0,T]\times\mathbf{T}^d)}\\ &+ \|f_{\epsilon_1} - f_{\epsilon_2}\|_{L^2([0,T]\times\mathbf{T}^d)} + \|\phi_{\epsilon_1} - \phi_{\epsilon_2}\|_{L^2([0,T]\times\mathbf{T}^d)}\,.\end{aligned}$$

All terms on the right tend to 0 as $\epsilon_1, \epsilon_2 \to 0$. For the first two terms we use Lemma 2.3 and the others are obvious.

If $N(\phi) < \infty$ but ϕ is only in $L^2([0,T]\times\mathbf{T}^d)$, we truncate it at level C and define

$$\phi_C(x) = \begin{cases} \phi & \text{if } |\phi| \le C, \\ C & \text{if } |\phi| \ge C. \end{cases}$$

By the previous lemma, for all $\psi \in C^\infty$,

$$\int_0^T dt \int dx\, \phi_C\, \nabla\cdot(\sigma\psi) = -\int\int_{|\phi|\le C} dxdt\, f\psi.$$

Then for $0 < C_1 < C_2 < \infty$,

$$\|\phi_{C_1} - \phi_{C_2}\|_{1,a}^2 = \int\int_{C_1\le|\phi|\le C_2} dxdt\, |f|^2 + \int_0^T \int dtdx\, |\phi_{C_1} - \phi_{C_2}|^2$$

In particular, ϕ_C has a limit in $H_{1,a}$ as $C \to \infty$ which is of course ϕ.

3. Existence and uniqueness for a class of degenerate parabolic equations.

If we are given the basic divergence form operator

$$L_t^0 = \nabla \cdot \nabla$$

there are several natural problems involving the operator that one can consider. For some suitable c one can study

$$\begin{cases} \frac{\partial u}{\partial t} + \nabla \cdot a \nabla u + \sigma c \cdot \nabla u = g \quad \text{for} \quad 0 \le t \le T, \\ u(T, x) = f(x) \end{cases} \tag{3.1}$$

Because of the maximim principle if $f, g \in L^\infty$ we can look for a solution $u \in L^\infty$.

We could also exploit the energy relation and consider, for suitable c_1, c_2, g and h, solutions in L^2 of

$$\begin{cases} \frac{\partial u}{\partial t} + \nabla \cdot a \nabla u + \sigma c_1 \cdot \nabla u + \nabla \cdot \sigma c_2 u = g + \nabla \cdot \sigma h \quad \text{for} \quad 0 \le t \le T, \\ u(T, x) = f(x) \end{cases} \tag{3.2}$$

The adjoint problem

$$\begin{cases} \frac{\partial p}{\partial t} = \nabla \cdot a \nabla p - \nabla \cdot \sigma c p \quad \text{for} \quad 0 \le t \le T, \\ p(0, x) = p_0(x) \end{cases} \tag{3.3}$$

is important in probability theory as a Fokker-Planck equation and p is a nonnegative probability density, usually with a total integral equal to 1.

We will have something to say about each one of these problems. The minimal regularity that our solutions satisfy will be that they are in $H_{1,a}$ and

$$\sup_{0 \le t \le T} \int_{\mathbf{T}^d} |u(t, x)|^2 \, dx < \infty.$$

All of our equations have the form

$$\frac{\partial u}{\partial t} = \nabla \cdot \sigma F + G \tag{3.4}$$

and this is to be interpreted in a suitable sense. We say that (3.4) is satisfied in the sense of distributions if, for any C^∞ test function,

$$\frac{d}{dt} \int_{\mathbf{T}^d} \phi(t, x) u(t, x) dx = \int_{\mathbf{T}^d} [-F \cdot \sigma \nabla \phi + \phi G + u \, \frac{\partial \phi}{\partial t}] dx \tag{3.5}$$

It does not make any difference if we limit ourselves to using C^∞ functions $\phi(x)$ that do not depend on t. We begin with a lemma that will be useful in dealing with such solutions.

Lemma 3.1. *Suppose $u_1, u_2, \cdots, u_k$ are functions such that for each $i = 1, \ldots, k$,*

$$\begin{cases} u_i \in L^\infty([0,T] \times \mathbf{T}^d) \cap H_{1,a} \\ \frac{\partial u_i}{\partial t} = \nabla \cdot \sigma f_i + g_i \end{cases}$$

in the sense of (3.5), with $f_i \in L^2([0,T] \times \mathbf{T}^d)$ and $g_i \in L^1([0,T] \times \mathbf{T}^d)$. Then for any smooth function $F : \mathbf{R}^k \to \mathbf{R}$, if $v = F(u) = F(u_1, \cdots, u_k)$,

$$\frac{\partial v}{\partial t} = \sum_{i=1}^k \nabla \cdot \frac{\partial F}{\partial x_i}(u)\, \sigma f_i - \sum_{i,j=1}^k \frac{\partial^2 F}{\partial x_i \partial x_j}(u) f_i \cdot \sigma \nabla u_j + \sum_{i=1}^k \frac{\partial F}{\partial x_i}(u) g_i$$

again in the sense of (3.5). We can assume without loss of generality that for every i, f_i is in the range of σ for almost all (t, x).

Proof. Let us take the case $k = 1$. We convolute u by the heat kernel G_ϵ at time ϵ^2 to get u_ϵ and let $v_\epsilon = F(u_\epsilon)$. It is not hard to see that

$$\begin{aligned} \frac{\partial v_\epsilon}{\partial t} &= F'(u_\epsilon)(\frac{\partial u}{\partial t} * G_\epsilon) = F'(u_\epsilon)[\nabla(\sigma f)_\epsilon + g_\epsilon] \\ &= \nabla F'(u_\epsilon)(\sigma f)_\epsilon + F'(u_\epsilon) g_\epsilon - F''(u_\epsilon) \nabla u_\epsilon \cdot (\sigma f)_\epsilon. \end{aligned}$$

We use a test function ϕ that depends only on x, and write

$$\frac{d}{dt} \int dx\, v_\epsilon \phi = - \int dx\, F'(u_\epsilon) \nabla \phi \cdot (\sigma f)_\epsilon + \int dx\, F'(u_\epsilon) \phi g_\epsilon - \int dx\, \phi F''(u_\epsilon) \nabla u_\epsilon \cdot (\sigma f)_\epsilon$$

We want to let $\epsilon \to 0$ and conclude that

$$\frac{d}{dt} \int dx\, v \phi = - \int dx\, F'(u) \nabla \phi \cdot (\sigma f) + \int dx\, F'(u) \phi g - \int dx\, \phi F''(u) \nabla u \cdot (\sigma f)$$

which establishes the lemma. The only tricky part is the convergence of $\nabla u_\epsilon \cdot (\sigma f)_\epsilon$ to $\nabla u \cdot \sigma f$ in $L^1([0,T] \times \mathbf{T}^d)$. To this end we write

$$\nabla u_\epsilon \cdot (\sigma f)_\epsilon = \sigma_\epsilon \nabla u_\epsilon \cdot (\sigma_\epsilon)^{-1} (\sigma f)_\epsilon$$

and Lemma 2.4 assures us that $\sigma_\epsilon \nabla u_\epsilon \to \sigma \nabla u$ in $L^2([0,T] \times \mathbf{T}^d)$. It suffices to check that $(\sigma_\epsilon)^{-1}(\sigma f)_\epsilon \to f$ in $L^2([0,T] \times \mathbf{T}^d)$. By the convexity of the map $\xi, A \to \langle \xi, A^{-1} \xi \rangle$ it follows that

$$\|(\sigma_\epsilon)^{-1}(\sigma f)_\epsilon\|_{L^2} \leq \|f\|_{L^2}$$

and with a little computation one can show the convergence of $(\sigma_\epsilon)^{-1}(\sigma f)_\epsilon$ to f in $L^2([0,T] \times \mathbf{T}^d)$.

Remark. In principle, if we assume that g_i are in L^2, there is no apriori reason why we need u_i to be in L^∞ for the lemma. The final formula makes sense if u_i are just in $H_{1,a}$. But we find that we have to strengthen the definition of (3.4) with a larger class of test functions in order to drop the L^∞ assumption. For bounded functions in $H_{1,a}$ the validity for the larger class follows already from the validity for smooth test functions, whereas for a general function in $H_{1,a}$ one has to assume it. The culprit is Lemma 2.3 where ϕ has to be L^∞ in order to imply convergence in L^2. In fact if $u_i \notin L^\infty$ the conclusion of the lemma is *false* as shown by the following counterexample: In dimension $d = 1$ take

$$\sigma(t,x) = t^{-\alpha}|x|$$

and

$$u(t,x) = t^{1/4}|x|^{-1/2+\sqrt{t}}.$$

If $\alpha < 1/2$ then σ satisfies (**A1**),

$$u \in H_{1,a}$$

and

$$\frac{\partial u}{\partial t} = \nabla \cdot \sigma f, \qquad f \in L^2([0,T] \times \mathbf{T}^d).$$

However for this example

$$\frac{\partial}{\partial t} u^2 \neq 2u \frac{\partial u}{\partial t}$$

in the sense of (3.5).

The simplest situation is equation (3.1). Here $F = \sigma \nabla u \in L^2$, and if we assume that $c \in L^\infty$, then $G = -\sigma c \cdot \nabla u + g \in L^2$. In this case it will suffice to interpret (3.4) in the sense of (3.5) with C^∞ test functions.

Lemma 3.2. *If $c \in L^\infty$, equation (3.1) has at most one bounded solution.*

Proof. Since we can take the difference of two solutions to get a new solution, it amounts to proving that any bounded solution of

$$\begin{cases} \frac{\partial u}{\partial t} + \nabla \cdot a \nabla u + \sigma c \cdot \nabla u = 0 \;\text{ for }\; 0 \le t \le T, \\ u(T,x) = 0 \end{cases}$$

is identically zero. From lemma 3.1

$$\frac{\partial \phi(u)}{\partial t} = -\nabla \cdot \phi'(u) a \nabla u - \phi'(u) \sigma c \cdot \nabla u = \phi''(u) \nabla u \cdot a \nabla u$$

It is not difficult to see that $\phi(u)$ can be replaced by a more general function $\phi(t,u)$ and this produces an extra term which is $\frac{\partial \phi}{\partial t}(t,u)$. Since u is bounded the choice $\phi(t,u) = e^{kt}u^2$ is allowed

$$\frac{\partial e^{kt}u^2}{\partial t} = -2e^{kt}[\nabla u a \nabla u + 2u\sigma c \nabla u - \nabla u \cdot a \nabla u] + ke^{kt}u^2$$

integrating over space we get

$$\begin{aligned} \frac{d}{dt} e^{kt} \int u^2(t,x)\,dx &= e^{kt} \int [-2u\sigma c \nabla u + 2 < \nabla u \,, a \nabla u > + k u^2]\,dx \\ &\ge 0 \end{aligned}$$

provided $k > \frac{1}{2}C^2$ where C is an upper bound for $\|c\|$. In particular $\int u^2(t,x)\,dx \le \int u^2(T,x)\,dx = 0$ and we are done.

For proving existence we have to approximate the equation by more regular ones.

Lemma 3.3. *Suppose that* (**A1**) *holds. Then there exists a family of positive definite matrices* $\{a_\epsilon\}$ *defined for* $\epsilon > 0$ *with positive definite square roots* $\{\sigma_\epsilon\}$ *such that for some* $c_\epsilon > 0$

$$
\begin{array}{rll}
(i) & c_\epsilon\,|\xi|^2 \le \xi\cdot a_\epsilon(t,x)\xi \le c_\epsilon^{-1}|\xi|^2, & (t,x)\in[0,T]\times\mathbf{T}^d,\ \xi\in\mathbf{R}^d,\\
(ii) & \|(\sigma_\epsilon)_{ij}-\sigma_{ij}\|_{L^2([0,T]\times\mathbf{T}^d)}\to 0, & i,j=1,\ldots,d,\\
(iii) & \|\dot{(\sigma_\epsilon)}_{ij}-\dot\sigma_{ij}\|_{L^2([0,T]\times\mathbf{T}^d)}\to 0 & i,j=1,\ldots,d.
\end{array}
$$

The dot denotes the partial derivative in any spatial direction. We can assume without loss of generality that σ_ϵ *is smooth for each* $\epsilon > 0$.

Proof. For instance

$$\sigma_\epsilon = (I+\epsilon\sigma)^{-1}(\sigma+\epsilon I)$$

clearly works. We can always mollify it somewhat to get smoothness. Another choice is to convolute a with $\mathcal{G}_\epsilon$ to smoothen it and take the positive definite square root. While it is not obvious, it is nevertheless true that $\|\nabla\sigma_\epsilon - \nabla\sigma\|_{L^2}^2 \to 0$. If we need it we can always add a small $\delta(\epsilon)I$ to a or a_ϵ to make it elliptic.

We now prove existence.

Lemma 3.4. *Suppose* (**A1**) *holds and* c, f *and* $g \in L^\infty$. *Then there is a* $u \in H_{1,a}$ *which is a solution to the equation (3.1), in the sense of (3.5) .*

Proof. For each $\epsilon > 0$, it is known that a regular solution u_ϵ of

$$\begin{cases} \frac{\partial u_\epsilon}{\partial t} + \nabla\cdot a_\epsilon\nabla u_\epsilon + \sigma_\epsilon c\cdot\nabla u_\epsilon = g \ \text{ for } \ 0\le t\le T,\\ u_\epsilon(T,x) = f(x)\end{cases} \tag{3.6}$$

exists satisfying

$$\begin{cases} \sup_\epsilon \|u_\epsilon\|_{L^\infty} \le C\\ \sup_\epsilon \int_0^T dt\int dx\,|\sigma_\epsilon\nabla u_\epsilon|^2 \le C\end{cases} \tag{3.7}$$

It is clear from the above estimates that we can choose a subsequence, along which $u_\epsilon \rightharpoonup u$ and $\sigma_\epsilon\nabla u_\epsilon \rightharpoonup v$ weakly in $L^2([0,T]\times\mathbf{T}^d)$. We will show that u is a solution of (3.1). Clearly $u \in L^\infty$. From

$$\frac{\partial u_\epsilon}{\partial t} = \nabla\cdot\sigma_\epsilon F_\epsilon + G_\epsilon \tag{3.8}$$

with $F_\epsilon = -\sigma_\epsilon\nabla u_\epsilon$ and $G_\epsilon = -c\cdot\sigma_\epsilon\nabla u_\epsilon + g$, by taking limits we get

$$\frac{\partial u}{\partial t} = \nabla\cdot\sigma F + G \tag{3.9}$$

with $F = -v$ and $G_\epsilon = -c\cdot v + g$. It remains only to show that $u \in H_{1,a}$ and $\sigma\nabla u = v$. But we have the relation

$$\int_0^T\int_{\mathbf{T}^d} u_\epsilon(t,x)\nabla\cdot\sigma_\epsilon\psi\,dxdt = -\int_0^T\int_{\mathbf{T}^d}\sigma_\epsilon\nabla u_\epsilon\cdot\psi dxdt \tag{3.10}$$

The strong convergence of σ_ϵ and its gradient along with the weak convergence in L^2 of u_ϵ allows us to pass to the limit to conclude that

$$\int_0^T \int_{\mathbf{T}^d} u(t,x)\nabla\cdot\sigma\psi\, dxdt = -\int_0^T \int_{\mathbf{T}^d} v\cdot\psi\, dxdt \tag{3.11}$$

holds for all smooth test functions. This proves that $u \in H_{1,a}$ and $\sigma\nabla u = v$. The boundary value is preserved, at least in the sense of a weak limit as $t \to T$ and we are done.

The next step is to establish the energy identity and some other estimates on the solution. The same argument that we used to prove uniqueness in lemma 3.2 yields the following identities for each $\epsilon > 0$ as well as for the limit.

$$\begin{aligned}
&\int_{\mathbf{T}^d} e^{kT}u_\epsilon^2(T,x)dx - \int_{\mathbf{T}^d} e^{kt}u_\epsilon^2(t,x)dx \\
&\quad= \int_t^T \int_{\mathbf{T}^d} e^{kt}\left[2\nabla u_\epsilon\cdot a_\epsilon\nabla u_\epsilon - 2u_\epsilon c.\sigma_\epsilon\nabla u_\epsilon + 2u_\epsilon g + ku_\epsilon^2\right]\, dxds \\
&\quad= 2\int_t^T \int_{\mathbf{T}^d} e^{kt}\|\sigma_\epsilon\nabla u_\epsilon - \frac{1}{2}cu_\epsilon\|^2\, dxds + \int_t^T \int_{\mathbf{T}^d} e^{kt}[k - \frac{1}{2}\|c\|^2]u_\epsilon^2\, dxds \\
&\qquad\quad + 2\int_t^T \int_{\mathbf{T}^d} u_\epsilon g\, dxds
\end{aligned}$$

We can rewrite this as

$$\begin{aligned}
\int_{\mathbf{T}^d} e^{kt}u_\epsilon^2(t,x)dx + 2\int_t^T \int_{\mathbf{T}^d} e^{ks}\|\sigma_\epsilon\nabla u_\epsilon - \frac{1}{2}cu_\epsilon\|^2\, dxds + \int_t^T \int_{\mathbf{T}^d} e^{ks}[k - \frac{1}{2}\|c\|^2]u_\epsilon^2\, dxds \\
= \int_{\mathbf{T}^d} e^{kT}f^2(x)dx + 2\int_t^T \int_{\mathbf{T}^d} e^{ks}u_\epsilon g\, dxds
\end{aligned} \tag{3.12}$$

and a similar equation with $\epsilon = 0$ in the limit. Since the left hand side is lower semicontinuous under weak convergence and the right hand side is continuous and equlaity holds in the limit, we conclude that

$$\lim_{\epsilon\to 0} \|u_\epsilon - u\|_{L^2([0,T]\times\mathbf{T}^d)} = 0$$

and

$$\lim_{\epsilon\to 0} \|\sigma_\epsilon\nabla u_\epsilon - \sigma\nabla u\|_{L^2([0,T]\times\mathbf{T}^d)} = 0$$

The mappings $t \to u_\epsilon(t,\cdot)$ are uniformly weakly continuous as a maps into $L^2(\mathbf{T}^d)$ and the above arguments actually tell us that for each $t \in [0,T]$,

$$\lim_{\epsilon\to 0} \|u_\epsilon(t,\cdot) - u(t,\cdot)\|_{L^2(\mathbf{T}^d)} = 0$$

Moreover, we can read off from equation (3.12), the equicontinuity of $||u_\epsilon(t,\cdot)\|^2$ as a family of function of t. We conclude that

$$\lim_{\epsilon\to 0} \sup_{0\le t\le T} \|u_\epsilon(t,\cdot) - u(t,\cdot)\|_{L^2(\mathbf{T}^d)} = 0$$

We combine these observations in the form of

Theorem 3.5. *Under the assumptions* ($\mathbf{A}1$), *if* $c, f, g \in L^\infty$, *there equation (3.1) exists a unique* $u \in H_{1,a} \cap L^\infty$ *that solves (3.1) in the sense of (3.5). Moreover the solution admits the following energy identity and estimate.*

$$\int_{\mathbf{T}^d} f^2(x)dx - \int_{\mathbf{T}^d} u^2(t,x)dx = \int_t^T \int_{\mathbf{T}^d} [2\nabla u \cdot a\nabla u - 2uc\cdot\sigma\nabla u + 2ug]\, dxds \tag{3.13}$$

$$\sup_{0\le t\le T} \|u(t,\cdot)\|^2_{L^2(\mathbf{T}^d)} + \int_0^T \int_{\mathbf{T}^d} |\sigma\nabla u|^2\, dxdt \le C_T \left[\|f\|^2_{L^2(\mathbf{T}^d)} + \|g\|^2_{L^2([0,T]\times\mathbf{T}^d)}\right] \tag{3.14}$$

where the constant C_T *depends on* T *and the* L^∞ *norm of* c.

Proof. Except for the estimate (3.14) everything else has already been established. From the identity (3.13), using Schwarz's inequality, we get

$$\begin{aligned}\sup_{0\le t\le T} \|u(t,\cdot)\|^2_{L^2(\mathbf{T}^d)} &+ \int_0^T \int_{\mathbf{T}^d} |\sigma\nabla u|^2\, dxdt \\ &\le \|f\|^2_{L^2(\mathbf{T}^d)} + C_T\left[\|f\|^2_{L^2(\mathbf{T}^d)} + \|g\|^2_{L^2([0,T]\times\mathbf{T}^d)}\right] + \frac{1}{2}\sup_{0\le t\le T}\|u(t,\cdot)\|^2_{L^2(\mathbf{T}^d)}\end{aligned}$$

and we are done.

We now turn to the study of equation (3.2). If we assume that everything in sight is smooth and the problem is elliptic then (3.2) clearly has a smooth solution. If we multiply by u and integrate we get the energy identity

$$\begin{aligned}\int_{\mathbf{T}^d} u^2(t,x)dx &+ \int_t^T \int_{\mathbf{T}^d} |\sigma\nabla u|^2\, dxds \\ &= \int_{\mathbf{T}^d} f^2 dx + \int_t^T \int_{\mathbf{T}^d} (uc_1 - uc_2 + h)\sigma\nabla u\, dxds - \int_t^T \int_{\mathbf{T}^d} g\, u\, dxds\end{aligned} \tag{3.15}$$

It is but one step from this to get an apriori estimate

$$\begin{aligned}\sup_{0\le t\le T} \|u(t,\cdot)\|^2_{L^2(\mathbf{T}^d)} &+ \int_0^T \int_{\mathbf{T}^d} |\sigma\nabla u|^2\, dxdt \\ &\le C_T\left[\|f\|^2_{L^2(\mathbf{T}^d)} + \|g\|^2_{L^2([0,T]\times\mathbf{T}^d)} + \|h\|^2_{L^2([0,T]\times\mathbf{T}^d)}\right]\end{aligned} \tag{3.16}$$

where the constant depends on T and L^∞ bounds for c_1, c_2

In an exactly similar fashion as before we can pass to the limit via solutions of a mollified equation involving smooth $a_\epsilon, \sigma_\epsilon, c_{1,\epsilon}, c_{2,\epsilon}, g_\epsilon, h_\epsilon$ and f_ϵ. When the dust settles what we have is

Theorem 3.6. *For any* c_1, c_2 *in* L^∞ *and* f, g, h *in* L^2 *there exists a function* $u \in H_{1,a}$ *satisfying the bounds (3.16), that is a solution of (3.2) in the sense of (3.5).*

The difficult question is that of uniqueness. The sticky point is lemma 3.1 which requires that the functions u_i be in L^∞ in order to prove the energy identity and we know by counterexamples that this is necessary. We can save the situation by demanding a little bit more from our potential solutions. If we have a function $u \in H_{1,a}$ with $\frac{du}{dt} \in H_{-1,a}$, what we have is a function $u \in H_{1,a}$ that satisfies (3.4) in the sense of (3.5). We could try smooth ϕ or test against $\phi \in H_{1,a}$ that satisfy

$$\frac{\partial \phi}{\partial t} = \nabla \cdot \sigma \xi + \eta$$

verifying

$$\frac{d}{dt} \int_{\mathbf{T}^d} u(t,x)\phi(t,x)dx = \int_{\mathbf{T}^d} \left[-F \cdot \sigma \nabla \phi + G\phi - \xi \cdot \sigma \nabla u + u\eta \right] dx$$

While it is sufficient to use smooth test functions for $u \in L^\infty$, for $u \in L^2$ we need a larger class. We can get away with using test functions $\phi \in L^\infty$. So let us adopt the convention that we will say that (3.4) holds in the *distributional* sense if we test with smooth functions and in the *weak* sense if we enlarge the class of test functions to include L^∞ functions. The distinction is real as can be seen from our counterexample: Dimension $d = 1$,

$$\sigma(t,x) = t^{-\alpha}|x|$$

with $0 < \alpha < 1/2$ and

$$u(t,x) = t^{1/4}|x|^{-1/2+\sqrt{t}}.$$

Then σ satisfies (**A1**) and $u \in H_{1,a}$ is a distributional solution but *not* a weak solution of

$$\frac{\partial u}{\partial t} - \nabla \cdot a \nabla u = \nabla \cdot \sigma h$$

with $h \in L^2$ and initial condition

$$u(0,x) = 0.$$

We will now prove existence and uniqueness of weak solutions, so for this example there is nonuniqueness of distributional solutions.

Just for the record,

Lemma 3.7. *If $u \in L^\infty$ and (3.4) holds in the distributional sense, then it holds in the weak sense.*

Proof. This is basically the same as lemma 3.1. Start with $u, v \in L^\infty$ satisfying

$$\frac{\partial u}{\partial t} = \nabla \cdot \sigma f + g, \qquad \frac{\partial v}{\partial t} = \nabla \cdot \sigma \xi + \eta$$

mollify both by $\mathcal{G}_\epsilon$, compute

$$\frac{d}{dt} \int_{\mathbf{T}^d} u_\epsilon(t,x) v_\epsilon(t,x)\, dx$$

integrate by parts the obvious terms and pass to the limit as $\epsilon \to 0$.

We now strengthen Theorem 3.6 to get the slightly stronger version of existence, and then prove uniqueness.

Theorem 3.8. *Let c_1, $c_2 \in L^\infty$ and f, g, $h \in L^2$. Then the equation (3.2) admits a solution in the weak sense. The weak solution is unique and satisfies the energy identity and estimate.*

Proof. We do not have to work any harder to construct a solution in the *weak* sense. We use the special chioce

$$a_\epsilon = a * \mathcal{G}_\epsilon$$

to solve our approximate problems. If we mollify our bounded test function v by $\mathcal{G}_\epsilon$ as well, by lemma 3.1 we get

$$\frac{\partial v_\epsilon}{\partial t} = \nabla \sigma_\epsilon \, \xi_\epsilon + \eta_\epsilon$$

and we calculate

$$\frac{d}{dt} \int_{\mathbf{T}^d} u_\epsilon(t,x) v_\epsilon(t,x) dx = \int_{\mathbf{T}^d} \left[-F_\epsilon \cdot \sigma_\epsilon \nabla v_\epsilon + G_\epsilon v_\epsilon - \xi_\epsilon \cdot \sigma_\epsilon \nabla u_\epsilon + \eta_\epsilon u_\epsilon \right] dx \tag{3.18}$$

where

$$F_\epsilon = -\sigma_\epsilon \nabla u_\epsilon - c_{2,\epsilon} u_\epsilon + h_\epsilon, \quad G_\epsilon = -c_{1,\epsilon} \cdot \sigma_\epsilon \nabla u_\epsilon + g_\epsilon$$

$$\xi_\epsilon = \xi * \mathcal{G}_\epsilon, \quad \text{and} \quad \eta_\epsilon = \eta * \mathcal{G}_\epsilon$$

As $\epsilon \to 0$, $F_\epsilon, G_\epsilon, \sigma_\epsilon \nabla u_\epsilon$ and u_ϵ converge weakly to $F = \sigma \nabla u - c_2 u + h$, $G = c_1 \sigma \nabla u + g$, $\sigma \nabla u$, and u respectively. On the other hand $\sigma_\epsilon \nabla v_\epsilon$, v_ϵ, ξ_ϵ and η_ϵ converge strongly to $\sigma \nabla v$, v, ξ and η respectively. We can therefore take the limit as $\epsilon \to 0$ in (3.18) and we are done.

We now turn to the proof of uniqueness as well as the energy identity. Let us look at the special case of (3.2) with $c_1 = c_2 = 0$.

$$\frac{\partial u}{\partial t} + \nabla \cdot a \nabla u = g + \nabla \dot{\sigma} h \qquad u(T,x) = f(x) \tag{3.19}$$

with f, g, h in $L^2(\mathbf{T}^d)$. We can now use the solution of

$$\frac{\partial v}{\partial s} = \nabla \cdot a \nabla v \quad \text{in} \quad [t,T] \times \mathbf{T}^d, \qquad v(t,x) = \phi(x)$$

in $L^\infty \cap H_{1.a}$, as a test functions to conclude that the difference w of two solutions of (3.19) in the *weak* sense, satisfies

$$\int_{\mathbf{T}^d} \phi(x) w(t,x) = \int_t^T \int_{\mathbf{T}^d} [-w \nabla \cdot a \nabla v + v \nabla \cdot a \nabla w] dx ds = 0$$

proving the uniqueness of such solutions in the special case. If we approximate f, g in L^2 by smooth functions f_n, g_n respectively and take $h = 0$ we can solve (3.19) with $f_n, g_n, 0$ replacing f, g, h. From the estimate (3.16) it follows that the solutions u_n have a limit in a very strong sense and the limit will be a solution of (3.19) in the *weak* sense. Because of uniqueness, the limit has to equal u. In particular u as the limit of u_n will satisfy the energy identity

$$\int_{\mathbf{T}^d} u^2(t_2,x)dx - \int_{\mathbf{T}^d} u^2(t_1,x)dx = 2\int_{t_1}^{t_2}\int_{\mathbf{T}^d}[|\sigma\nabla u|^2 + ug]dxdt$$

We can replace g by $g + \nabla\sigma h$ so long as h is smooth. Again we can approximate $h \in L^2$ by smooth h, and we conclude that any solution in the *weak* sense of (3.19) will have to satisfy

$$\int_{\mathbf{T}^d} u^2(t_2,x)dx - \int_{\mathbf{T}^d} u^2(t_1,x)dx = 2\int_{t_1}^{t_2}\int_{\mathbf{T}^d}[|\sigma\nabla u|^2 + ug - h\nabla u]dxdt \qquad (3.21)$$

Now if we take any solution of (3.2) in the *weak* sense, we can think of it as a solution in the weak sense of

$$\frac{\partial u}{\partial t} + \nabla\cdot a\nabla u = \tilde{g} + \nabla\cdot\sigma\tilde{h} \qquad u(T,x) = f(x) \qquad (3.20)$$

with

$$\tilde{g} = g - c_1\cdot\sigma\nabla u \qquad \text{and} \qquad \tilde{h} = h - \nabla\cdot\sigma c_2 u$$

and (3.21) will hold with $\tilde{g}$ and $\tilde{h}$ replacing g and h. This is nothing else but the energy identity for (3.2). Once we have the energy identity for all solutions of (3.2) in the *weak* sense, uniqueness is immediate.

Remark. One of the consequences of uniqueness is that if $u \in H_{1,a}$, and

$$\frac{\partial u}{\partial t} = \nabla\cdot\sigma F + G$$

in the *weak* sense, with $F, G \in L^2$ then we can rewrite

$$\frac{\partial u}{\partial t} + \nabla\cdot a\nabla u = \nabla\cdot\sigma[F + \sigma\nabla u] + G$$

and as we saw in the earlier argument u is a limit of bounded solutions. In other words there are bounded functions u_n such that

$$\frac{\partial u_n}{\partial t} = \nabla\cdot\sigma F_n + G_n$$

with u_n, F_n and G_n converging in L^2 respectively to u, F and G. In particular lemma 3.1 is now valid and we can use test functions ϕ that satisfy

$$\frac{\partial\phi}{\partial t} = \nabla\cdot\sigma\xi + \eta$$

in the *weak* sense.

Remark. At this point one may have the impression that to get the *weak* solution one has to be careful in the choice of a_ϵ used to approximate the problem. In fact it is not necessary. If we use arbitrary a_ϵ satisfying lemma 3.3, we can solve at the same time equations of the form

$$\frac{\partial v_\epsilon}{\partial t} = \nabla \cdot a_\epsilon \nabla v_\epsilon \qquad v_\epsilon(0,x) = f(x)$$

and $v_\epsilon \to v$ very nicely, as $\epsilon \to 0$ according to Theorem 3.3. where v solves

$$\frac{\partial v}{\partial t} = \nabla \cdot a \nabla v \qquad v(0,x) = f(x)$$

and it is easy to check that such v's can be used as test functions. But that is enough to establish uniqueness.

If we want to go beyond bounded c in equation (3.1), it is not very natural to replace the condition $c \in L^\infty$ by $c \in L^2$. If we take the simplest case of $L_t = L = \Delta + b(x).\nabla$, and impose the condition that $b(x) \in L^p$, there is an easy counterexample of a smooth solution to $u_t - Lu \equiv 0$ provided the dimension $d > p$. As for relaxing the condition $\nabla\sigma \in L^2$, even in $d = 1$ the operator $L = D_x|x|^\alpha D_x$ has problems at 0 as soon as $\alpha < 1$. It is easy to see that if we approximate by $D_x a(x) D_x$ with smooth $a(\cdot)$ from below the Greens functions $G(x,y) = 0$ for $x < 0, y > 0$ or $x > 0, y < 0$ for all the approximations. On the other hand if we approximate by $D_x(\epsilon + |x|^\alpha)D_x$, $G(x,y) > 0$ for all x, y even in the limit.

We now turn to the study of equation (3.3) which as we said earlier is important in probability theory. In the context that motivated this study, where the operator L_t comes from a specific solution of a non-linear problem we are given a particular solution of equation (3.3). In dealing with simple exclusion models the densities are all bounded by 1 and after renormalization they remain bounded. We are concerned with related existence and uniqueness results that are of interest in probability theory and are implied by the existence of a particular solution. We will deal exclusively with bounded densities and therefore lemma 3.1 applies. We will use relative entropy as our main tool for proving uniqueness. Let us recall the definition. If u_1 and u_2 are nonnegative integrable functions with $\int u_1 dx = \int u_2 dx = 1$,

$$h(u_1/u_2) = \int dx\, u_1 \log(u_1/u_2).$$

Lemma 3.9. *Let* (**A1**) *hold. Suppose that u_i, $i = 1,2$ are bounded, non-negative, solutions of the forward equations*

$$\frac{\partial u_i}{\partial t} = L_t^* u_i = \nabla \cdot a \nabla u_i - \nabla \cdot \sigma c_i u_i, \qquad u_i(0) = u_i^0,$$

satisfying

$$\begin{cases} \int_0^T dt \int dx \frac{\nabla u_i \cdot a \nabla u_i}{u_i} < \infty, \ i = 1,2, \\ \int_0^T dt \int dx\, |c_i|^2 u_i < \infty, \ i = 1,2, \\ \int_0^T dt \int dx\, |c_2 - c_1|^2 u_1 < \infty, \\ h(u_1^0/u_2^0) < \infty. \end{cases}$$

Then the relative entropy at time $T > 0$ is finite and

$$\begin{aligned}
&h(u_1(T)/u_2(T)) - h(u_1^0/u_2^0) \\
&\leq -\frac{1}{2}\int_0^T dt \int dx \left(\frac{\nabla u_1}{u_1} - \frac{\nabla u_2}{u_2}\right) \cdot a \left(\frac{\nabla u_1}{u_1} - \frac{\nabla u_2}{u_2}\right) u_1 \\
&+ \frac{1}{2}\int_0^T dt \int dx \, (c_2 - c_1)^2 u_1.
\end{aligned}$$

Proof. We note first that our equations can be cast in the form

$$\frac{\partial u_i}{\partial t} = \nabla \cdot \sigma F_i.$$

with F_i in $L^2([0,T] \times \mathbf{T}^d)$ We wish to apply Lemma 3.1 for the function $H(u_1, u_2) = u_1 \log \frac{u_1}{u_2}$. Due the singularity near 0 we have to do it with $H_\delta(u_1, u_2) = u_1 \log \frac{u_1+\delta}{u_2+\delta}$ and then let $\delta \to 0$. This can be done and after an integration by parts one obtains

$$\begin{aligned}
h(u_1(T)/u_2(T)) - h(u_1^0/u_2^0) = &- \int_0^T dt \int dx \left(\frac{\nabla u_1}{u_1} - \frac{\nabla u_2}{u_2}\right) \cdot a \left(\frac{\nabla u_1}{u_1} - \frac{\nabla u_2}{u_2}\right) u_1 \\
&- \int_0^T dt \int dx \left(\frac{\nabla u_1}{u_1} - \frac{\nabla u_2}{u_2}\right) \cdot \sigma(c_2 - c_1) u_1.
\end{aligned} \tag{3.21}$$

The Lemma follows by Schwarz's inequality.

A consequence of the lemma is

Corollary 3.10. *If condition* (**A**1) *holds, a nonnegative bounded solution of*

$$\frac{\partial u}{\partial t} = \nabla \cdot a \nabla u - \nabla \cdot \sigma c u$$

in the class satisfying

$$\begin{cases} \int_0^T \int_{\mathbf{T}^d} \frac{|\sigma \nabla u|^2}{u} dx dt < \infty \\ \int_0^T \int_{\mathbf{T}^d} |c|^2 u dx dt < \infty \end{cases}$$

is unique.

Before we prove the next theorem on the existence and uniqueness of solutions for (3.3) in certain situations we prove the following lemma that we need.

Lemma 3.11. *Suppose that* (**A1**) *holds and $c \in L^2\left([0,T] \times \mathbf{T}^d\right)$. Then for each bounded non-negative u^0 there exists a weak solution of*

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla \cdot a \nabla u + c \cdot \sigma \nabla u, \quad \text{in} \quad [0,T] \times \mathbf{T}^d \\ u(0) = u^0, \end{cases}$$

satisfying $0 \leq u \leq \|u^0\|_\infty$ and

$$\int_0^T dt \int dx \, \frac{\nabla u \cdot a \nabla u}{u} \leq C \tag{3.22}$$

with $C < \infty$ depending only on $\|u^0\|_\infty$ and $\|c\|_{L^2([0,T]\times\mathbf{T}^d)}$.

Proof. Let c^n be a truncation of c at level n. For each integer n by reversing time in theorem 3.8 we obtain a unique weak solution $u_n \geq 0$, of

$$\frac{\partial u_n}{\partial t} = \nabla \cdot a \nabla u_n + \sigma c^n \cdot \nabla u_n, \qquad u_n(0) = u^0 \tag{3.23}$$

satisfying $0 \leq u_n \leq \|u^0\|_\infty$. If we take $\phi(u) = u \log u$ in lemma (3.1) and integrate by parts (we can replace u by $u+\delta$ and let $\delta \to 0$ at the end), we get

$$\int_0^T dt \int dx \, \frac{|\sigma \nabla u_n|^2}{u_n} \leq C,$$

and $\|u_n\|_{H_{1,a}} \leq C$ where $C < \infty$ depends only on $\|u^0\|_\infty$ and $\|c\|_{L^2([0,T]\times\mathbf{T}^d)}$. From the definition, for any $\phi \in C_0^\infty\left([0,T) \times \mathbf{T}^d\right)$,

$$\begin{aligned} 0 = &- \int_0^T dt \int dx \, u_n \frac{\partial \phi}{\partial t} - \int dx \, f \phi(0,x) \\ &+ \int_0^T dt \int dx \, \nabla u_n \cdot a \nabla \phi - \int_0^T dt \int dx \, c^n \cdot \sigma \nabla u_n \, \phi. \end{aligned} \tag{3.24}$$

We can extract subsequnces $u_n \rightharpoonup u$ and $\sigma \nabla u_n \rightharpoonup v$ weakly in $L^2([0,T] \times \mathbf{T}^d)$. We can identify $v = \sigma \nabla u$ as in the proof of Lemma 3.4. By lower semicontinuity we obtain (3.22). Since $c^n \to c$ strongly in L^2 we can take the limit as $n \to \infty$ in (3.24) to verify that u is the desired weak solution.

Let us remark once again that there is in general no uniqueness in this context. The final theorem in this section is

Theorem 3.12. *Suppose that* (**A1**) *holds and in addition that there exists a bounded, non-negative weak solution ρ of the equation $\partial_t \rho = L_t^* \rho = \nabla \cdot a \nabla \rho - \nabla \cdot \sigma c \rho$, $\rho(0,x) = \rho^0(x)$, satisfying*

$$\begin{cases} \int_0^T dt \int dx \, \frac{\nabla \rho \cdot a \nabla \rho}{\rho} < \infty, \\ \int_0^T dt \int dx \, |c|^2 \rho < \infty. \end{cases}$$

Then, for any $0 \leq R^0 \leq C \rho^0$ for some $C < \infty$, there exists a unique weak solution R of the equation $\partial_t R = L_t^* R$, $R(0,x) = R^0(x)$ satisfying $0 \leq R \leq C \rho$ on $[0,T] \times \mathbf{T}^d$ and

$$\int_0^T dt \int dx \, \frac{\nabla R \cdot a \nabla R}{R} < \infty.$$

Proof. First of all note that the uniqueness part of the lemma follows immediately from Corollary 3.10. We now prove the existence part.

Step 1. Suppose that $\rho \geq \delta$ for some $\delta > 0$, and

$$\int_0^T dt \int dx \, |c|^2 < \infty.$$

Let

$$g = 2\sigma\frac{\nabla\rho}{\rho} - c,$$

Since $g \in L^2\left([0,T]\times\mathbf{T}^d\right)$, Theorem 3.5 gives us a bounded non-negative weak solution u of the equation

$$\frac{\partial u}{\partial t} = \nabla\cdot a\nabla u + \sigma g\cdot\nabla u, \qquad u(0,x) = R^0/\rho^0.$$

satisfying

$$\int_0^T dt\int dx\,\frac{\nabla u\cdot a\nabla u}{u} < \infty.$$

Now,

$$\begin{aligned}&\int_0^T dt\int dx\left(\frac{\partial R}{\partial t}\phi + \nabla R\cdot a\nabla\phi + \sigma c\cdot R\nabla\phi\right)\\ &= \int_0^T dt\int dx\left(\frac{\partial u}{\partial t}\psi + \nabla u\cdot a\nabla\psi - \sigma g\cdot\nabla u\,\psi + \frac{\partial\rho}{\partial t}\xi + \nabla\rho\cdot a\nabla\xi + \sigma c\cdot\rho\nabla\xi\right)\end{aligned} \tag{3.25}$$

where $\psi = \rho\phi$ and $\xi = u\phi$. If ϕ is a smooth test function, then since both ρ and u are elements of $H_{1,a}\cap L^\infty$, so are ψ and ξ. Therefore the right hand side, and consequently the left hand side of (3.25) vanishes for any smooth ϕ. It follows that $R = \rho u$ is a weak solution of the equation

$$\frac{\partial R}{\partial t} = \nabla\cdot a\nabla R - \nabla\cdot\sigma c R, \qquad R(0) = R^0.$$

Furthermore, since u and ρ are bounded,

$$\int_0^T dt\int dx\,\frac{\nabla R\cdot a\nabla R}{R} \le \int_0^T dt\int dx\,\frac{\nabla u\cdot a\nabla u}{u}\rho + \int_0^T dt\int dx\,\frac{\nabla\rho\cdot a\nabla\rho}{\rho}u < \infty.$$

This completes the proof in case $\int dt\int dx\,|c|^2 < \infty$.

Step 2. In the general case let

$$\rho_\delta = \rho + \delta$$

and note that ρ_δ is a weak solution of the forward equation with c replaced by c_δ given by

$$c_\delta = c\rho/\rho_\delta.$$

This ρ_δ satisfies the assumptions of the lemma and furthermore $\rho_\delta \ge \delta$ and

$$\int_0^T dt\int dx\,c_\delta^2 < \infty.$$

Let $R^0 \le C\rho^0$. By *Step 1* there exists a weak solution R_δ of the forward equation

$$\frac{\partial R_\delta}{\partial t} = \nabla\cdot a\nabla R_\delta - \nabla\cdot\sigma c_\delta R_\delta, \qquad R_\delta(0) = R^0.$$

satisfying

$$\int_0^T dt \int dx \, \frac{\nabla R_\delta \cdot a \nabla R_\delta}{R_\delta} < \infty.$$

By (?),

$$\begin{aligned} \int_0^T dt \int dx \frac{\nabla R_\delta \cdot a \nabla R_\delta}{R_\delta} &\le 4 \left[h(R_\delta(T)/\rho_\delta(T)) - h(R^0/\rho_\delta^0) \right] \\ &\quad + 2 \int_0^T dt \int dx \frac{\nabla \rho \cdot a \nabla \rho}{\rho}. \end{aligned} \tag{3.26}$$

The right hand side is bounded independent of δ.

Since R_δ and u_δ are uniformly bounded we can extract a subsequences $R_\delta \rightharpoonup R$ and $u_\delta \rightharpoonup u$ weak*-L^∞. In other words, for any $\phi \in L^1\left([0,T] \times \mathbf{T}^d\right)$, $\int_0^T dt \int dx \, \phi R_\delta \to \int_0^T dt \int dx \, \phi R$ and similarly for u_δ. But then since ρ is bounded $\int_0^T dt \int dx \, \phi \rho_\delta u_\delta \to \int_0^T dt \int dx \, \phi \rho u$. Therefore $R = \rho u$. Also by (3.26), as in the proof of Lemma 3.3 we can find a subsequence $\sigma \nabla R_\delta \rightharpoonup \sigma \nabla R$ in L^2. For any $\phi \in C_0^\infty\left([0,T) \times \mathbf{T}^d\right)$,

$$\begin{aligned} 0 = &- \int_0^T dt \int dx \, R_\delta \frac{\partial \phi}{\partial t} - \int dx \, R^0 \phi(0,x) \\ &+ \int_0^T dt \int dx \, \nabla R_\delta \cdot a \nabla \phi - \int_0^T dt \int dx \, R_\delta c_\delta \cdot \sigma \nabla \phi. \end{aligned} \tag{3.27}$$

Write

$$\int_0^T dt \int dx \, R_\delta c_\delta \cdot \sigma \nabla \phi = \int_0^T dt \int dx \, u_\delta \rho c \cdot \sigma \nabla \phi.$$

Note that $\rho c \cdot \sigma \nabla \phi \in L^1\left([0,T] \times \mathbf{T}^d\right)$. Therefore we can let $\delta \to 0$ to get

$$\int_0^T dt \int dx \, u \rho c \cdot \sigma \nabla \phi = \int_0^T dt \int dx \, R c \cdot \sigma \nabla \phi.$$

We can take the limit of the other terms in (3.27) as in the proof of the previous theorems and find that R is the desired weak solution.

Remarks. We make a few comments regarding some earlier work.

1. The special case of $a \equiv I$ has been studied in detail by Eric Carlen in [C]

2. In the time homogeneous case the energy inequality holds for weak solutions of (3.41) using only smooth test functions by the following lemma which is a special case of a general theorem of Lions and Magenes [LM], and therefore such solutions are unique.

Lemma. *Let $V \subset H \subset V'$ be separable Hilbert spaces with V and V' being in duality relative to the inner product of H. Suppose we have a curve $u(t) : 0 \le t \le T$ satisfying $u \in L^2(0,T;V)$ and $u' \in L^2(0,T;V')$. Then u is almost everywhere equal to a function continuous from $[0,T]$ into H and for $0 \le t_1 \le t_2 \le T$,*

$$\|u(t_2)\|_H^2 - \|u(t_1)\|_H^2 = 2 \int_{t_1}^{t_2} \langle u(t), u'(t) \rangle dt$$

3. There are a few previous results of existence in the time homogeneous case. Let

$$\mathcal{E}(u,v) = \int dx \nabla\phi \cdot a\nabla\psi.$$

If $a \in W^{1,2}$ then L_0 with $b = 0$ maps C^∞ into L^2 so $\mathcal{E}$ comes from an operator and therefore it has a minimal self-adjoint extension, the Freidrichs extension. If only $\sigma \in W^{1,2}$ this is no longer the case, however Albeverio, Hoegh-Krohn and Streit [AHS] proved that if $\sigma \in W^{1,2}$, $\mathcal{E}$ with domain C_0^∞ is closable on L^2. The closure $\bar{\mathcal{E}}$ is then the quadratic form of a self-adjoint operator L_0. In particular, for each $u_0 \in L^2$ there exists a weak solution u of

$$\frac{\partial u}{\partial t} - \nabla \cdot a\nabla u = 0, \quad \text{in } [0,T] \times \mathbf{T}^d, \qquad u(0,x) = u^0(x).$$

The closure is a regular Dirichlet form, so for almost every $x \in \mathbf{T}^d$ there exists a diffusion process P with generator L_0 and $P(x(0) = x) = 1$. Detailed properties of P can be found in [F].

The argument in [AHS] helps explain the condition (**A1**). For $\phi \in C_0^\infty(\mathbf{T}^d;\mathbf{R})$ and $\psi \in C_0^\infty(\mathbf{T}^d;\mathbf{R}^d)$ let

$$\lambda(\phi) = \int dx\,(\sigma\nabla\phi)\cdot\psi.$$

Integrating by parts, and applying Schwarz's inequality,

$$|\lambda| \le (\|\psi\|_\infty + \|\nabla\psi\|_\infty)\int dx\,\left(|\nabla\cdot\sigma|^2 + \|\sigma\|^2\right),$$

so if $\int dx\,\left(|\nabla\cdot\sigma|^2 + \|\sigma\|^2\right) < \infty$, then λ is bounded on L^2. Therefore $(\sigma\nabla)^*$ is densely defined, and consequently $\sigma\nabla$ is closable. Finally if $\sigma\nabla$ is closable then the quadratic form $\mathcal{E}(\phi,\psi) = \int dx\,\sigma\nabla\phi\cdot\sigma\nabla\psi$ is closable.

4. The difference in difficulty between existence and uniqueness is made very apparent by the situation in one dimension. As we saw already, uniqueness fails for the operator $L = D_x|x|^\alpha D_x$ if $\alpha < 1$. On the other hand, $\mathcal{E}$ is closable under much more general conditions. A point x is called *regular* for $a(x)$ if a^{-1} is locally integrable at x and nonregular points are called *singular*. The following theorem is well known: A proof can be found in [F].

Theorem. *In one dimension, the form $\mathcal{E}$ is closable if and only if $a(x)$ vanishes almost everywhere on its singular set.*

5. Let $m \in L^1\left(\mathbf{T}^d\right)$. One can consider the Dirichlet form

$$\mathcal{E}(u,v) = \int m(x)dx\,\nabla\phi\cdot a\nabla\psi.$$

The corresponding operator

$$Lu = \nabla\cdot a\nabla u + \frac{\nabla m}{m}\cdot a\nabla u \tag{3.28}$$

is reversible on $L^2(m(x)dx)$ instead of $L^2(dx)$. Certainly there is no reason to prefer one to the other. However, since our methods are not based on Dirichlet forms, operators of the form (3.28) fit easily into the theory. This is a case in which the conditions of Theorem 3.12 are very easy to check. If m is non-negative, bounded and

$$\int dx \frac{\nabla m \cdot a \nabla m}{m} < \infty.$$

then m itself is an apriori weak solution of the forward equation $\partial_t m - L^* m = 0$ satisfying the conditions of the Theorem. So if $\sigma_{ij} \in W^{1,2},\ i,j = 1,\ldots,d$ then L^* generates a nice semigroup on $L^\infty(m(x)dx)$.

4. Very weak solutions.

A *very weak* solution of the forward equation

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla \cdot a\nabla u - \nabla \cdot \sigma c u, & \text{in } [0,T] \times \mathbf{T}^d \\ u(0) = u^0 \end{cases} \tag{4.1}$$

is a function $u \in L^\infty$ satisfying

$$\begin{aligned} 0 = & -\int_0^T dt \int dx u \frac{\partial \phi}{\partial t} - \int dx u^0 \phi(0,x) \\ & - \int_0^T dt \int dx u \left(\nabla \cdot a \nabla \phi\right) + \int_0^T dt \int dx u \sigma c \cdot \nabla \phi \end{aligned} \tag{4.2}$$

for all $\phi \in C_0^\infty\left([0,T) \times \mathbf{T}^d\right)$. In order for the last term in (4.2) to make sense we will assume that

$$\int_0^T dt \int dx \, |c|^2 u < \infty. \tag{4.3}$$

If $(\mathbf{A1})$ holds then the rest of (4.2) makes sense for any smooth ϕ since

$$\nabla \cdot a\nabla\phi = \left(\nabla \cdot a\right) \nabla\phi + a\nabla \cdot \nabla\phi,$$

and $(\mathbf{A1})$ implies in particular that $\nabla \cdot a$ and a are in $L^1\left([0,T] \times \mathbf{T}^d\right)$.

Such questions come up naturally when we study the question of existence and uniqueness of solutions to the martingale problem corresponding to the operator

$$L_t = \nabla \cdot a\nabla u + \sigma c \cdot \nabla u.$$

It is natural to write the operator in nondivergence form and ask questions of uniqueness for weak solutions of the adjoint equation, having defined the weak solution by using only smooth test functions. In this context it is not natural to assume *a priori* that u has any regularity.

Theorem 4.1. *Suppose that* $(\mathbf{A2})$ *holds. Assume that* $u \in L^\infty\left([0,T] \times \mathbf{T}^d\right)$ *is a very weak solution of the forward equation (4.1), and (4.3) holds. Then (i)* $u \in H_{1,a}$, *and* u *is a weak solution of (4.1) (ii) If, in addition,* u *is non-negative, then we also have*

$$\int_0^T dt \int dx \frac{\langle \nabla u, a\nabla u\rangle}{u} < \infty.$$

Proof. Suppose we have a very weak solution of the forward equation (4.1). Let $\mathcal{G}_\epsilon$ be the fundamental solution of the heat equation at time $t = \epsilon^2$ and $u_\epsilon = u * \mathcal{G}_\epsilon$. Then it follows that

$$\frac{\partial u_\epsilon}{\partial t} = \nabla \cdot \left(a\nabla u\right)_\epsilon - \nabla \cdot \left(\sigma c u\right)_\epsilon \tag{4.4}$$

Multiply equation (4.4) by u_ϵ, integrate over space and time and integrate by parts to obtain,

$$\begin{aligned}&\int dx\, u_\epsilon^2(T) - \int dx\, u_\epsilon^2(0)\\ &= -\int_0^T dt \int dx\, \nabla u_\epsilon \cdot (a\nabla u)_\epsilon + \int_0^T dt \int dx\, (\sigma c u)_\epsilon \cdot \nabla u_\epsilon.\end{aligned}\tag{4.5}$$

Consider the principal term

$$\begin{aligned}\int dx\, \nabla u_\epsilon \cdot (a\nabla u)_\epsilon = &\int dx\, (\nabla u_\epsilon)\cdot a(\nabla u_\epsilon)\\ &+ \int dx\, \nabla u_\epsilon(x) \cdot \int dy\, \left(a(x+y) - a(x)\right) \nabla u(x+y) \mathcal{G}_\epsilon(y).\end{aligned}\tag{4.6}$$

An integration by parts gives

$$\begin{aligned}&\int dx\, \nabla u_\epsilon(x) \int dy\, \left(a(x+y) - a(x)\right) \nabla u(x+y)\mathcal{G}_\epsilon(y)\\ &= \int dx\, \nabla u_\epsilon(x) \int dy\, \nabla\cdot a(x+y)\, u(x+y)\mathcal{G}_\epsilon(y)\\ &+ \int dx\, \nabla u_\epsilon(x) \int dy\, \left(a(x+y) - a(x)\right) u(x+y)\nabla\mathcal{G}_\epsilon(y)\end{aligned}$$

We use Schwarz's inequality to bound the first term on the right hand side by

$$\frac{1}{4}\int dx\, \nabla u_\epsilon \cdot a_\epsilon \nabla u_\epsilon + \|u\|_{L^\infty}^2 \int dx\, (\nabla\cdot a)a^{-1}(\nabla\cdot a).$$

The second term above is bounded by assumption, by some constant $B_1 < \infty$. By Lemma 2.4,

$$\int dx\, \nabla u_\epsilon \cdot a_\epsilon \nabla u_\epsilon = \int dx\, \left|(\sigma\nabla u)_\epsilon\right|^2 + o(1).$$

The second term on the right hand side of (4.6) can be handled similarly. We prefer to do the convolution on $\mathbf{R}^d$ rather than on $\mathbf{T}^d$.

$$\begin{aligned}&\int dx\, \nabla u_\epsilon(x) \int_{\mathbf{T}^d} dy\, \left(a(x+y) - a(x)\right) u(x+y)\nabla\mathcal{G}_\epsilon(y)\\ &\int dx\, \nabla u_\epsilon(x) \int_{\mathbf{R}^d} dy\, \left(a(x+y) - a(x)\right) u(x+y)\nabla\mathcal{G}_\epsilon(y)\\ &= \int dx\, \nabla u_\epsilon(x) \int_{\mathbf{R}^d} dy \int_0^1 y\cdot\nabla a(x+ty)dt\, u(x+y)\nabla\mathcal{G}_\epsilon(y)\end{aligned}$$

We use Schwarz's inequality to dominate this by

$$\begin{aligned}&\frac{1}{4}\int_0^1 dt \int dx \int_{\mathbf{R}^d} dy\, \nabla u_\epsilon(x)\cdot a(x+ty)\nabla u_\epsilon(x)\mathcal{G}_\epsilon(y)\\ &+ \|u\|_{L^\infty}^2 \int_0^1 dt \int dx \int_{\mathbf{R}^d} dy\, y\cdot\nabla a(x+yt)\nabla\mathcal{G}_\epsilon a^{-1}(x+ty) y\cdot\nabla a(x+ty)\nabla\mathcal{G}_\epsilon(y)\mathcal{G}_\epsilon^{-1}(y)\end{aligned}\tag{4.7}$$

By Lemma 2.4,

$$\int_0^1 dt \int dx \int_{\mathbf{R}^d} dy\, \nabla u_\epsilon(x) \cdot a(x+ty) \nabla u_\epsilon(x) \mathcal{G}_\epsilon(y) = \int dx\ \left|(\sigma\nabla u)_\epsilon\right|^2 + o(1).$$

The integrand in the last term in (4.11) is dominated by a dimensional constant times

$$\sum_{i,j,k,\ell} \left(a_{jk,i} a^{-1}_{j\ell} a_{\ell k,i}\right)(x+ty)\left(y_i \mathcal{G}_{\epsilon,k}(y)\right)^2 \mathcal{G}_\epsilon^{-1}(y).$$

For any $f \in L^1$ it is straightforward to show that $\int_0^1 dt \int dx \int dy f(x+ty) \left(y_i \frac{\mathcal{G}_{\epsilon,k}}{\mathcal{G}_\epsilon}(y)\right)^2 \mathcal{G}_\epsilon(y)$ is bounded uniformly in ϵ. Therefore by (**A2**) the last term in (4.7) is bounded by $B_2 < \infty$ independent of ϵ. For the lower order part of (4.5),

$$\begin{aligned} \int dx\ (\sigma c u)_\epsilon \cdot \nabla u_\epsilon &\le \frac{1}{4}\int dx\, \nabla u_\epsilon a_\epsilon \nabla u_\epsilon + \int dx\, |c|^2 u \\ &= \frac{1}{4}\int dx\, |(\sigma\nabla u)_\epsilon|^2 + o(1) + B_3, \end{aligned}$$

where $B_3 < \infty$. From (4.5) we conclude that

$$\int dx\, u_\epsilon^2(T) - \int dx\, u_\epsilon^2(0) = -\frac{1}{4}\int dx\ \left|(\sigma\nabla u)_\epsilon\right|^2 + B_1 + B_2 + B_3 + o(1).$$

Since u is uniformly bounded, we conclude that

$$\int dx\ \left|(\sigma\nabla u)_\epsilon\right|^2 \le C$$

for some C independent of ϵ. Letting $\epsilon \to 0$ and using the lower semicontinuity of the norm, the theorem is proved.

Suppose in addition u is non-negative, we want to show that $\int dt \int dx \frac{\langle \nabla u, a \nabla u\rangle}{u} < \infty$. We have

$$\frac{\partial u}{\partial t} = \nabla \cdot \sigma F$$

with

$$F = \sigma\nabla u - cu \in L^2([0,T] \times \mathbf{T}^d)$$

We can apply Lemma 3.1 to conclude that

$$\begin{aligned} \int dx h(u(T)) - \int dx h(u(0)) &= \int_0^T dt \int dx h'(u(t,x)) \nabla \cdot \sigma F \\ &= -\int_0^T dt \int dx h''(u(t,x)) \langle F, \sigma \nabla u\rangle \\ &= -\int_0^T dt \int dx h''(u(t,x)) [\langle \nabla u, a \nabla u\rangle - \langle cu, \sigma\nabla u\rangle] \end{aligned}$$

We can take $h_\delta(u) = (u+\delta)\log(u+\delta)$ to obtain

$$\int dx h_\delta(u(T)) - \int dx h_\delta(u(0)) = -\int_0^T dt \int dx \frac{\langle \nabla u, a\nabla u\rangle}{u+\delta} + \int_0^T dt \int dx \ \frac{\langle c, \sigma\nabla u\rangle u}{u+\delta}$$

We can estimate by Schwarz's inequlity

$$|\langle c, \sigma\nabla u\rangle u| \leq |c|^2 u^2 + \frac{1}{4}\langle \nabla u, a\nabla u\rangle$$

and therefore

$$\left| \int_0^T dt \int dx \ \frac{c\cdot\sigma\nabla u}{u+\delta} \right| \leq \int_0^T dt \int dx \frac{|c|^2 u^2}{u+\delta} + \frac{1}{4}\int_0^T dt \int dx \frac{\langle \nabla u, a\nabla u\rangle}{u+\delta}$$

This yields the bound

$$\int_0^T dt \int dx \frac{\langle \nabla u, a\nabla u\rangle}{u+\delta} \leq C\left[\int_0^T dt \int dx \frac{|c|^2 u^2}{u+\delta} + \int dx h_\delta(u(0)) - \int dx h_\delta(u(T))\right] \quad (4.8)$$

with a constant C not depending on δ. We can let $\delta \to 0$ in (4.8) to establish

$$\int_0^T dt \int dx \frac{\langle \nabla u, a\nabla u\rangle}{u} < \infty.$$

We can now state the uniqueness theorem for *very weak solutions.*

Theorem 4.2. *Suppose $\rho_i, i = 1,2$ are two bounded, non-negative, very weak solutions satisfying*

$$\begin{cases} \frac{\partial\rho_i}{\partial t} = \nabla\cdot a\nabla\rho_i - \nabla\cdot\sigma c\rho_i \\ \rho_1(0,x) = \rho_2(0,x) \quad \text{on} \quad \mathbf{T}^d \end{cases}$$

with

$$\int_0^T \int dx |c|^2 \rho_i < \infty$$

then $\rho_1 \equiv \rho_2$.

Proof. Since we know now that *very weak solutions* are really *weak solutions* we can use Corollary 3.10.

5. Existence and uniqueness for a class of degenerate diffusion processes.

In this section we will construct diffusion processes corresponding to the operator L_t with diffusion matrix satisfying (**A1**) and obtain uniqueness if (**A2**) holds as well. Now for such wild coefficients, existence of a process for any starting point x is just not in the cards. So we give the following slightly non-standard definition: Let $\Omega = C\left([0,T];\mathbf{T}^d\right)$, $\mathcal{F}$ the Borel σ-field, and $\mathcal{F}_t = \sigma(x(s):\ 0 \le s \le t),\ 0 \le t \le T$. Given a bounded density function ρ^0, a probability measure P on $(\Omega, \mathcal{F})$ is *a solution of the martingale problem for L_t with initial distribution f* if for every $\phi \in C^\infty(\mathbf{T}^d)$,

$$(i)\quad E^P\left[\phi(x_0)\right] = \int dx\,\phi f,$$

$$(ii)\quad \phi(x_t) - \int_0^t [L_u\phi](x_u)du \ \text{ is a martingale with respect to } P \text{ and } \mathcal{F}_t.$$

Lemma 5.1. *Let $a^\alpha : [0,T] \times \mathbf{T}^d \to \mathbf{R}^d \otimes \mathbf{R}^d$ be non-negative definite symmetric matrix valued and $c^\alpha : [0,T] \times \mathbf{T}^d \to \mathbf{R}^d$ be a family depending on a parameter $\alpha \in \mathcal{A}$. Let σ^α be the symmetric non-negative definite square root of a^α. For each $\alpha \in \mathcal{A}$ let P^α be a solution of the martingale problem with initial distribution π^α and generator*

$$L^\alpha = \nabla \cdot a^\alpha \nabla + \sigma^\alpha c^\alpha \cdot \nabla.$$

Let $\pi^{\alpha,t}$ be the marginals of P^α. Suppose that for each $i,j = 1,\dots,d$, $(\sigma^\alpha_{ij})^2$ and $[\nabla \cdot a^\alpha + \sigma^\alpha c^\alpha]_{ij}$ are uniformly integrable with respect to $dt\ \pi^{\alpha,t}(dx)$ on $[0,T] \times \mathbf{T}^d$. Then the family $P^\alpha,\ \alpha \in \mathcal{A}$ is tight.

Proof. To show tightness we will prove that for every $\epsilon > 0$,

$$\lim_{\delta\to 0} \sup_{\alpha\in\mathcal{A}} P^\alpha\Big(\sup_{\substack{0\le s<t\le T\\ |t-s|<\delta}} |x(t) - x(s)| > \epsilon\Big) = 0. \tag{5.1}$$

Under P^α we have

$$x(t) - x(s) = \int_s^t \sigma^\alpha d\beta + \int_s^t \left(\nabla \cdot a^\alpha + \sigma^\alpha c^\alpha\right) du \tag{5.2}$$

where β is a standard d-dimensional Brownian motion and the right hand side is computed on $\mathbf{R}^d$, then projected onto the torus. Let

$$y_{ij}(t) = \int_0^t \sigma^\alpha_{ij} d\beta_j.$$

For each i, j we can write $y_{ij}(t)$ as a random time change of another Brownian motion $\hat\beta(\cdot)$ in $\mathbf{R}^1$,

$$y_{ij}(t) = \hat\beta\left(\int_0^t (\sigma^\alpha_{ij})^2 du\right).$$

Since Brownian motions are almost surely continuous the desired estimate (5.1) for the first term on the right hand side of (5.2) will follow if we can show

$$\lim_{\delta\to 0}\sup_{\alpha\in\mathcal{A}} P^\alpha\left(\sup_{\substack{0\le s<t\le T\\ |t-s|<\delta}}\left|\int_s^t (\sigma_{ij}^\alpha)^2\,du\right|>\epsilon\right)=0. \tag{5.3}$$

If we replace $(\sigma_{ij}^\alpha)^2$ by $(\sigma_{ij}^\alpha)^2\wedge\ell$ for some $\ell>0$ then (5.3) is immediate. But then

$$P^\alpha\left(\sup_{\substack{0\le s<t\le T\\ |t-s|<\delta}}\left|\int_s^t \left[(\sigma_{ij}^\alpha)^2-(\sigma_{ij}^\alpha)^2\wedge\ell\right]du\right|>\epsilon\right)\le\epsilon^{-1}E^{P^\alpha}\left[\int_0^T \left[(\sigma_{ij}^\alpha)^2-(\sigma_{ij}^\alpha)^2\wedge\ell\right]du\right].$$

The right hand side vanishes in the limit of large ℓ, uniformly in α, by the uniform integrability of $(\sigma_{ij}^\alpha)^2$ with respect to $dt\;\pi^{\alpha,t}(dx)$. The second term on the right hand side of (5.2) can be handled in a similar fashion. In fact if we examine the preceding argument we see that the key point is the uniform integrability of each coordinate of $(\nabla\cdot a^\alpha+\sigma^\alpha c^\alpha)$ with respect to $dt\;\pi^{\alpha,t}(dx)$.

Lemma 5.2. *Suppose that* (**A1**) *holds. Then there exists a solution P_0 of the martingale problem for*

$$L_t=\nabla\cdot a\nabla$$

with initial distribution $f\equiv 1$.

Proof. For each $\epsilon>0$ let P_ϵ be the solution of the martingale problem for

$$L_t^\epsilon=\nabla\cdot a_\epsilon\nabla$$

with initial distribution $f\equiv 1$ where a_ϵ is chosen according to Lemma 3.1. Note that all marginals are given by $\rho_\epsilon\equiv 1$ and the conditions of Lemma 5.1 then follow from (**A1**). By Lemma 5.1 there is a weak limit P_0. We have to show that P_0 solves the martingale problem for L_t. This amounts to the statement that for any $\phi\in C^\infty(\mathbf{T}^d)$ and any bounded continuous $F\in\mathcal{F}_s$,

$$E^{P_\epsilon}\left[\left(\phi(x_t)-\phi(x_s)-\int_s^t[L_u^\epsilon]\phi(x_u)du\right)F\right]\to E^{P_0}\left[\left(\phi(x_t)-\phi(x_s)-\int_s^t[L_u\phi](x_u)du\right)F\right].$$

By the Markov property it suffices to take $F(x(\cdot))$ of the form $f(x_s)$. The boundary terms present no problem and therefore one only has to verify that for bounded continuous f

$$E^{P_\epsilon}\left[\int_s^t[(\nabla\cdot a_\epsilon\nabla)\phi](x_u)du\;f(x_s)\right]\to E^{P_0}\left[\int_s^t[\nabla\cdot a\nabla\phi](x_u)du\;f(x_s)\right].$$

which is not difficult.

Theorem 5.3. *Suppose that* (**A1**) *holds and that there exists a bounded non-negative weak solution ρ of the forward equation $\partial_t \rho - L_t^* \rho = 0$, $0 \le t \le T$, satisfying*

$$\begin{cases} \int_0^T dt \int dx \frac{\nabla \rho \cdot a \nabla \rho}{\rho} < \infty, \\ \int_0^T dt \int dx \, |c|^2 \rho < \infty. \end{cases}$$

Then for any $0 \le f(x) \le C\rho(0,x)$ with $\int dx \, f = 1$ there exists a solution P of the martingale problem for L_t with initial density f and having marginal densities $R(t,x)$, the unique weak solution of $\partial_t R - L_t^* R = 0$, $R(0) = f$ satisfying $0 \le R \le C\rho$ and $\int_0^T dt \int dx \, \frac{\nabla R \cdot a \nabla R}{R} < \infty$.

Proof. By Theorem ?, for any $0 \le f(x) \le C\rho(s,x)$ there exists a weak solution $u(t,x) = \left[P_{s,t}^* f\right](x)$ of the forward equation after time s with $\int_s^T dt \int dx \, \frac{\nabla u \cdot a \nabla u}{u} < \infty$ and

$$P_{s,t}^* : L^\infty(\rho(s)) \to L^\infty(\rho(t))$$

is bounded. For any n and any partition $0 \le t_1 < t_2 < \cdots < t_n \le T$, let $P^{t_1,\ldots,t_n}$ be the probability measure on $(\mathbf{T}^d)^n$ given as follows: For any bounded functions, $f_i : \mathbf{T}^d \to \mathbf{R}$, $i = 1, \ldots, n$,

$$\int f_1(x_1) \cdots f_n(x_n) dP^{t_1,\ldots,t_n} = \int dx \, f_n(x) [P_{t_{n-1},t_n}^* f_{n-1} P_{t_{n-2},t_{n-1}}^* \cdots P_{t_1,t_2}^* f_1 P_{0,t_1}^* f](x).$$

Let P be the unique probability measure on $(\mathbf{T}^d)^{[0,T]}$ such that

$$E^P \left[f_1(x_{t_1}) \cdots f_n(x_{t_n})\right] = \int f_1(x_1) \cdots f_n(x_n) dP^{t_1,\ldots,t_n}.$$

Let P_0 be a solution of the martingale problem for $\nabla \cdot a \nabla$ with initial density $f \equiv 1$. Such a solution exists by the previous theorem. We will show that the relative entropy satisfies

$$H(P/P_0) \le \int dx \, f \log f + \frac{C}{4} \int_0^T dt \int dx |c|^2 \rho. \tag{5.4}$$

Then P is a probability measure on $C\left([0,T]; \mathbf{T}^d\right)$ and by construction P solves the martingale problem. To prove (5.4) we note that, because

$$H(P/P_0) = \sup h\left(P^{t_1,\ldots,t_n} / P_0^{t_1,\ldots,t_n}\right).$$

where the supremum runs over all partitions of $[0,T]$ it is sufficient to show that

$$h\left(P^{t_1,\ldots,t_n} / P_0^{t_1,\ldots,t_n}\right) \le \int dx \, f \log f + \frac{C}{4} \int_0^T dt \int dx |c|^2 \rho. \tag{5.5}$$

for all partitions. By Lemma 3.5,

$$h\left(P^{t_1,\ldots,t_n} / P_0^{t_1,\ldots,t_n}\right) \le h\left(P^{t_1,\ldots,t_{n-1}} / P_0^{t_1,\ldots,t_{n-1}}\right) + \frac{1}{4} \int_{t_{n-1}}^{t_n} dt \int dx \, |c|^2 R$$

from which we establish (5.5) by induction.

Remark. If one is interested in the more standard concept of solutions of the martingale problem starting from a point x at time s, then Theorem 5.3 provides us with the result that for each s, there exists a solution starting from x at time s for almost every x with respect to $\rho(s,x)dx$. This is the most that can be expected under our conditions.

Now we consider the problem of uniqueness of our process. While we have an existence and uniqueness theorem for weak solutions under assumptiom (**A1**), this is not of much use in proving uniqueness of solutions to the martingale problem. Even if we start with a solution to the martingale problem with marginals having densities in class $H_{1,a}$ the conditionals are not necessarily in the same regularity class. Assumption (**A2**) will be important in this context.

Suppose we have an operator of the form

$$L_t = \nabla \cdot a\nabla + \sigma c \cdot \nabla$$

with a satisfying (**A2**). We will limit overselves to the class $\mathcal{P}$ of processes P that satisfy the following conditions:

(i) There exist a constant C such that all the one dimensional marginals of P have densities bounded by C.

(ii) If $\rho(t,x)$ is the marginal density of P then,

$$\int_0^T \int dx|c|^2\rho(t,x) = E^P\left[\int_0^T |c(t,x(t))|^2\,dt\right] < \infty.$$

We can now state and prove our main theorem.

Theorem 5.4. *Assume that* (**A2**) *is valid. For any bounded initial density $\rho_0(x)$ the solution to the martingale problem for*

$$L_t = \nabla \cdot a\nabla + \sigma c \cdot \nabla$$

is unique in the class $\mathcal{P}$. If we assume that we have a non-negative solution $\rho(t,x)$ that satisfies

$$\begin{cases} \frac{\partial\rho}{\partial t} = L_t^*\rho = \nabla\cdot a\nabla\rho - \nabla\cdot\sigma c\rho \\ \int_0^T dt \int dx\|c\|^2\rho < \infty \end{cases}$$

then a solution to the martingale problem for L_t exists (uniquely) in the class $\mathcal{P}$ and has the solution $\rho(t,x)$ given above for its marginal densities.

Proof. We need only uniqueness to complete the proof. We will prove by induction the following statement. If P_1 and P_2 are two solutions to the martingale problem in class $\mathcal{P}$ and if $0 = t_1 < t_2 < \cdots < t_n \le T$ are n time points the distribution of P_1 and P_2 agree on the sigma field $\sigma\{x(t_i), i = 1,\ldots,n\}$. Assume that the statement is true for $0 = t_1 < t_2 < \cdots < t_n \le T$ and we want to prove it for $0 = t_1 < t_2 < \cdots < t_n < t_{n+1} \le T$. In other words for any $A \in \sigma\{x(t_i), i = 1,\ldots,n\}$ and Borel set $B \subset \mathbf{T}^d$ we would like to show that

$$P_1[A \cap \{x(t_{n+1}) \in B\}] = P_2[A \cap \{x(t_{n+1}) \in B\}]. \tag{5.6}$$

Clearly we can assume without loss of generality that $P_i(A) > 0$. Define Q_1 and Q_2 by

$$Q_i(E) = \frac{P_i(A \cap E)}{P_i(A)} \quad \text{for} \quad i = 1, 2.$$

Then one can check that on the interval $[t_n, T]$, Q_1 and Q_2 are in class $\mathcal{P}$ and are again solutions to the martingale problem. From the uniqueness of the marginals we conclude that

$$Q_1[x(t_{n+1}) \in B] = Q_2[x(t_{n+1}) \in B]$$

which is equivalent to (5.6). This establishes the induction.

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