RELAXATION TO EQUILIBRIUM OF CONSERVATIVE DYNAMICS I : ZERO RANGE PROCESSES

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Abstract. Under mild assumptions we prove that the decay rate to equilibrium in L^2 of zero range dynamics on the d dimensional integer lattice is $t^{-d/2}$ with logarithmic corrections.

0. Introduction.

In this article we present a method to estimate the decay to equilibrium in the variance sense of conservative interacting particle systems in infinite volume. Although such issues are interesting for a wide variety of models, we will concentrate here on a particular class of models in order to present the method in a simple setting. These are the symmetric zero-range models, and the key simplifying feature is that the invariant measures are of product form.

By analogy with the heat equation in $\mathbb{R}^d$, which appears in the diffusive scaling limit of these models, and with the noninteracting case, one expects an algebraic decay to equilibrium for such models on $\mathbb{Z}^d$ at rate $O(t^{-d/2})$. Indeed, by a careful choice of test functions, one can show that the decay could not in general be faster. Upper bounds, on the other hand, have proved more difficult to obtain. We will derive a bound of the form $O(t^{-d/2}(\log t)^{d+3})$. The logarithmic corrections here are not physical, but a spurious consequence of the method which involves entropy estimates.

One should note the sharp contrast between the algebraic decay for conservative systems and the well known exponential decay displayed by nonconservative systems. In the first case, the slow decay is a consequence of the need to transport mass across large distances in order to equilibrate, while in the latter case distant regions equilibrate more or less independently. This manifests itself in the behaviour near zero of the spectrum of the generator for the process in infinite volume. For the conservative system the spectrum is continuous at zero, while the nonconservative system has a gap at the bottom of the spectrum. It appears also in the different

1991 *Mathematics Subject Classification.* Primary 60K35; secondary 82A05.
Key words and phrases. interacting particle system, spectral gap, relaxation to equilibrium.

decay rates for the process on finite regions. On boxes of linear size ℓ, the decay rate for the conservative system is exponential, but with a rate $O(\ell^{-2})$, either in terms of the spectral gap or the logarithmic Sobolev inequality [Q, DS, LY, LSV, Y]. These estimates have been used heavily in the hydrodynamical limit [Q, VY]. The nonconservative dynamics on the other hand, decays exponentially with a rate independent of the size of the box. In fact the dependence of the exponential decay rate on the size of the box in the conservative system is a key ingredient of the present proof of algebraic decay in infinite volume

In systems with conservation law one studies the density-density correlation functions

$$< \eta_t(x); \eta_0(0) >$$

where $\eta_t(x)$ is the number of particles at x at time t . These can be thought of as representing the response at position x at time t to a small initial disturbance at position 0. Physically the disturbance should diffuse out, so we expect that at least for large t and x the density-density correlation functions decay as

$$\chi(4\pi t \det D)^{-d/2} \exp\left\{-\frac{x \cdot D^{-1} x}{4t}\right\}$$

where χ is the compressibility and D is the bulk diffusion coefficient given by the Green-Kubo formula (see [S]). Such a picture can be made rigorous at various levels. The simplest is linear response theory, or equilibrium fluctuations, which deal with small pertubations of equilibrium, and large space and time scales. More difficult is the hydrodynamics limit, where the space and time scales are still large, but the deviations from equilibrium are no longer small. Finally, in the present paper we consider such models without rescaling and show an algebraic decay, but of course at the loss of precise constants, and the Gaussian factor.

The traditional approach for algebraic decay for heat equations is via Nash estimates [N, FS, Da]. The Nash inequality on $\mathbb{R}^d$ states that

$$\|f\|_{L^2}^2 \le C D(f)^{d/(d+2)} \|f\|_{L^1}^{4/(d+2)}$$

where the Dirichlet form is given by

$$D(f) = \int_{\mathbb{R}^d} |\nabla f|^2 dx.$$

To use this estimate, recall that the energy inequality of the standard heat equation gives for the solution f_t of the heat equation $\partial_t f = \Delta f$,

$$\frac{d}{dt}\|f_t\|_{L^2}^2 \le -C D(f_t)\,.$$

Since the heat kernel is a contraction in L^1, we can bound $\|f_t\|_{L^1}$ by its initial value at time $t = 0$. Hence the Nash estimate gives a lower bound of the Dirichlet form in terms of its L^2 norm. Combining Nash's estimate with the energy estimate we obtain an inequality involving only $\|f_t\|_{L^2}^2$ and the initial value $\|f_0\|_{L^1}$, which can then be integrated to obtain a decay estimate for $\|f_t\|_{L^2}^2$, with the optimal exponent in t. To generalize this idea, it may appear that the key ingredient is an generalization of the Nash inequality. However, the contractivity of the heat kernel in the L^1 norm plays a central role. As it stands, the Nash inequality is unlikely to be true in the infinite dimensional setting since the L^1 (or any L^p) norm on the right hand side is too weak to control the variance. One can generalize the Nash inequality by replacing the L^1 norm by a suitably chosen norm, $\|\!| \cdot \|\!|$: For any mean zero function

$$\|f\|_{L^2}^2 \le CD(f)^{d/(d+2)} \|\!| f \|\!|^{4/(d+2)} .$$

On the other hand, we do not know of any norm other than the standard L^p norm contracting (or uniformly bounded in time) under the zero-range or lattice gas dynamics. In the first appendix we prove such a Nash inequality with a seemingly natural choice of the norm $\|\!| \cdot \|\!|$. Unfortunately we can only prove that this norm is uniformly bounded in time for the symmetric simple exclusion process. In fact, this is the only special case where Nash's idea has been carried through to the infinite dimensional setting [BZ] (with a stronger choice of the norm $\|\!| \cdot \|\!|$). However for this model certain special techniques become available and therefore simple proofs of the decay are available. In the second appendix we give elementary proofs of the $t^{-d/2}$ decay for the symmetric simple exclusion process using coupling. We should emphasize that the simplification in the case of the symmetric simple exclusion process comes mainly from its very special duality property, and not so much from the fact that the invariant measures are Bernoulli. For example, at the present time Nash's ideas cannot be extended to models with speed change even when the invariant measures are Bernoulli.

Next we comment on the sense in which decay to equilibrium is measured in this article. Of the few monotone functionals available, the most natural in which to study the decay are the L^2 and L^∞ norms. In L^p the expected decay rate is $t^{-d(p-1)/2p}$ and for $1 < p < 2$ this can be obtained from the trivial L^1 bound and the L^2 decay by interpolation.

The case of L^∞ is more interesting. The processes under study have a family of extremal invariant measures ν_ρ parametrized by the density. A natural statement of ergodicity is that

$$P_t f(\eta) - E^{\nu_\rho}[f] \to 0 \quad \text{as} \quad t \to \infty \tag{0.1}$$

if the configuration η has a density

$$\rho = \lim_{L\to\infty} (2L+1)^{-d} \sum_{|x|\le L} \eta(x). \tag{0.2}$$

We are not aware of any results in this direction except for the simple exclusion process where a fairly complete picture can be obtained using duality [L2]. If we wanted to go further and understand the rate of convergence in (0.1), we would need to make assumptions about the rate of convergence in (0.2). One way to eliminate the dependence on the rate of convergence of (0.2) is to allow the choice of ρ in (0.1) to depend on t. So we choose $\rho(t)$ carefully and study

$$P_t f(\eta) - E^{\nu_{\rho(t)}}[f] \tag{0.3}$$

Note that $\rho(t)$ should be independent of f for otherwise there is nothing to prove. We can normalize the choice of $\rho(t)$ by requiring equality in (0.3) with $f = \eta_0$. In the case of symmetric simple exclusions one can compute $P_t f$ explicitly if $f = \eta_x$ and the answer is

$$P_t f = \sum_y p_t(y-x)\eta_y$$

where $p_t(y-x)$ is the heat kernel on the lattice. Hence $\rho(t) = \sum_y p_t(y)\eta_y$ and in particular if $f = \eta_x$,

$$P_t f(\eta) - E^{\nu_{\rho(t)}}[f] = \sum_y [p_t(y-x) - p_t(y)]\eta_y. \tag{0.4}$$

We fix $x \neq 0$ and study the behaviour of the right hand side as $t \to \infty$. Even if we require that the convergence of (0.2) is as good as possible, say $(2L+1)^{-d}\sum_{|x|\le L}\eta(x) = \rho$ for all L large enough, we can find configurations η so that the right hand side of (0.4) is as large as $t^{-1/2}$ for all $t \ge 1$ in any dimension. Of course, for typical η the decay will be faster. Under any equilibrium measure ν_ρ, the right hand side of (0.4) is of order $t^{-(d+2)/4}$ in root mean square. From the law of the iterated logarithm one expects it to be of order $t^{-(d+2)/4}(\log\log t)^{1/2}$ almost surely with respect to any ν_ρ.

On the other hand, it is shown in [HR] that for noninteracting random walks, if one starts with one particle at each site, then for any local function there are constants c_1 and c_2 so that

$$c_1 t^{-d/2} \le |P_t f(\eta) - E^{\nu_\rho}[f]| \le c_2 t^{-d/2}.$$

One can see from all this that any L^∞ estimate would have to depend quite subtly on the initial data.

The main body of the paper is concerned with the $t^{-d/2}(\log t)^{d+3}$ upper bound on the variance for a wide class of reversible zero-range models. Our method is very general and applies to lattice gases with mixing assumptions, to be presented in detail in a subsequent paper. It shares with the Nash inequality the use of the spectral gap estimate on finite cubes as a key input. But otherwise the idea is quite different. The main observation of the new approach is the recovery of translational invariance of the forward equation. This is controlled by an entropy argument. Another important technical input is a cutoff estimate which shows that disturbances move at speed less than $O(\sqrt{t}\log t)$. The proof of the main result is contained in sections 2-4. The method does have several drawbacks. One is the logarithmic corrections. Another is that the constant in the decay estimate is not very explicit, though it is clearly finite for any local function.

1. Notation and Results.

We consider zero-range models described as follows. Particles are distributed on the lattice $\mathbb{Z}^d$ with η_x denoting the number of particles at site x. Configurations will be called η and the state space is the set $\mathbb{N}^{\mathbb{Z}^d}$ of such configurations. We also choose jump rates $g\colon \mathbb{N} \to \mathbb{R}_+$ such that $g(0) = 0 < g(k)$ for $k \geq 1$. The dynamics of the process is described as follows. If there are η_x particles at site x then at rate $g(\eta_x)$ one of them jumps to nearest neighbor site y. This takes place independently of all the other particles, and the new configuration $\sigma^{x,x+y}\eta$ obtained from η in this way is given by

$$(\sigma^{x,y}\eta)_z = \begin{cases} \eta_z & \text{if } z \neq x, y, \\ \eta_x - 1 & \text{if } z = x, \\ \eta_y + 1 & \text{if } z = y\ . \end{cases} \tag{1.1}$$

The dynamics we have described is a Markov process on the state space $\mathbb{N}^{\mathbb{Z}^d}$ whose generator acts on functions that depend only on a finite number of coordinates as

$$(\mathcal{L}f)(\eta) \ = \ \sum_{x\sim y} g(\eta_x)[f(\sigma^{x,y}\eta) - f(\eta)]\ , \tag{1.2}$$

where $x \sim y$ denotes nearest neighbors.

To ensure that the process is well defined, we make the following Lipschitz assumption on the jump rates (cf. [A]):

(L) $G_1^\star = \sup_{n\geq 0} |g(n+1) - g(n)| < \infty$.

Denote by $Z\colon \mathbb{R}_+ \to \mathbb{R}_+$ the partition function defined by

$$Z(\varphi) \ = \ \sum_{k\geq 0} \frac{\varphi^k}{g(1)\cdots g(k)}$$

and by φ^* the radius of convergence of Z. In order to avoid some degeneracies we will also assume that the partition function Z diverges as one approaches the boundary of its domain of definition :

$$\lim_{\varphi\to\varphi^*} Z(\varphi) \ = \ \infty\ . \tag{1.3}$$

For $0 \le \varphi < \varphi^*$, let $\bar\nu_\varphi$ denote the product measure on $\mathbb{N}^{\mathbb{Z}^d}$ with marginals given by :

$$\bar\nu_\varphi(\eta_x = j) \;=\; \frac{1}{Z(\varphi)} \frac{\varphi^j}{g(1)\cdots g(j)} \tag{1.4}$$

for $j \in \mathbb{N}$, $x \in \mathbb{Z}^d$. The dynamics we have described conserves the total number of particles and the set of measures $\bar\nu_\varphi$ represent a full set of extremal invariant measures. Let $\rho(\varphi) = E_{\bar\nu_\varphi}[\eta_0]$ be the density of particles for the measure $\bar\nu_\varphi$. From assumption (1.3) it follows that ρ: $[0, \varphi^*) \to [0, \infty)$ is a smooth strictly increasing bijection. Since $\rho(\varphi)$ has a physical meaning as the density of particles, instead of parameterizing the above family of measures by φ, we use the density ρ as parameter and we write

$$\nu_{\rho_0} \;=\; \bar\nu_{\rho^{-1}(\rho_0)}$$

for $\rho \in [0, \infty)$. With this convention,

$$\varphi(\rho) \;=\; E_{\nu_\rho}[g(\eta_0)]\ .$$

Moreover, φ is a smooth function whose derivative is bounded above by $G_1^\star$ and below by a strictly positive constant on each compact set of $\mathbb{R}_+$ (cf. [KL]). Because each nearest neighbor jump is chosen by the particles with equal probability, the process is reversible with respect to each ν_ρ and the corresponding Dirichlet form is given by

$$D(\nu_\rho, f) := -E_{\nu_\rho}\Big[f\mathcal{L}f\Big] = \frac{1}{2}\sum_{x\sim y} E_{\nu_\rho}\Big[g(\eta_x)\Big(f(\sigma^{x,y}\eta) - f(\eta)\Big)^2\Big].$$

We will also consider the process restricted to a box of side length ℓ. Jumps to sites outside the box are simply excluded. The corresponding generator will be denoted by $\mathcal{L}_\ell$. We now make the following additional assumption which guarantee that $\mathcal{L}_\ell$ has a gap of order ℓ^{-2} uniformly in the density [LSV].

(H) There exists $\delta > 0$ and $k_0 \ge 1$ such that $g(m) - g(n) \ge \delta$ for all $m - n \ge k_0$.

Note that from **(H)** and **(L)** it follows that there exists an $\epsilon > 0$ such that for all $k \in \mathbb{N}$,

$$\epsilon k \;\le\; g(k) \;\le\; \epsilon^{-1} k. \tag{1.5}$$

Fix a density $\rho > 0$ and denote by P_t the semigroup associated to the generator $\mathcal{L}$ and by Var (ν, u) the variance, with respect to a probability measure ν, of a function u in $L^2(\nu)$. The main theorem of this article states that under assumptions (L) and (H), the process relaxes to equilibrium in $L^2(\nu_\rho)$ at rate $t^{-d/2}$ with logarithmic corrections.

Theorem 1.1 . *For every bounded cylinder function u, there exists a finite constant $C(u,d,\rho)$ depending only on the dimension d, the density ρ and u such that*

$$\mathrm{Var}\ (\nu_\rho, [P_t u]) \ \le\ C(u,d,\rho)\frac{(\log t)^{d+3}}{t^{d/2}}\ .$$

2. Proof of the main result.

Fix $\rho \in [0,\infty)$ and u a cylinder function which is mean zero with respect to ν_ρ, i.e. $E_{\nu_\rho}[u]=0$. We will also use $<\cdot>$ to denote expectation with respect to ν_ρ. Set $u_t = P_t u$. In other words, u_t is the solution of the backward equation

$$\partial_t u_t \ =\ \mathcal{L} u_t\ , \qquad u_0\ =\ u\ .$$

In particular,

$$\frac{d}{ds} s^{d/2} E_{\nu_\rho}\big[u_s^2\big] \ =\ -2 s^{d/2} D(\nu_\rho, u_s)\ +\ (d/2)\, s^{(d-2)/2} E_{\nu_\rho}\big[u_s^2\big]\ . \tag{2.1}$$

Fix $t>0$ and two positive integers a, b. Let n_0 be such that

$$ab^{n_0} \le t < ab^{n_0+1}$$

For $0\le n\le n_0$, set $t_n = ab^n$ and $t_{n_0+1}=t$. We shall estimate the integral of the right hand side of (2.1) on each interval $[t_n, t_{n+1}]$.

For a positive integer L, denote by Λ_L^0 the cube of length $2L+1$ centered at the origin

$$\Lambda_L^0 \ =\ \{-L,\dots,L\}^d\ .$$

Denote by $\mathcal{F}_{\Lambda_L^0}$ the σ-algebra generated by $\{\eta_x\,, x\in\Lambda_L^0\}$ and by $A_L h$ the conditional expectation of a $L^2(\nu_\rho)$ function h given $\mathcal{F}_{\Lambda_L^0}$:

$$A_L h \ =\ E_{\nu_\rho}\Big[h\,\Big|\,\mathcal{F}_{\Lambda_L^0}\Big]\ . \tag{2.2}$$

Lemma ♣ys2 below shows that we may replace the L^2 norm of u_s on the right hand side of (2.1) by the L^2 norm of $A_L u_s$ provided s is large enough and L is chosen as $\gamma\sqrt{s}\log s$ for sufficiently large γ. In preparation, fix an interval $[t_n, t_{n+1}]$ and let $L_n = \sqrt{t_{n+1}}\log t_{n+1}^{d+2}$. The time integral on the interval $[t_n,t_{n+1}]$ of the second term on the right hand side of (2.1) is equal to

$$\frac{d}{2}\int_{t_n}^{t_{n+1}} s^{(d-2)/2} E_{\nu_\rho}[(u_s - A_{L_n}u_s)^2]\,ds\ +\ \frac{d}{2}\int_{t_n}^{t_{n+1}} s^{(d-2)/2} E_{\nu_\rho}[(A_{L_n}u_s)^2]\,ds\ .$$

The proof of the following cutoff estimate will be presented in Section 4.

Lemma 2.1 . *Let $t_n = ab^n$ and $L_n = \sqrt{t_{n+1}} \log t_{n+1}^{d+2}$ and assume that $ab \ge 4|supp\ u|^2$. There exists a finite constant $C(d,\rho)$ depending only on the density ρ and the dimension d such that*

$$E_{\nu_\rho}[(A_{L_n} u_s - u_s)^2] \ \le\ C(d,\rho) s^{-(d+2)/2} E_{\nu_\rho}[u_s^2]$$

for all $t_n \le s \le t_{n+1}$.

It follows from this lemma that the time integral over the interval $[t_n, t_{n+1}]$ of the right hand side of (2.1) is bounded above by

$$\begin{aligned} -2\int_{t_n}^{t_{n+1}} s^{d/2} D(\nu_\rho, u_s)\, ds \ &+\ \frac{d}{2}\int_{t_n}^{t_{n+1}} ds\, s^{(d-2)/2} < (A_{L_n} u_s)^2 > \\ +\ C(\rho, d)\int_{t_n}^{t_{n+1}} s^{-2}\, ds &< u^2 > \end{aligned} \tag{2.3}$$

for some finite constant $C(\rho, d)$ that depends only on the dimension d and the density ρ.

For a finite subset Λ of $\mathbb{Z}^d$, denote by ν_ρ^Λ the product measure on $\mathbb{N}^\Lambda$ with marginals equal to the marginals of ν_ρ. For each $K \ge 0$, let $\nu_{\Lambda,K}$ stand for the canonical measure on $\mathbb{N}^\Lambda$, this is the product measure ν_ρ^Λ conditioned that the total number of particles on Λ is K

$$\nu_{\Lambda,K}(\cdot) \ =\ \nu_\rho^\Lambda(\cdot\,|\sum_{x\in\Lambda}\eta_x = K)\ .$$

Note that the right hand side does not depend on the particular choice of the parameter ρ.

For a subset Ω of $\mathbb{Z}^d$, a cube $\Lambda \subset \Omega$, a probability measure ν on $\mathbb{N}^\Omega$ and a function f in $L_2(\nu)$, denote by $D_\Lambda(\nu, f)$ the Dirichlet form of f on the cube Λ :

$$D_\Lambda(\nu, f) \ =\ -\int f\, \mathcal{L}_\Lambda f\, d\nu\ . \tag{2.4}$$

In the case where $\Lambda = \mathbb{Z}^d$, we denote $D_\Lambda(\nu, f)$ simply by $D(\nu, f)$. From [LSV] we have the following spectral gap estimate.

Theorem 2.2 . *Under the assumption (L) and (H) on the jump rate $g(\cdot)$, there exists an universal constant R_0 such that for all $\ell \ge 2$, $K \ge 0$, if Λ_ℓ denotes a cube of side length ℓ,*

$$E_{\nu_{\Lambda_\ell,K}}\Big[\Big(f - E_{\nu_{\Lambda_\ell,K}}[f]\Big)^2\Big] \ \le\ R_0 \ell^2 D_{\Lambda_\ell}(\nu_{\Lambda_\ell,K}, f)\ .$$

The second step in the proof of Theorem 1.1 consists in applying the spectral gap for the dynamics restricted to finite boxes in order to replace $A_{L_n} u_s$ by a function that depends only on the density of particles on boxes of length $O(\sqrt{s})$.

Let $\ell_n = \sqrt{(2/bdR_0)t_{n+1}}$. To simply the presentation we suppress the subindex n of ℓ and L. Let $\mathcal{R} = \{(2\ell+1)x\,, x \in \mathbb{Z}^d\}$ and consider an enumeration of this set : $\mathcal{R} = \{x_1, x_2, \dots\}$

such that $|x_j| \le |x_k|$ for $j \le k$. Let $\Omega_j = x_j + \Lambda_\ell^0$ and let $M_j = M_j(\eta)$ be the total number of particles in Ω_j for the configuration η

$$M_j \;=\; \sum_{x\in\Omega_j} \eta_x \;.$$

Let q denote the total number of cubes with nonvoid intersection with Λ_L^0. Note $q = O((L/\ell)^d)$. For each $j = 1, 2, \dots$, denote by $\mathbf{M}_j$ the vector $(M_1, \dots, M_j)$.

For a function h in $L^2(\nu_\rho)$, denote by $B_{\ell,L}h$ the conditional expectation of h given $\mathbf{M}_q$

$$B_{\ell,L}h \;=\; E_{\nu_\rho}\Big[h \,\Big|\, M_1, \dots, M_q\Big] \;. \tag{2.5}$$

If ℓ and L are chosen in such a way that $(2L+1)/(2\ell+1)$ is an odd number,

$$B_{\ell,L}A_L h = B_{\ell,L}h.$$

Since we may modify the definition of L, increasing it if necessary, without changing our estimates, we can assume that $(2L+1)/(2\ell+1)$ is odd.

Lemma 2.3 . *For any $u \in L^2(\nu_\rho)$,*

$$E_{\nu_\rho}\Big[\Big(A_L u - B_{\ell,L}A_L u\Big)^2\Big] \;\le\; R_0 \ell^2 D_{\Lambda_L^0}(\nu_\rho, u).$$

Proof. Fix a $\mathcal{F}_{\Lambda_L^0}$-measurable function h. For $1 \le j \le q$ denote by $\mathcal{G}_j$ the decreasing sequence of σ-algebras generated by $\{M_1, \cdots, M_j\}$ and $\{\eta_x,\, x \in \Omega_{j+1} \cup \cdots \cup \Omega_q\}$. Let $h_0 = h$ and for $1 \le j \le q$, let

$$h_j \;=\; E_{\nu_\rho}\Big[h \,\Big|\, \mathcal{G}_j\Big] \;.$$

With this notation we have that $B_{\ell,L}h = h_q$ and that

$$E_{\nu_\rho}\Big[(h - B_{\ell,L}h)^2\Big] \;=\; \sum_{j=0}^{q-1} E_{\nu_\rho}\Big[(h_{j+1} - h_j)^2\Big] \;.$$

Fix $0 \le j \le q-1$ and recall the definition of the canonical measures $\nu_{\Lambda,K}$. Taking conditional expectation with respect to $\mathcal{G}_{j+1}$, we have that

$$E_{\nu_\rho}\Big[(h_{j+1} - h_j)^2\Big] \;=\; E_{\nu_\rho}\Big[E_{\nu_\rho}\Big[(h_{j+1} - h_j)^2 \big|\mathcal{G}_{j+1}\Big]\Big] \;=\; E_{\nu_\rho}\Big[\mathrm{Var}\;(\nu_{\Omega_{j+1}, M_{j+1}}, h_j)\Big] \;.$$

By Lemma 2.2,

$$E_{\nu_\rho}\Big[\mathrm{Var}\;(\nu_{\Omega_{j+1}, M_{j+1}}, h_j)\Big] \;\le\; R_0 \ell^2 E_{\nu_\rho}\Big[D_{\Omega_{j+1}}(\nu_{\Omega_{j+1}, M_{j+1}}, h_j)\Big] \;.$$

By the convexity of the Dirichlet form, this expression is bounded above by $R_0\ell^2 D_{\Omega_{j+1}}(\nu_\rho, h)$. To conclude the proof of the lemma it only remains to sum over j. $\Box$

It follows from this lemma, the convexity of the Dirichlet form and the choice of $\ell = \ell_n$ that

$$\frac{d}{2}\int_{t_n}^{t_{n+1}} ds\, s^{(d-2)/2} < (A_L u_s - B_{\ell,L}u_s)^2 > \;\le\; \int_{t_n}^{t_{n+1}} s^{d/2} D(\nu_\rho, u_s)\, ds\ .$$

Since $< (A_L u_s)^2 > = < (A_L u_s - B_{\ell,L}u_s)^2 > + < (B_{\ell,L}u_s)^2 >$, (2.3) is now bounded above by

$$\begin{aligned} &-\int_{t_n}^{t_{n+1}} s^{d/2} D(\nu_\rho, u_s)\, ds \;+\; \frac{d}{2}\int_{t_n}^{t_{n+1}} ds\, s^{(d-2)/2} < (B_{\ell,L}u_s)^2 > \\ &\quad +\; C(\rho, d)\int_{t_n}^{t_{n+1}} s^{-2}\, ds < u^2 > \end{aligned} \tag{2.6}$$

for some finite constant $C(\rho, d)$.

We next take advantage of the translation invariance of the dynamics to replace $< (B_{\ell,L}u_s)^2 >$ in formula (2.6) by the average $|\Lambda^0_{\ell/4}|^{-1}\sum_{x\in\Lambda^0_{\ell/4}} < (B_{\ell,L}\tau_x u_s)^2 >$. Here $\{\tau_x,\, x\in\mathbb{Z}^d\}$ is the group of translations.

Fix x in $\Lambda^0_{\ell/4}$ and set $v = \tau_x u$, $v_t = P_t v$. Since the dynamics and the invariant measure ν_ρ are translation invariant, $v_t = \tau_x u_t$, $< v_t^2 > = < u_t^2 >$ and $D(\nu_\rho, v_t) = D(\nu_\rho, u_t)$. In particular, since x belongs to $\Lambda^0_{\ell/4}$ and a, b were chosen so that $ab \ge 4|\text{supp } u|^2$, it follows from Corollary 2.1 and Lemma 2.3 that the time integral over the interval $[t_n, t_{n+1}]$ of the left hand side of (2.1) is bounded above by

$$\begin{aligned} &-\int_{t_n}^{t_{n+1}} s^{d/2} D(\nu_\rho, u_s)\, ds \\ &\quad + \frac{d}{2}\frac{1}{|\Lambda^0_{\ell/4}|}\sum_{x\in\Lambda^0_{\ell/4}}\int_{t_n}^{t_{n+1}} ds\, s^{(d-2)/2} < (B_{\ell,L}\tau_x u_s)^2 > \; + C(\rho,d)\int_{t_n}^{t_{n+1}} s^{-2}\, ds < u^2 > . \end{aligned} \tag{2.7}$$

To conclude the proof of Theorem 1.1, it remains to estimate the second term of this formula. By Schwarz's inequality, it is bounded by

$$\begin{aligned} &\frac{d}{|\Lambda^0_{\ell/4}|^2}\sum_{x,y\in\Lambda^0_{\ell/4}}\int_{t_n}^{t_{n+1}} s^{(d-2)/2} E_{\nu_\rho}\Big[\Big(B_{\ell,L}\tau_x\{u_s - \tau_y u_s\}\Big)^2\Big]\, ds \\ &\quad + \frac{d}{|\Lambda^0_{\ell/4}|}\sum_{x\in\Lambda^0_{\ell/4}}\int_{t_n}^{t_{n+1}} s^{(d-2)/2} E_{\nu_\rho}\Big[\Big(B_{\ell,L}\frac{1}{|\Lambda^0_{\ell/4}|}\sum_{y\in\Lambda^0_{\ell/4}}\tau_{x+y}u_s\Big)^2\Big] . \end{aligned} \tag{2.8}$$

Recall the definition of the canonical measures $\nu_{\Lambda,K}$ and the product measure ν_ρ^Λ and the decomposition of Λ^0_L into subcubes $\Omega_1, \dots, \Omega_q$ of side length $2\ell+1$ with $\mathbf{M}_q = (M_1, \dots, M_q)$

the number of particles in each. Fix a vector $\mathbf{M}_q$ and let $f = f_{\ell,L,\mathbf{M}_q}$ be the Radon-Nikodym derivative given by

$$f(\eta) \;=\; f_{\ell,L,\mathbf{M}_q}(\eta) \;=\; \frac{d\nu_{\Omega_1,M_1}\otimes\cdots\otimes d\nu_{\Omega_q,M_q}}{d\nu_\rho^{\Lambda_L^0}}\ . \tag{2.9}$$

Since ν_ρ is translation invariant and reversible, and since the dynamics is translation invariant, we have that

$$B_{\ell,L}\tau_x h_s \;=\; E_{\nu_\rho}\Big[\tau_x h_s \,\Big|\, \mathbf{M}_q\Big] \;=\; \int \tau_x h_s(\eta) f(\eta)\nu_\rho(d\eta) \;=\; E_{\nu_\rho}[\tau_x h\, f_s]\ . \tag{2.10}$$

Therefore

$$B_{\ell,L}\frac{1}{|\Lambda^0_{\ell/4}|}\sum_{y\in\Lambda^0_{\ell/4}}\tau_{x+y}u_s \;=\; E_{\nu_\rho}\Big[\frac{1}{|\Lambda^0_{\ell/4}|}\sum_{y\in\Lambda^0_{\ell/4}}(\tau_{x+y}u)\, f_s\Big]\ .$$

In particular, by Schwarz inequality,

$$E_{\nu_\rho}\Big[\Big(B_{\ell,L}\frac{1}{|\Lambda^0_{\ell/4}|}\sum_{y\in\Lambda^0_{\ell/4}}\tau_{x+y}u_s\Big)^2\Big] \;\le\; E_{\nu_\rho}\Big[\Big(\frac{1}{|\Lambda^0_{\ell/4}|}\sum_{y\in\Lambda^0_{\ell/4}}\tau_{x+y}u\Big)^2\Big].$$

Since u is a cylinder function and ν_ρ a product measure, there exists a finite constant $C = C(d,u,\rho)$ such that

$$E_{\nu_\rho}\Big[\Big(B_{\ell,L}\frac{1}{|\Lambda^0_{\ell/4}|}\sum_{y\in\Lambda^0_{\ell/4}}\tau_{x+y}u_s\Big)^2\Big] \;\le\; C\ell^{-d}\ . \tag{2.11}$$

The next lemma provides an estimate for the first term in (2.8). It will be proved in the next section.

Lemma 2.4 . *There exists a finite constant C depending only on ρ, b, d, and u such that*

$$\frac{d}{|\Lambda^0_{\ell/4}|^2}\sum_{x,y\in\Lambda^0_{\ell/4}}\int_{t_n}^{t_{n+1}} s^{(d-2)/2}E_{\nu_\rho}\Big[\Big(B_{\ell,L}\tau_x\{u_s-\tau_y u_s\}\Big)^2\Big]\, ds \;\le\; C(\log t_{n+1})^{d+1} \tag{2.12}$$

provided a is taken large enough.

With this estimate we can complete the proof of Theorem 1.1.

Proof of Theorem 1.1. In view of (2.7), (2.8), (2.11) and Lemma 2.4, the time integral of the left hand side of (2.1) over the interval $[t_0, t_{n_0+1} = t]$ is bounded above by

$$-\int_{t_0}^t s^{d/2}D(\nu_\rho,u_s)\, ds \;+\; C\int_{t_0}^t s^{-2}E_{\nu_\rho}[u_s^2]ds \;+\; C\sum_{j=0}^{n_0}(\log t_{j+1})^{d+2}$$

for some finite constant C depending only on ρ, b, d, and u. Since $E_{\nu_\rho}[u_s^2]$ is decreasing, the second term contributes a finite constant depending on ρ, d and u, but independent of time. A simple computation shows that the third term is bounded above by $C(\log t_{n_0+1})^{d+3}$ with a new finite constant C. Therefore,

$$E_{\nu_\rho}[u_t^2] \;+\; \frac{1}{t^{d/2}}\int_{t_0}^{t} s^{d/2} D(\nu_\rho, u_s)\, ds \;\leq\; \frac{C(\log t)^{d+3}}{t^{d/2}}$$

which proves Theorem 1.1. □

3. Entropy estimates.

Denote by $H(f)$ the relative entropy of $f d\nu_\rho$ with respect to ν_ρ :

$$H(f) = E_{\nu_\rho}[f \log f].$$

Since ν_ρ is an invariant measure for the dynamics, $H(f_s)$ is decreasing in time. We now obtain a bound for $H(f)$ which therefore immediately bounds $H(f_s)$ for all times $s \geq 0$. Recall that we are assuming that $(2\ell+1)$ divides $(2L+1)$.

Lemma 3.1 . *Let $f_{\ell,L,\mathbf{M}_q}$ be defined as in (2.9). There exists a constant $C = C(\rho, b, d, R_0)$ such that*

$$E_{\nu_\rho}\Big[H(f_{\ell,L,\mathbf{M}_q})\Big] \;\leq\; C(L/\ell)^d \log \ell$$

provided ℓ is chosen sufficiently large. Here the expectation is over the random vector $\mathbf{M}_q$.

Proof. Recall the definition of the product measures ν_ρ^Λ. Let $\rho_j = K_j/(2\ell+1)^d$ and $m_j = M_j/(2\ell+1)^d$. The entropy $H(f)$ can be written as

$$H(f) \;=\; \int \log \frac{f d\nu_\rho^{\Lambda_L^0}}{d\nu_{\rho_0}^{\Omega_1}\otimes\cdots\otimes d\nu_{\rho_q}^{\Omega_q}} f d\nu_\rho \;+\; \int \log \frac{d\nu_{\rho_0}^{\Omega_1}\otimes\cdots\otimes d\nu_{\rho_q}^{\Omega_q}}{d\nu_\rho^{\Lambda_L^0}} f d\nu_\rho \;.$$

By definition of f, this expression is equal to

$$\sum_{j=1}^{q} \log \frac{1}{\nu_{\rho_j}^{\Omega_j}[M_j = K_j]} \;+\; \sum_{j=1}^{q} \Big\{ M_j \log \frac{\varphi(\rho_j)}{\varphi(\rho)} \;-\; |\Lambda_\ell^0| \log \frac{Z(\varphi(\rho_j))}{Z(\varphi(\rho))} \Big\} \;. \tag{3.1}$$

By the uniform local central limit theorem for zero range distributions (cf. [LSV] Theorem 6.1), $\nu_{\rho_j}^{\Omega_j}[M_j = K_j]$ is bounded below by $C_0/\sqrt{|\Omega_j|\sigma(\rho_j)^2}$ for some universal constant C_0, provided ℓ is large enough. The expected value, with respect to ν_ρ, of the first term of the previous formula is thus bounded below by

$$(1/2)\sum_{j=1}^{q} E_{\nu_\rho}\Big[\log C|\Omega_j|\sigma(\rho_j)^2\Big] \;.$$

It follows from assumptions (**L**) and (**H**) that there exists universal constants C_1 and C_2 such that $0 < C_1 \leq \sigma(\rho)^2/\varphi(\rho) \leq C_2 < \infty$ (cf. (5.2) in [LSV]). Since also $0 < C_3 \leq \rho/\varphi(\rho) \leq C_4 < \infty$

and $\log \alpha \le \alpha$, by definition of $q = O((L/\ell)^d)$, the previous expression is bounded above by $C(\rho, b, d, R_0)(L/\ell)^d \log \ell$ provided ℓ is sufficiently large.

Let $F(\theta) = \theta \log\{\varphi(\theta)/\varphi(\rho)\} - \log\{Z(\varphi(\theta))/Z(\varphi(\rho))\}$. The expected value, with respect to ν_ρ, of the second term of (3.1) is equal to

$$\sum_{j=0}^{q} |\Lambda_\ell^0| \, E_{\nu_\rho}[F(\rho_j)] \, .$$

It is easy to check that F and its first derivative vanish at ρ : $F(\rho) = F'(\rho) = 0$, that $\lim_{\theta\to 0} F(\theta) = C(\rho)$ and that there exists a finite constant C such that $F(\theta) \le C\theta \log\theta$ for θ large. In particular, there exists a finite constant $C_1(\rho)$ depending only on ρ such that $F(\theta) \le C_1(\theta - \rho)^2$ for all $\theta \ge 0$. The previous sum is thus bounded above by

$$C_1(\rho)|\Lambda_\ell^0| \sum_{j=1}^{q} E_{\nu_\rho}[(\rho_j - \rho)^2] \;\le\; C_2(\rho)(L/\ell)^d \, .$$

This concludes the proof of the lemma. $\square$

The proof of the following perturbation result is standard. However, we were not able to find an explicit reference so the proof is included for completeness.

Lemma 3.2 . *Let $(\Omega, P, \mathcal{F})$ be a probability space and let $\langle f, g\rangle = \int fgdP$ denote the standard inner product on $L^2(\Omega, P, \mathcal{F})$. Let A be a non-negative definite symmetric operator on $L^2(\Omega, P, \mathcal{F})$ which has 0 as a simple eigenvalue with corresponding eigenfunction the constant function 1, and second smallest eigenvalue $\delta > 0$ (the spectral gap). Let V be a f unction of mean zero, $\langle 1, V\rangle = 0$ and assume that V is esssentially bounded. Denote by λ_ϵ the principal eigenvalue of $-A + \epsilon V$ given by the variational formula*

$$\lambda_\epsilon = \sup_{\|f\|_2 = 1} \langle f, (-A + \epsilon V) f\rangle \, .$$

Then for $\epsilon < \delta(2\|V\|_\infty)^{-1}$,

$$0 \le \lambda_\epsilon \le \frac{\epsilon^2 \left\langle V, A^{-1} V\right\rangle}{1 - 2\|V\|_\infty \epsilon\delta^{-1}} \, .$$

Proof. The lower bound follows immediately by setting $f = 1$ in the variational formula. Let G be any function with $\|G\|_2 = 1$ and $\langle 1, G\rangle \ge 0$. Since $\langle 1, V\rangle = 0$ we have

$$\lambda_\epsilon = \epsilon 2\langle V, G\rangle + \epsilon\langle V, [G-1]^2\rangle - \langle G, AG\rangle + \langle G, (\lambda_\epsilon + A - \epsilon V)G\rangle.$$

By Schwarz's inequality we can control the first two terms on the right hand side for any $\beta > 0$ by

$$\epsilon 2\langle V, G\rangle + \epsilon\langle V, [G-1]^2\rangle \le \frac{\epsilon^2}{\beta}\langle V, A^{-1}V\rangle + \beta\langle G, AG\rangle + 2\epsilon\|V\|_\infty(1 - \langle 1, G\rangle).$$

Also by Schwarz's inequality, $\langle 1, G\rangle \le 1$ so by the spectral gap

$$1 - \langle 1, G\rangle \le \langle G, G\rangle - \langle 1, G\rangle^2 \le \delta^{-1}\langle G, AG\rangle.$$

Therefore for any $\beta > 0$,

$$\lambda_\epsilon \le \frac{\epsilon^2}{\beta}\langle V, A^{-1}V\rangle + (\beta + 2\epsilon\|V\|_\infty\delta^{-1} - 1)\langle G, AG\rangle + \langle G, (\lambda_\epsilon + A - \epsilon V)G\rangle.$$

Consider an optimising sequence $\{G_{\epsilon,n}, n = 1, 2, \dots\}$ in the variational formula and, without loss of generality, assume that $\langle 1, G_{\epsilon,n}\rangle \ge 0$ for all n. Choose $\beta = 1 - 2\epsilon\|V\|_\infty\delta^{-1}$ and $G = G_{\epsilon,n}$ in the previous bound. Letting $n \to \infty$ we have $\langle G_{\epsilon,n}, (\lambda_\epsilon + A - \epsilon V)G_{\epsilon,n}\rangle \to 0$ and the upper bound follows. $\square$

Lemma 3.3 . *Let u be any local function. Let Λ_u be the smallest cube centered at the origin containing the support of $u - \tau_e u$ for all unit vectors e in $\mathbb{Z}^d$. There exists a finite constant $C = C(u,\rho)$ depending only on u and ρ such that for any unit vector e in $\mathbb{Z}^d$,*

$$\Big(E_{\nu_\rho}\Big[f(u - \tau_e u)\Big]\Big)^2 \;\le\; C(u,\rho)\, D_{\Lambda_u}(\nu_\rho, \sqrt{f})\,.$$

Proof. Consider the zero-range process corresponding to the Dirichlet form $D_{\Lambda_u}(\nu_\rho, \sqrt{f})$. This process has a spectral gap of magnitude Γ that depends on the size of the support of u. Assume $\beta \le (1/8)(\|u\|_\infty\Gamma)^{-1}$. In this case by Lemma 3.2,

$$E_{\nu_\rho}\Big[f(u - \tau_e u)\Big] \;-\; \frac{1}{\beta}D_{\Lambda_u}(\nu_\rho, \sqrt{f}) \;\le\; 2\beta < (u - \tau_e u), (-L_{\Lambda_u})^{-1}(u - \tau_e u) >\,,$$

where L_{Λ_u} is the generator restricted to the box Λ_u (with reflecting boundary conditions). On the other hand, for $\beta \ge (1/8)(\|u\|_\infty\Gamma)^{-1}$, $E_{\nu_\rho}\Big[f(u - \tau_e u)\Big]$ is bounded above by $2\|u\|_\infty \le 16\beta\Gamma\|u\|_\infty^2$. Optimising over β we obtain the lemma. $\square$

Lemma 3.4 . *There exists a finite constant $C = C(d,u,\rho)$ depending only on u, ρ and the dimension d, such that for n sufficiently large,*

$$\frac{1}{|\Lambda_n|^2}\sum_{x,y\in\Lambda_n}\Big(E_{\nu_\rho}\Big[\tau_x f(u - \tau_y u)\Big]\Big)^2 \;\le\; C(d,u,\rho)\, n^{2-d} D_{\Lambda_{3n}}(\nu_\rho, \sqrt{f})$$

Proof. Define a canonical path $0 = x_0, x_1, \dots, x_m = y$ where $m = \sum_{j=1}^d |y_j|$ from the origin to y by nearest neighbor steps, i.e., $\tilde{e}_i = x_{i+1} - x_i$ are unit vectors, by first moving towards y in the first coordinate direction, then in the second coordinate direction, etc. Then,

$$u - \tau_y u = \sum_{i=0}^{m-1} \tau_{x_i} u - \tau_{x_{i+1}} u.$$

Therefore for any f, by the previous lemma and Schwarz's inequality,

$$\left(E_{\nu_\rho}\Big[f(u-\tau_y u)\Big]\right)^2 = \left(\sum_{i=0}^{m-1} E_{\nu_\rho}\Big[\tau_{-x_i} f(u-\tau_{\tilde e_i} u)\Big]\right)^2 \le C(d,u,\rho)m\sum_{i=0}^{m-1} D_{\Lambda_u}(\nu_\rho,\sqrt{\tau_{-x_i}f}).$$

Since $m \le 2dn$, as long as n is larger than the side length of Λ_u we have that $D_{\Lambda_u}(\nu_\rho,\sqrt{\tau_x f}) = D_{\tau_{-x}\Lambda_u}(\nu_\rho,\sqrt{f})$ and by explicit counting, that

$$\frac{1}{|\Lambda_n|^2}\sum_{x,y\in\Lambda_n}\left(E_{\nu_\rho}\Big[\tau_x f(u-\tau_y u)\Big]\right)^2 \;\le\; C(d,u,\rho)\,n^{2-d} D_{\Lambda_{3n}}(\nu_\rho,\sqrt{f})$$

with a new constant $C(d,u,\rho)$. □

Proof of Lemma 2.4. Recalling (2.10), we obtain that the left hand side of (2.12) is bounded above by

$$C(d,u,\rho)E_{\nu_\rho}\Big[\int_{t_n}^{t_{n+1}} D_{\Lambda_L}(\nu_\rho,\sqrt{f_s})\Big]\,ds \;\le\; C(d,u,\rho)E_{\nu_\rho}\Big[H(f)\Big]\,.$$

for some finite constant $C(d,u,\rho)$. Lemma 2.4 follows from this coupled with the estimate on the entropy $H(f)$ obtained in Lemma 3.1. □

4. Cutoff estimate.

In this section we prove the cutoff estimate stated in Lemma 2.1. The proof will be developed in several lemmas which follow. First we need some notation. Recall that for each positive integer j, Λ_j^0 is a cube of side length $2j+1$ centered at the origin in $\mathbb{Z}^d$, $\mathcal{F}_j$ is the σ-algebra generated by the variables $\{\eta_x, x\in\Lambda_j^0\}$ and $A_j f = E_{\nu_\rho}[f|\mathcal{F}_j]$ is the conditional expectation given those variables. $\mathcal{L}$ is the generator of our process which can be written as

$$\mathcal{L} = \sum_b \mathcal{L}_b$$

where the sum is over nearest neighbor bonds $b=(x,y)$ and

$$(\mathcal{L}_b u)(\eta) = g(\eta_x)[u(\sigma^{x,y}\eta)-u(\eta)] + g(\eta_y)[u(\sigma^{y,x}\eta)-u(\eta)].$$

Likewise we can also write the Dirichlet form $D(\nu_\rho,u)$ as the sum $\sum_b D_b(\nu_\rho,u)$ where $D_b(\nu_\rho,u)$ is the piece of the Dirichlet form corresponding to jumps over the bond b,

$$D_b(\nu_\rho,u) \;= \;<u,(-\mathcal{L}_b)u>\;.$$

We shall say that a nearest neighbor bond $b=(x,y)$ belongs to a subset Λ of $\mathbb{Z}^d$ if both ends x and y belong to Λ and that it belongs to the boundary $\partial\Lambda$ of Λ if one and only one of the ends belongs to Λ. In this case we always denote by x the end that belongs to Λ and by y the end that does not belong to Λ.

Note that if $u_t = P_t u$ is evolving by the dynamics then for each $j \geq 1$,

$$\partial_t A_j u_t \;=\; \partial_t E_{\nu_\rho}\Big[u_t \,\big|\, \mathcal{F}_j\Big] \;=\; E_{\nu_\rho}\Big[\partial_t u_t \,\big|\, \mathcal{F}_j\Big] \;=\; E_{\nu_\rho}\Big[\mathcal{L} u_t \,\big|\, \mathcal{F}_j\Big] \,.$$

and therefore

$$\frac{d}{dt} < (A_j u_t)^2 >= 2 < A_j u_t, \mathcal{L} u_t > .$$

The plan of attack is to control an appropriate combination of the $< (A_j u)^2 >$. With this in mind we first provide an estimate for this last term.

Lemma 4.1 . *There exists a finite constant $C(\rho)$ such that for all $\beta \geq 0$,*

$$\begin{aligned} < A_j u, (-\mathcal{L})u > \leq \sum_{b \in \Lambda_j^0} D_b(\nu_\rho, u) + (1 + \frac{\beta}{2}) \sum_{b \in \partial \Lambda_j^0} D_b(\nu_\rho, u) \\ + \frac{C(\rho)}{\beta} \Big[< (A_{j+1} u)^2 > - < (A_j u)^2 > \Big]. \end{aligned}$$

Proof. We can write $< A_j u, (-\mathcal{L})u >$ as the sum of interior terms and boundary terms

$$\sum_{b \in \Lambda_j^0} < A_j u, (-\mathcal{L}_b) u > \; + \; \sum_{b \in \partial \Lambda_j^0} < A_j u, (-\mathcal{L}_b) u > \; . \tag{4.1}$$

Note that for interior bonds b the conditional expectation A_j commutes with $\mathcal{L}_b$ and so, by convexity, each interior term is controlled by the Dirichlet term,

$$< A_j u, (-\mathcal{L}_b) u > \leq D_b(\nu_\rho, u), \quad b \in \Lambda_j^0.$$

Next we consider the case $b = (x, y) \in \partial \Lambda_j^0$, and now we have to face the fact that A_j and $\mathcal{L}_b$ do not commute. It is convenient to write the bilinear form corresponding to $\mathcal{L}_b$ as

$$< u, (-\mathcal{L}_b) v >= \varphi < \nabla_b u, \nabla_b v >$$

where $\varphi = \varphi(\rho)$ and

$$(\nabla_b u)(\eta) = u(\eta + \mathfrak{d}_x) - u(\eta + \mathfrak{d}_y).$$

Here $\mathfrak{d}_x$ represents the configuration with one particle at x and none elsewhere, and addition is componentwise at each site. Since $y \notin \Lambda_j^0$,

$$(\nabla_b A_j u)(\eta) = A_j u(\eta + \mathfrak{d}_x) - A_j u(\eta).$$

By changing variables we can write $A_j u(\eta + \mathfrak{d}_x)$ as

$$\int \{u(\eta\circ\zeta + \mathfrak{d}_x) - u(\eta\circ\zeta + \mathfrak{d}_y)\}\nu_\rho^{\mathbb{Z}^d\setminus\Lambda_j^0}(d\zeta) \;+\; \int u(\eta\circ\zeta)\widetilde{V}(\zeta_y)\nu_\rho^{\mathbb{Z}^d\setminus\Lambda_j^0}(d\zeta)\;,$$

where $(\eta\circ\zeta)_z$ is equal to η_z for $z\in\Lambda_j^0$ and ζ_z for $z\notin\Lambda_j^0$. The Jacobian $\widetilde{V}$ is given by $\widetilde{V}(k) = g(k)/\varphi$ for each $k\geq 0$. Notice that for each site $y\notin\Lambda_j^0$, $\widetilde{V}(\zeta_y)$ is a $\nu_\rho^{\mathbb{Z}^d\setminus\Lambda_j^0}$ mean one cylinder function. Define $V_y(\eta) = \widetilde{V}(\eta_y) - 1$. The second term of (4.1) can be written as

$$\varphi \sum_{b\in\partial\Lambda_j^0} \Big\{ <\nabla_b u, A_j \nabla_b u> \;+\; <\nabla_b u, A_j(uV_y)> \Big\}\;.$$

By Schwarz inequality and the convexity of the Dirichlet form, this expression is bounded above by

$$(1+\frac{\beta}{2})\sum_{b\in\partial\Lambda_j^0} D_b(\nu_\rho, u) \;+\; \frac{\varphi}{2\beta}\sum_{b\in\partial\Lambda_j^0} <\{A_j(uV_y)\}^2> \tag{4.2}$$

for each $\beta>0$.

Consider a collection $\{\psi_i\,, 1\leq i\leq m\}$ of orthogonal vectors in a Hilbert space $\mathcal{H}$ with inner product denoted by $\ll\cdot,\cdot\gg$. It is easy to see that for every ψ in $\mathcal{H}$,

$$\sum_{i=1}^m \Big(\ll\psi,\psi_i\gg \Big)^2 \;\leq\; \max_{1\leq i\leq m} \ll\psi_i,\psi_i\gg \ll\psi,\psi\gg\;. \tag{4.3}$$

Since for each bond b in $\partial\Lambda_j^0$, V_y is $\mathcal{F}_{j+1}$ measurable and has mean zero with respect to $E_{\nu_\rho}[\cdot|\mathcal{F}_j]$, we have that

$$\begin{aligned} A_j(uV_y) \;&=\; E_{\nu_\rho}\Big[uV_y\big|\mathcal{F}_j\Big] \;=\; E_{\nu_\rho}\Big[(A_{j+1}u)V_y\big|\mathcal{F}_j\Big] \\ &=\; E_{\nu_\rho}\Big[(A_{j+1}u - A_ju)V_y\big|\mathcal{F}_j\Big] \;=\; A_j(\{A_{j+1}u - A_ju\}V_y)\;. \end{aligned}$$

Since the functions $\{V_y,\, b\in\partial\Lambda_j^0\}$ are orthogonal with respect to $E_{\nu_\rho}[\cdot|\mathcal{F}_j]$, by inequality (4.3), the second term of (4.2) is bounded above by

$$\frac{\varphi}{2\beta}\max_{b\in\partial\Lambda_j^0} <V_y^2><(A_{j+1}u - A_ju)^2> \leq \frac{C(\rho)}{\beta}\Big\{ <(A_{j+1}u)^2> - <(A_ju)^2> \Big\}$$

for some finite constant $C(\rho)$ depending on ρ only. This completes the proof of the lemma. □

For positive integers $k<K$ and $\beta>0$, define $\mathfrak{A} = \mathfrak{A}_{k,K,\beta}$ on $L^2(\nu_\rho)$ by

$$\mathfrak{A}u \;=\; \alpha_{k+1} <(A_ku)^2> \;+\; \sum_{j=k}^{K-1} \alpha_{j+1} <(A_{j+1}u - A_ju)^2> \;+\alpha_{K+1} <(u - A_Ku)^2>\;,$$

where

$$\alpha_j = \exp\{j/2\beta k\}.$$

Lemma 4.2 . *There exists a finite constant $C(\rho)$ such that for each k, K and β satisfying $\beta k \ge 2$, for each $t \ge 0$,*

$$\mathfrak{A}_{k,K,\beta} u_t \;\le\; \exp\Big\{ \frac{C(\rho) t}{\beta^2 k^2} \Big\} \mathfrak{A}_{k,K,\beta} u .$$

Proof. Notice that $\mathfrak{A} u$ may be rewritten as

$$\mathfrak{A} u \;=\; \alpha_{K+1} < u^2 > \;-\; \sum_{j=k+1}^{K} (\alpha_{j+1} - \alpha_j) < (A_j u)^2 > \;.$$

In particular,

$$\frac{d}{dt} \mathfrak{A} u_t \;=\; -2 \alpha_{K+1} D(\nu_\rho, u_t) \;-\; 2 \sum_{j=k+1}^{K} (\alpha_{j+1} - \alpha_j) < A_j u_t, \mathcal{L} u_t > .$$

By the previous lemma, the right hand side is bounded above by

$$\begin{aligned} &- 2\alpha_{K+1} D(\nu_\rho, u_t) \;+\; 2 \sum_{j=k+1}^{K} (\alpha_{j+1} - \alpha_j) \sum_{b \in \Lambda_j^0} D_b(\nu_\rho, u_t) \\ &\quad +\; 2\beta k \sum_{j=k+1}^{K} (\alpha_{j+1} - \alpha_j) \sum_{b \in \partial \Lambda_j^0} D_b(\nu_\rho, u_t) \\ &\qquad +\; \frac{C(\rho)}{\beta k} \sum_{j=k+1}^{K} (\alpha_{j+1} - \alpha_j) \Big\{ < (A_{j+1} u_t)^2 > - < (A_j u_t)^2 > \Big\} . \end{aligned} \tag{4.4}$$

From the definition of the sequence α_j, we have that $\alpha_{j+1} \ge \beta k (\alpha_{j+1} - \alpha_j)$. It follows from this inequality and a summation by parts that the Dirichlet part of the previous expression is negative. Applying the inequality for the α's again, the third line of (4.4) is bounded above by

$$\frac{C(\rho)}{(\beta k)^2} \sum_{j=k+1}^{K} \alpha_{j+1} \Big\{ < (A_{j+1} u_t)^2 > - < (A_j u_t)^2 > \Big\} \;\le\; \frac{C(\rho)}{(\beta k)^2} \mathfrak{A} u_t .$$

The lemma follows by Gronwall's inequality. □

Proof of Lemma 2.1. Let $\beta = 1$, $k = k_n = \sqrt{t_{n+1}}$ and $K = L_n$ in Lemma 4.2. Since $n \ge 0$, by definition, $t_{n+1} \ge ab$ so that $\beta k \ge 2$ and $k \ge \operatorname{supp} u$. It follows from this last property that $\mathfrak{A} u_0$ is equal to $\alpha_{k+1} < u^2 >$. By definition of $\mathfrak{A} u_s$ and Lemma 4.2,

$$\alpha_{L_n+1} < (A_{L_n} u_s - u_s)^2 > \le \; \mathfrak{A} u_s \;\le\; \exp\Big\{ \frac{C(\rho) s}{k^2} \Big\} \mathfrak{A} u_0 \;=\; \exp\Big\{ \frac{C(\rho) s}{k^2} \Big\} \alpha_{k+1} < u^2 > \;.$$

Therefore,

$$< (A_{L_n} u_s - u_s)^2 > \le \; \frac{\alpha_{k+1}}{\alpha_{L_n+1}} \exp\Big\{ \frac{C(\rho) s}{k^2} \Big\} < u^2 > \;.$$

To conclude the proof of the lemma, all that remains is to use the definition of α_j and recall that $t_n \le s \le t_{n+1}$. □

5. Appendix I. Nash's inequality.

In this appendix we state and prove Nash-type inequalities for zero range processes satisfying assumptions (L) and (H).

For each site x and each cylinder function f, denote by $\partial f / \partial \eta_x$ the dependence of f on the addition of a particle at x

$$\frac{\partial f}{\partial \eta_x} \;=\; f(\eta + \mathfrak{d}_x) \;-\; f(\eta) \;, \tag{5.1}$$

where $\mathfrak{d}_x$ stands for the configuration with no particle but one at x. On the space $\mathcal{C}_b$ of bounded cylinder functions, consider the triple norm $|\!|\!| \cdot |\!|\!|$ defined by

$$|\!|\!| f |\!|\!| \;=\; \sum_{x \in \mathbb{Z}^d} \Big\| \frac{\partial f}{\partial \eta_x} \Big\|_2 \;. \tag{5.2}$$

where $\| \cdot \|_2$ stands for the usual L^2 norm with respect to ν_ρ.

Theorem 5.1 . (**Nash's Inequality**). *Under assumptions (H) and (L) on the rate jump $g(\cdot)$, there exists a finite constant C_0 depending only on ϵ and ρ such that for all f in $L_2(\nu_\rho)$,*

$$\text{Var }(\nu_\rho, f) \;\le\; C_0\, D(\nu_\rho, f)^{\frac{d}{d+2}}\, |\!|\!| f |\!|\!|^{\frac{4}{d+2}} \;.$$

It is well known that the decay to equilibrium in $L^2(\nu_\rho)$ at rate $t^{-d/2}$ would follow easily from the above Nash's estimate if one could bound $|\!|\!| P_t u |\!|\!|$ uniformly in time (cf. Theorem 2.2 in [Li]). Unfortunately such a result seems to be be very difficult to prove for any interacting system except the symmetric simple exclusion process, for which these methods give full results, with no logarithmic corrections (see also [BZ]). We sketch the argument.

Suppose that there exists a finite constant C depending only on ρ such that for all f with $|\!|\!| f |\!|\!| < \infty$,

$$|\!|\!| P_t f |\!|\!| \;\le\; C |\!|\!| f |\!|\!| \tag{5.3}$$

for all $t \ge 0$. An elementary computation shows that

$$\frac{d}{dt} \text{Var }(\nu_\rho, P_t f) \;=\; -2 D(\nu_\rho, P_t f) \;.$$

Applying Nash inequality and estimate (5.3), we obtain that

$$\frac{d}{dt} \text{Var }(\nu_\rho, P_t f) \;\le\; -C |\!|\!| f |\!|\!|^{-4/d} \text{Var }(\nu_\rho, P_t f)^{(d+2)/d} \;.$$

Solving the differential inequality would lead to the $t^{-d/2}$ decay estimate

$$\text{Var}(\nu_\rho, P_t f) \le C t^{-d/2} |\!|\!| f |\!|\!|^2 .$$

We now proceed to prove Nash's inequality. We can normalize f so that $E_{\nu_\rho}[f] = 0$. For any positive integer ℓ, divide the lattice into nonoverlapping cubes of length ℓ : let $\Lambda_\ell = \{1, \dots, \ell\}^d$ and consider a collection $\{\Omega_k,\, k \geq 1\}$ such that $\bigcup_{k\geq 1} \Omega_k \;=\; \mathbb{Z}^d$, $\Omega_k \cap \Omega_j \;=\;$ for $j \neq k$, and for all $k \geq 1$, $\Omega_k = x_k + \Lambda_\ell$ for some x_k in $\mathbb{Z}^d$. The cubes are enumerated $\Omega_1, \Omega_2, \dots$. For each $k \geq 1$, let $M_k(\eta)$ denote the number of particles of configuration η in the box Ω_k,

$$M_k(\eta) \;=\; \sum_{x\in\Omega_k} \eta_x \; .$$

We will often denote by $\mathbf{M}$ the configuration $\{M_k,\, k \geq 1\}$, which is also to be thought of as a random variable under ν_ρ. Let $\bar{f}\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{R}$ denote the expectation of f conditioned on the number of particles in each box $\{\Omega_k,\, k \geq 1\}$

$$\bar{f}(\mathbf{M}) \;=\; E\Big[f \,|\, \sum_{x\in\Omega_k} \eta_x = M_k,\, k \geq 1 \Big] \; .$$

Note that we have suppressed the dependence on ℓ both on the sequence $\{\Omega_j,\, j \geq 1\}$ and on $\bar{f}$. By the spectral gap for the zero range process and the martingale argument of Lemma 2.3, there exists a finite constant R_0 so that for any positive integer ℓ, for any f,

$$E_{\nu_\rho}\big[f^2\big] \;\leq\; R_0 \ell^2 D(\nu_\rho, f) \;+\; E_{\nu_\rho}\Big[\bar{f}(\mathbf{M})^2\Big] \; .$$

We will show in Proposition ♣q1 below that there is a finite constant C depending only on the ϵ defined in (1.5), R_0 and ρ so that for any f,

$$E_{\nu_\rho}\Big[\bar{f}(\mathbf{M})^2\Big] \;\leq\; C\Big\{\ell^{-d} |||f|||^2 \;+\; \ell^2 D(\nu_\rho, f)\Big\} \; . \tag{5.4}$$

It follows from the two previous inequalities that there exists a finite constant C, such that for any positive integer ℓ,

$$E_{\nu_\rho}\big[f^2\big] \;\leq\; C\Big\{\ell^2 D(\nu_\rho, f) \;+\; \ell^{-d} |||f|||^2\Big\} \; .$$

Nash's inequality follows by minimizing over ℓ. □

Proposition 5.2 . *There exists a finite constants C depending only on ρ, ϵ and R_0 so that*

$$E_{\nu_\rho}\Big[\bar{f}(\mathbf{M})^2\Big] \;\leq\; C\Big\{\ell^{-d} |||f|||^2 \;+\; \ell^2 D(\nu_\rho, f)\Big\} \; .$$

The proof of this proposition is divided in several lemmas. We start introducing some notation. For each $j \geq 1$ and configuration $\mathbf{M} = \{M_k,\, k \geq 1\}$, denote by $\mathbf{M}_j$ the first j coordinates of $\mathbf{M}$: $\mathbf{M}_j = (M_1, \dots, M_j)$. Let $\mathcal{T}\colon \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{N}^{\mathbb{N}}$ be the map that sends a configuration η to $\mathbf{M}(\eta)$. Denote by m_ρ the product measure $\nu_\rho \mathcal{T}^{-1}$ on $\mathbb{N}^{\mathbb{N}}$. Let $m_\rho^1(\cdot)$ stand for the one site marginal of m_ρ. Consider the martingale $\{\bar{f}_j,\, j \geq 0\}$ given by $\bar{f}_0 = E_{m_\rho}[\bar{f}] = 0$ and

$$\bar{f}_j \;=\; E_{m_\rho}\Big[\bar{f} \,|\, \mathbf{M}_j\Big]$$

for $j \geq 1$.

Lemma 5.3 . *For any ν_ρ-mean zero function f,*

$$E_{m_\rho}\Big[\bar f(\mathbf{M})^2\Big] = (1/2)\sum_{j\ge 0}\sum_{M\ge 0}\sum_{N\ge 0} m^1_\rho(M) m^1_\rho(N) E_{m_\rho}\Big[\Big\{\bar f_{j+1}(\mathbf{M}_j, M) - \bar f_{j+1}(\mathbf{M}_j, N)\Big\}^2\Big] . \tag{5.5}$$

Proof. Since f has mean zero, we may rewrite $E_{m_\rho}[\bar f^2]$ as

$$E_{m_\rho}[\bar f^2] = \sum_{j\ge 0} E_{m_\rho}\Big[(\bar f_{j+1} - \bar f_j)^2\Big] = \sum_{j\ge 0} E_{m_\rho}\Big[E_{m_\rho}\Big[(\bar f_{j+1} - \bar f_j)^2 \,|\, \mathbf{M}_j\Big]\Big] ,$$

The conditional expectation in the last term is just the variance of $\bar f_{j+1}$ with respect to the measure $m_\rho(\cdot\,|\,\mathbf{M}_j)$. Since m_ρ is a product measure this variance reduces to

$$\sum_{M\ge 0} m^1_\rho(M)\Big\{\bar f_{j+1}(\mathbf{M}_j, M) - \bar f_j(\mathbf{M}_j)\Big\}^2 = (1/2)\sum_{M\ge 0}\sum_{N\ge 0} m^1_\rho(M) m^1_\rho(N)\Big\{\bar f_{j+1}(\mathbf{M}_j, M) - \bar f_{j+1}(\mathbf{M}_j, N)\Big\}^2 .$$

This concludes the proof of the lemma. □

Let $\Gamma_j = \cup_{k\le j}\Omega_k$. For each vector $\mathbf{M}_j = (M_1, \dots, M_j)$ denote by $\nu_{\mathbf{M}_j}$ the probability measure on $\mathbb{N}^{\mathbb{Z}^d}$ whose restriction on each cube Ω_k, $1\le k\le j$, is the canonical measure with M_k particles and whose marginals on Ω_k, $k\ge j+1$, are the product measure $\nu_\rho^{\Omega_k}$:

$$\nu_{\mathbf{M}_j} = \otimes_{k=1}^{j} \nu_{\Omega_k, M} \;\; \otimes_{k\ge j+1} \nu_\rho^{\Omega_k} .$$

Lemma 5.4 . *For every $\mathbf{M}$ in $\mathbb{N}^{\mathbb{N}}$ and $j\ge 1$,*

$$\bar f_{j+1}(\mathbf{M}_j, M+1) - \bar f_{j+1}(\mathbf{M}_j, M) = E_{\nu_{(\mathbf{M}_j, M)}}\Big[\frac{1}{|\Lambda_\ell|}\sum_{x\in\Omega_{j+1}} \frac{\partial f}{\partial \eta_x} h(\eta_x)\Big] + E_{\nu_{(\mathbf{M}_j, M)}}\Big[f ; \frac{1}{|\Lambda_\ell|}\sum_{x\in\Omega_{j+1}} h(\eta_x)\Big] ,$$

where

$$h(\eta_x) = h_{\ell, M}(\eta_x) = \frac{(\eta_x+1)/g(\eta_x+1)}{E_{\nu_{\Omega_{j+1}, M}}[(\eta_x+1)/g(\eta_x+1)]} . \tag{5.6}$$

In this formula and below, for two functions f and g, $E_\nu[f;g]$ stands for the covariance of f and g with respect to the measure ν : $E_\nu[f;g] = E_\nu[fg] - E_\nu[f]E_\nu[g]$.

Note that by virtue of (1.5), the cylinder function h is uniformly bounded above and below

$$\epsilon^2 \le h(\eta_0) \le \epsilon^{-2} .$$

Proof of Lemma 5.4. By definition, $\bar f_{j+1}(\mathbf{M}_j, M+1)$ is equal to $E_{\nu_\rho}[f|\mathbf{M}_j, M_{j+1} = M+1]$. A simple computation shows that

$$\bar f_j(\mathbf{M}_j) \;=\; E_{\nu_{\mathbf{M}_j}}[f]$$

for each $j \ge 1$ and $\mathbf{M}_j$ in $\mathbb{N}^j$. In particular,

$$\bar f_{j+1}(\mathbf{M}_j, M+1) \;=\; E_{\nu_{(\mathbf{M}_j, M+1)}}[f] \;=\; \frac{1}{M+1} \sum_{x \in \Omega_{j+1}} E_{\nu_{(\mathbf{M}_j, M+1)}}[\eta_x f]$$

because the measure $\nu_{(\mathbf{M}_j, M+1)}$ is concentrated on configurations with $M+1$ particle on Ω_{j+1}. Performing a change of variables $\xi = \eta - \mathfrak{d}_x$ we obtain that

$$\frac{1}{M+1} \sum_{x \in \Omega_{j+1}} E_{\nu_{(\mathbf{M}_j, M+1)}}\Big[\eta_x f(\eta)\Big] \;=\; \frac{1}{|\Lambda_\ell|} \sum_{x \in \Omega_{j+1}} E_{\nu_{(\mathbf{M}_j, M)}}\Big[h(\eta_x) f(\eta + \mathfrak{d}_x)\Big]\;,$$

where

$$h(\eta_x) \;=\; (1+\eta_x)\frac{|\Lambda_\ell|}{M+1}\frac{\nu_{(\mathbf{M}_j, M+1)}(\eta)}{\nu_{(\mathbf{M}_j, M)}(\xi)} \;=\; \frac{1+\eta_x}{g(1+\eta_x)}\frac{|\Lambda_\ell|}{M+1}\frac{\varphi \bar\nu_\varphi\{\eta,\, \sum_{x\in\Lambda_\ell} \eta_x = M\}}{\bar\nu_\varphi\{\eta,\, \sum_{x\in\Lambda_\ell}\eta_x = M+1\}}\;. \tag{5.7}$$

Notice that this last expression does not depend on the particular choice of φ, and that we omitted the dependence of h on M and $|\Lambda_\ell|$. Moreover, taking $f = 1$ gives $E_{\nu_{(\mathbf{M}_j, M)}}[h(\eta_x)] = E_{\nu_{\Omega_{j+1}, M}}[h(\eta_x)] = 1$ for any x in Ω_{j+1}. We may therefore rewrite $h(\eta_x)$ as in (5.6). To conclude the proof of the lemma, it remains to add and subtract the intermediary term

$$\frac{1}{|\Lambda_\ell|} \sum_{x \in \Omega_{j+1}} E_{\nu_{(\mathbf{M}_j, M)}}\Big[h(\eta_x) f(\eta)\Big]$$

and to recall that $h(\eta_x)$ has mean one with respect to $\nu_{(\mathbf{M}_j, M)}$. $\square$

Lemma 5.5 . *There exist a finite constant C depending only on ϵ, ρ and R_0 such that*

$$\begin{aligned} &\sum_{M,N\ge 0} m^1_\rho(M) m^1_\rho(N) E_{m_\rho}\Big[\Big(\bar f_{j+1}(\mathbf{M}_j, M) - \bar f_{j+1}(\mathbf{M}_j, N)\Big)^2\Big] \\ &\qquad \le\; C\Big\{\frac{1}{|\Lambda_\ell|}\Big\| \sum_{x\in\Omega_{j+1}} \Big|\frac{\partial f}{\partial \eta_x}\Big|\,\Big\|_2^2 \;+\; \ell^2 D_{\Omega_{j+1}}(\nu_\rho, f)\Big\}\;. \end{aligned} \tag{5.8}$$

Proof. Rewrite the left hand side of (5.8) as

$$2 \sum_{0\le M<N} m^1_\rho(M) m^1_\rho(N) E_{m_\rho}\Big[\Big\{\sum_{K=M}^{N-1} \bar f_{j+1}(\mathbf{M}_j, K+1) - \bar f_{j+1}(\mathbf{M}_j, K)\Big\}^2\Big]\;.$$

By Lemma 5.4 and Schwarz inequality, this expression is bounded above by

$$
\begin{aligned}
& 4 \sum_{0\le M<N} m^1_\rho(M) m^1_\rho(N) E_{m_\rho}\Big[\Big\{ \sum_{K=M}^{N-1} E_{\nu_{(\mathbf{M}_j,K)}}\Big[\frac{1}{|\Lambda_\ell|} \sum_{x\in\Omega_{j+1}} \frac{\partial f}{\partial \eta_x} h(\eta_x)\Big]\Big\}^2\Big] \\
& \quad + 4 \sum_{0\le M<N} m^1_\rho(M) m^1_\rho(N) E_{m_\rho}\Big[\Big\{ \sum_{K=M}^{N-1} E_{\nu_{(\mathbf{M}_j,K)}}\Big[f ; \frac{1}{|\Lambda_\ell|} \sum_{x\in\Omega_{j+1}} h(\eta_x)\Big]\Big\}^2\Big] \, .
\end{aligned} \tag{5.9}
$$

We shall estimate these two terms separately.

By Schwarz inequality, the expectation appearing in the first term of (5.9) is bounded above by

$$
\begin{aligned}
& \frac{C(\epsilon)}{|\Lambda_\ell|^2}(N-M) \sum_{K=M}^{N-1} E_{m_\rho}\Big[E_{\nu_{(\mathbf{M}_j,K)}}\Big[\Big\{ \sum_{x\in\Omega_{j+1}} \Big|\frac{\partial f}{\partial \eta_x}\Big|\Big\}^2\Big]\Big] \\
& \qquad = \frac{C(\epsilon)}{|\Lambda_\ell|^2}(N-M) \sum_{K=M}^{N-1} E_{\nu_\rho^{\Omega_{j+1},K}}\Big[\Big\{ \sum_{x\in\Omega_{j+1}} \Big|\frac{\partial f}{\partial \eta_x}\Big|\Big\}^2\Big]
\end{aligned}
$$

because h is bounded by ϵ^{-2}. Here $\nu_\rho^{\Omega_{j+1},K}$ is the probability measure on $\mathbb{N}^{\mathbb{Z}^d}$ given by

$$
\nu_\rho^{\Omega_{j+1},K} \;=\; \nu_{\Omega_{j+1},K} \otimes \nu_\rho^{\Omega^c_{j+1}} \, . \tag{5.10}
$$

Therefore, the first term in (5.9) is less than or equal to

$$
\frac{C(\epsilon)}{|\Lambda_\ell|} \sum_{K\ge 0} m_\rho(K) E_{\nu_\rho^{\Omega_{j+1},K}}\Big[\Big\{ \sum_{x\in\Omega_{j+1}} \Big|\frac{\partial f}{\partial \eta_x}\Big|\Big\}^2\Big] \frac{1}{|\Lambda_\ell|} \sum_{M\le K<N} \frac{m^1_\rho(M) m^1_\rho(N)}{m^1_\rho(K)} (N-M) \, .
$$

The second summation is carried over all pairs $\{M,N\}$, while K is fixed. By Lemma ♣qy3 below, this expression is bounded above by

$$
\frac{C(\epsilon,\rho)}{|\Lambda_\ell|} \sum_{K\ge 0} m_\rho(K) E_{\nu_\rho^{\Omega_{j+1},K}}\Big[\Big\{ \sum_{x\in\Omega_{j+1}} \Big|\frac{\partial f}{\partial \eta_x}\Big|\Big\}^2\Big] \;=\; \frac{C(\epsilon,\rho)}{|\Lambda_\ell|} \Big\| \sum_{x\in\Omega_{j+1}} \Big|\frac{\partial f}{\partial \eta_x}\Big| \Big\|_2^2 \, .
$$

We now turn to the second term of (5.9). In order to estimate it, observe that since $\sum_{x\in\Omega_{j+1}} h(\eta_x)$ depends on the configuration η only through the sites in Ω_{j+1}, the covariance

$$
E_{\nu_{(\mathbf{M}_j,K)}}\Big[f ; \frac{1}{|\Lambda_\ell|} \sum_{x\in\Omega_{j+1}} h(\eta_x)\Big] \;=\; E_{\nu_{(\mathbf{M}_j,K)}}\Big[F_{j+1} ; \frac{1}{|\Lambda_\ell|} \sum_{x\in\Omega_{j+1}} h(\eta_x)\Big] \, ,
$$

where F_{j+1} is the expectation of f given $\mathbf{M}_j$ and $\{\eta_x, \, x\in\Omega_{j+1}\}$:

$$
F_{j+1}(\mathbf{M}_j, \{\eta_x, \, x\in\Omega_{j+1}\}) \;=\; E_{\nu_\rho}\Big[f \,\Big|\, \mathbf{M}_j, \{\eta_x, \, x\in\Omega_{j+1}\}\Big] \, .
$$

Since $\nu_{(\mathbf{M}_j,K)}$ is a product measure and both $F_{j+1}(\mathbf{M}_j,\cdot)$ and $\sum_{x\in\Omega_{j+1}} h(\eta_x)$ are measurable with respect to $\sigma\{\eta_x,\, x\in\Omega_{j+1}\}$, the previous covariance is equal to

$$E_{\nu_{\Omega_{j+1},K}}\Big[F_{j+1}(\mathbf{M}_j,\cdot); \frac{1}{|\Lambda_\ell|}\sum_{x\in\Omega_{j+1}} h(\eta_x)\Big]\ .$$

By Schwarz inequality and the spectral gap for the generator of the zero range process restricted to a cube of size ℓ, this last covariance is bounded by

$$\ell\Big\{R_0 D_{\Omega_{j+1}}(\nu_{\Omega_{j+1},K}, F_{j+1}(\mathbf{M}_j,\cdot))\; E_{\nu_{\Omega_{j+1},K}}\Big[\frac{1}{|\Lambda_\ell|}\sum_{x\in\Omega_{j+1}} h(\eta_x); \frac{1}{|\Lambda_\ell|}\sum_{x\in\Omega_{j+1}} h(\eta_x)\Big]\Big\}^{1/2}$$

for some universal constant C. In Lemma ♣qy5 below, we shall prove that the variance of the average of translations of h is bounded above by $C|\Lambda_\ell|^{-1}$ for some universal constant C. In this formula, $D_{\Omega_{j+1}}(\nu_{\Omega_{j+1},K}, F_{j+1}(\mathbf{M}_j,\cdot))$ stands for the Dirichlet form of $F_{j+1}(\mathbf{M}_j,\cdot)$ with respect to the measure $\nu_{\Omega_{j+1},K}$ as defined in (2.4). A simple computation shows that this Dirichlet form $D_{\Omega_{j+1}}(\nu_{\Omega_{j+1},K}, F_{j+1}(\mathbf{M}_j,\cdot))$ is bounded above by the Dirichlet form of f with respect to $\nu_{(\mathbf{M}_j,K)}$ on Ω_{j+1} :

$$D_{\Omega_{j+1}}(\nu_{\Omega_{j+1},K}, F_{j+1}(\mathbf{M}_j,\cdot)) \;\le\; D_{\Omega_{j+1}}(\nu_{(\mathbf{M}_j,K)}, f)\ .$$

Therefore, by Schwarz inequality, the second term of (5.9) is bounded above by

$$\begin{aligned} 4R_0\ell^2 &\sum_{0\le M<N} m^1_\rho(M)m^1_\rho(N)\frac{N-M}{|\Lambda_\ell|}E_{m_\rho}\Big[\sum_{K=M}^{N-1} D_{\Omega_{j+1}}(\nu_{(\mathbf{M}_j,K)}, f)\Big] \\ &= 4R_0\ell^2 \sum_{0\le M<N} m^1_\rho(M)m^1_\rho(N)\frac{N-M}{|\Lambda_\ell|}\sum_{K=M}^{N-1} D_{\Omega_{j+1}}(\nu_\rho^{\Omega_{j+1},K}, f)\ .\end{aligned}$$

In this last formula, $\nu_\rho^{\Omega_{j+1},K}$ is the measure defined by (5.10). We may rewrite this last sum as

$$4R_0\ell^2\sum_{K\ge 0} m^1_\rho(K) D(\nu_\rho^{\Omega_{j+1},K}, f)\frac{1}{|\Lambda_\ell|}\sum_{0\le M\le K<N}\frac{m^1_\rho(M)m^1_\rho(N)}{m^1_\rho(K)}(N-M)\ .$$

Here, as before, the second sum is carried over all pairs $\{M,N\}$. In virtue of Lemma ♣qy3 below, this expression is bounded above by

$$C(R_0,\rho)\ell^2\sum_{K\ge 0} m^1_\rho(K) D(\nu_\rho^{\Omega_{j+1},K}, f) \;=\; C(R_0,\rho)\ell^2 D_{\Omega_{j+1}}(\nu_\rho, f)\ .$$

This concludes the proof of the lemma. $\square$

We are now ready to prove Proposition 5.2.

Proof of Proposition 5.2. By Lemma 5.3 and 5.5, $E_{m_\rho}[\bar{f}(\mathbf{M})^2]$ is bounded above by

$$C\sum_{j\ge 0}\Big\{\frac{1}{|\Lambda_\ell|}\Big\|\sum_{x\in\Omega_{j+1}}\Big|\frac{\partial f}{\partial\eta_x}\Big|\Big\|_2^2 \;+\; \ell^2 D_{\Omega_{j+1}}(\nu_\rho^{\Omega_{j+1}},f)\Big\}$$

for some constant C depending only on a, A, R_0 and ρ. Since this expression is trivially bounded by

$$\frac{C}{|\Lambda_\ell|}\Big\|\sum_{j\ge1}\sum_{x\in\Omega_j}\Big|\frac{\partial f}{\partial\eta_x}\Big|\Big\|_2^2 \;+\; C\ell^2 D(\nu_\rho,f) \;\le\; \frac{C}{|\Lambda_\ell|}\Big\{\sum_{x\in\mathbb{Z}^d}\Big\|\frac{\partial f}{\partial\eta_x}\Big\|_2\Big\}^2 \;+\; C\ell^2 D(\nu_\rho,f)\;,$$

the proposition is proved. $\square$

We conclude this section deriving two estimates used in the proof of Lemma 5.5.

Lemma 5.6 . *Recall the definition of the probability measures m_ρ. For every $\rho>0$, there exists a finite constant $C=C(\rho)$ such that*

$$\limsup_{\ell\to\infty}\;\sup_K\frac{1}{|\Lambda_\ell|}\sum_{M\le K<N}\frac{m_\rho^1(M)m_\rho^1(N)}{m_\rho^1(K)}\{N-M\}\;\le\;C\;.$$

Proof. By definition, for each positive integer M, $m_\rho^1(M)=\nu_\rho\{\sum_{x\in\Lambda_\ell}\eta_x=M\}$. In particular, by the local central limit theorem for i.i.d. random variables, for each $K\ge0$ and ℓ large enough,

$$\begin{aligned}&\frac{1}{|\Lambda_\ell|}\sum_{M\le K<N}\frac{m_\rho^1(M)m_\rho^1(N)}{m_\rho^1(K)}\{N-M\}\\ &\quad\le\;\frac{C}{|\Lambda_\ell|}\sum_{M\le K<N}\frac{\exp\Big\{-\frac{(M-\rho|\Lambda_\ell|)^2}{2|\Lambda_\ell|\sigma(\rho)^2}\Big\}\exp\Big\{-\frac{(N-\rho|\Lambda_\ell|)^2}{2|\Lambda_\ell|\sigma(\rho)^2}\Big\}}{\exp\Big\{-\frac{(K-\rho|\Lambda_\ell|)^2}{2|\Lambda_\ell|\sigma(\rho)^2}\Big\}}\frac{N-M}{|\Lambda_\ell|^{1/2}\sigma(\rho)}\end{aligned}$$

for some finite constant C depending only on ρ. In this formula, $\sigma(\rho)^2$ stands for the variance of η_0 under ν_ρ. Denote $K-\rho|\Lambda_\ell|$ by $B=B(\rho,\ell,K)$ and assume, without loss of generality, that $B>0$. Changing variables $\tilde M=M-\rho|\Lambda_\ell|$, $\tilde N=N-\rho|\Lambda_\ell|$, we may rewrite the last expression as

$$\begin{aligned}&\frac{C}{|\Lambda_\ell|}e^{B^2/2|\Lambda_\ell|\sigma(\rho)^2}\sum_{M\le B<N}e^{-(M^2+N^2)/2|\Lambda_\ell|\sigma(\rho)^2}\frac{N-M}{|\Lambda_\ell|^{1/2}\sigma(\rho)}\\ &\quad\le\;\frac{C}{|\Lambda_\ell|}e^{B^2/2|\Lambda_\ell|\sigma(\rho)^2}\sum_{M\le B<N}e^{-(M^2+N^2)/2|\Lambda_\ell|\sigma(\rho)^2}\;.\end{aligned}$$

Here we adopted the convention that the constant C may change of value from line to line. This last sum may be rewritten as

$$
\begin{aligned}
&\frac{C}{|\Lambda_\ell|^{1/2}} \sum_{M\le B} e^{-M^2/2|\Lambda_\ell|\sigma(\rho)^2} \frac{1}{|\Lambda_\ell|^{1/2}} \sum_{B<N} e^{-(N^2-B^2)/2|\Lambda_\ell|\sigma(\rho)^2} \\
&\quad \le \frac{C}{|\Lambda_\ell|^{1/2}} \sum_{M\le B} e^{-M^2/2|\Lambda_\ell|\sigma(\rho)^2} \frac{1}{|\Lambda_\ell|^{1/2}} \sum_{0<N} e^{-N^2/2|\Lambda_\ell|\sigma(\rho)^2} .
\end{aligned}
$$

Since this expression converges to a finite constant as $\ell \uparrow \infty$, the lemma is proved. □

Lemma 5.7 . *For every bounded cylinder function H, there exists a finite constant C such that*

$$
E_{\nu_{\Lambda_\ell,K}}\Big[\frac{1}{|\Lambda_\ell|}\sum_{x\in\Lambda_\ell} H(\eta_x); \frac{1}{|\Lambda_\ell|}\sum_{x\in\Lambda_\ell} H(\eta_x)\Big] \;\le\; \frac{C}{|\Lambda_\ell|} \, .
$$

Proof. Under assumptions (H1) and (H2) on the rate jump $g(\cdot)$, [LSV] (cf. Theorem 6.1) proved a local central limit theorem, uniform over the parameter $K/|\Lambda_\ell|$ for the random variables $\sum_{x\in\Lambda_\ell}\eta_x$ under the measure ν_ρ. A uniform expansion for $E_{\nu_{\Lambda_\ell,K}}[u]$, where u is a bounded cylinder function, is easily deduced from this uniform local central limit theorem (cf. [LSV] Corollary 6.1 and [KL] Corollary AII.2.7) : There exists a finite constant C depending on $\|u\|_\infty$ and the size of the support of u such that

$$
\sup_{K\ge 0} \Big| E_{\nu_{\Lambda_\ell,K}}[u] - E_{\nu_{\rho(K,\ell)}}[u] \Big| \;\le\; \frac{C}{|\Lambda_\ell|} \, .
$$

In this formula $\rho(K,\ell) = K/|\Lambda_\ell|$. The proof of this statement can be found in [KL]. The lemma follows from this estimate and some simple algebraic computations. □

6. Appendix II. Symmetric simple exclusion process.

In this model particles are distributed on $\mathbb{Z}^d$ with at most one particle per site. Each particle performs a continuous time symmetric random walk with jump law $p(\cdot)$. However jumps to already occupied sites are excluded. The state space is $\{0,1\}^{\mathbb{Z}^d}$ and the generator is given by

$$
Lf(\eta) = \sum_{x,y} p(y-x)\,[\eta_x(1-\eta_y) + \eta_y(1-\eta_x)]\,(f(\sigma^{x,y}\eta) - f(\eta))
$$

where $(\sigma^{x,y}\eta)_x = \eta_y$, $(\sigma^{x,y}\eta)_y = \eta_x$ and $(\sigma^{x,y}\eta)_z = \eta_z$ if $z \neq x, y$. Note that the term $[\eta_x(1-\eta_y) + \eta_y(1-\eta_x)]$ can be dropped from this expression without changing the meaning. The invariant measures are product measures μ_ρ, $0 \le \rho \le 1$ with marginals $\mu_\rho(\eta_x = 1) = \rho$ and the process is reversible [generator is symmetric] with respect to this family of measures.

Fix $\rho \in [0,1]$ and let P_t denote the semigroup on $L^2(\mu_\rho)$ corresponding to the process. For $0 \le p \le \infty$ and $0 \le q < \infty$ define the following seminorms,

$$|||f|||_{p,q} = \left(\sum_{x \in \mathbb{Z}^d} \left\| \frac{\partial f}{\partial \eta_x} \right\|_p^q \right)^{1/q}$$

where $\|f\|_p$ is the standard L_p norm, $\|f\|_p = \left(E_{\mu_\rho}[|f|^p]\right)^{1/p}$. In the present context

$$\frac{\partial f}{\partial \eta_x} = f(\sigma^x \eta) - f(\eta)$$

where σ^x is given by $(\sigma^x\eta)_x = 1 - \eta_x$ and $(\sigma^x\eta)_y = \eta_y$, $y \neq x$.

Theorem 6.1 . *For any $\rho \in [0,1]$ and $q \in [1,2)$, there exists a finite positive constant $C_q(d)$ such that for all functions f,*

$$Var(\mu_\rho, [P_t f]) \le C_q(d) t^{-d(\frac{1}{q}-\frac{1}{2})} |||f|||_{2,q}^2 , \qquad \text{for all } t > 0. \tag{6.1}$$

Remark. The theorem was proved previously by Bertini and Zagarlinski using Nash type inequalities with the norm $|||\cdot|||_{\infty,q}$ on the right hand side. Note that this follows from the above since $|||f|||_{2,q} \le |||f|||_{\infty,q}$ always. In fact the stronger result above can also be obtained using Nash inequalities as in the previous section. However one should note that these methods work only for the symmetric simple exclusion process because it is only for this process that one has a uniform bound in time on these norms. Perhaps not coincidentally, for this process one has simple proofs of the decay estimate using coupling which we now present.

Proof of Theorem 6.1. Let f be a function on $\{0,1\}^{\mathbb{Z}^d}$, let $x_1, x_2, \dots$ be an enumeration of $\mathbb{Z}^d$, let $\mathcal{F}_k$ be the $\sigma-$ field genearated by $\eta_{x_k}, \eta_{x_{k+1}}, \dots$ and define $f_k = E_{\mu_\rho}[f \mid \mathcal{F}_k]$. Then

$$\mathrm{Var}(\mu_\rho, f) = E_{\mu_\rho}[\sum_{k=0}^{\infty} E_{\mu_\rho}[f_k^2 \mid \mathcal{F}_{k+1}] - (E_{\mu_\rho}[f_k \mid \mathcal{F}_{k+1}])^2].$$

So that for any such function we have

$$\mathrm{Var}(\mu_\rho, f) \le \sum_{x \in \mathbb{Z}^d} \left\| \frac{\partial f}{\partial \eta_x} \right\|_2^2 = |||f|||_{2,2}^2. \tag{6.2}$$

In fact this is just a statement of the well known spectral gap for the Glauber dynamics with respect to the product measure μ_ρ.

By definition

$$\frac{\partial [P_t f]}{\partial \eta_x}(\eta) = E_{\sigma^x \eta}[f(\eta(t))] - E_\eta[f(\eta(t))],$$

where $E_\eta[\cdot]$ denotes the expectation with respect to the process starting at η. Now consider the following coupling (η, η') of the symmetric simple exclusion process with itself. The η process starts at η and the η' process starts at $\sigma_x\eta$. The generator of the coupled process is

$$\mathbb{L}f(\eta,\eta') = \sum_{x,y} p(y-x)\left(f(\sigma^{x,y}\eta, \sigma^{x,y}\eta') - f(\eta,\eta')\right).$$

Let $\mathbb{E}_{(x,\eta)}$ denote expectation with respect to this coupled process. Note that in this coupling it is true that for all times η and η' differ at exactly one site, which we call $x(t)$, the position of the 'second class particle'. We have

$$E_{\sigma^x\eta}[f(\eta(t))] = \mathbb{E}_{(x,\eta)}[f(\sigma^{x(t)}(\eta(t)))]$$

so that for every f we have the following formula

$$\frac{\partial[P_t f]}{\partial \eta_x} = \mathbb{E}_{(x,\eta)}\left[\frac{\partial f}{\partial \eta_{x(t)}}(\eta(t))\right]. \tag{6.3}$$

Note that we have reparametrized the coupled process as $(x(t), \eta(t))$ where $(\eta(t), \eta'(t)) = (\eta(t), \sigma^{x(t)}\eta(t))$. The generator of the (x,η) process is simply computed as $\tilde{\mathbb{L}} = L_1 + L_2$ where L_1 is the generator of the symmetric simple exclusion process with jumps to x disallowed,

$$L_1 f(x,\eta) = \sum_{y,z\neq x} p(y-z)\left(f(x,\sigma^{y,z}\eta) - f(x,\eta)\right),$$

and L_2 corresponds to jumps involving the second class particle at x,

$$L_2 f(x,\eta) = \sum_y p(y-x)\left(f(y,\sigma^{x,y}\eta) - f(x,\eta)\right).$$

Let

$$a(x,\eta,t) = \frac{\partial[P_t f]}{\partial \eta_x}(\eta).$$

Then, by (6.3) we have $a(x,\eta,t) = \mathbb{E}_{(x,\eta)}[a(x(t),\eta(t),0)]$ and therefore

$$\partial_t a(x,\eta,t) = L_1 a(x,\eta,t) + L_2 a(x,\eta,t)\ .$$

Note that L_1 alone is symmetric with respect to the product measure μ_ρ. Note also that μ_ρ is invariant under the map $\sigma^{x,y}$ for any fixed x and y. Therefore if we take the expectation over η with respect to the measure μ_ρ, we have

$$\begin{aligned}\partial_t \sum_x E[(a(x,\eta))^2] = &-\sum_x \sum_{y,z\neq x} p(y-z) E\left[\left(a(x,\sigma^{y,z}\eta) - a(x,\eta)\right)^2\right] \\ &-\sum_{x,y} p(y-x) E\left[\left(a(y,\sigma^{x,y}\eta) - a(x,\eta)\right)^2\right].\end{aligned} \tag{6.4}$$

Here and below to keep notation simple we shall sometimes omit the time dependence of $a(x,\eta,t)$. By the triangle inequality,

$$\left\{E\left[\left(a(y,\sigma^{x,y}\eta)-a(x,\eta)\right)^2\right]\right\}^{1/2} \ge \left\{E\left[\left(a(y,\sigma^{x,y}\eta)\right)^2\right]\right\}^{1/2} - \left\{E\left[\left(a(x,\eta)\right)^2\right]\right\}^{1/2}.$$

Define

$$g_x(t) = \left\{E\left[\left(a(x,\eta,t)\right)^2\right]\right\}^{1/2} = \left\|\frac{\partial P_t f}{\partial \eta_x}\right\|_2.$$

Since our product measure is invariant under $\sigma^{x,y}$ we also have $\left\{E\left[\left(a(y,\sigma^{x,y}\eta)\right)^2\right]\right\}^{1/2} = g_y$. We have thus shown that

$$\partial_t \sum_x (g_x(t))^2 \le -\sum_{x,y} p(y-x)\left(g_y(t)-g_x(t)\right)^2$$

which we can write as

$$\partial_t \|P_t f\|_{2,2}^2 \le -\mathbb{D}(P_t f)\,, \tag{6.5}$$

where we have introduced the notation $\mathbb{D}(f) = \sum_{x,y} p(y-x)\left(g_y-g_x\right)^2$.

We now want to show that our semigroup is a contraction in $\|\cdot\|_{2,q}$ for each $q\in[1,2]$. Recall that $\|P_t f\|_{2,1} = \sum_x \left\{E[(a(x,\eta,t))^2]\right\}^{\frac12}$. As earlier

$$\begin{aligned}\partial_t \sum_x \left\{E[(a(x,\eta))^2]\right\}^{\frac12} =& \sum_x \left\{E[(a(x,\eta))^2]\right\}^{-\frac12}\Big\{-\sum_{y,z\neq x} p(y-z)E\left[(a(x,\sigma^{y,z}\eta)-a(x,\eta))^2\right]\\ &+\sum_y p(y-x)E\left[a(x,\eta)\left(a(y,\sigma^{x,y}\eta)-a(x,\eta)\right)\right]\Big\}.\end{aligned} \tag{6.6}$$

We drop the first term on the left hand side which is negative. By Schwarz's inequality the left hand side of (6.6) is dominated by

$$\sum_{x,y} p(y-x)\left[-\left\{E\left[(a(x,\eta))^2\right]\right\}^{1/2} + \left\{E\left[(a(y,\sigma^{x,y}\eta)^2\right]\right\}^{1/2}\right].$$

After changing variables $\sigma^{x,y}\eta \mapsto \eta$ one can see that this expression vanishes identically. This proves the contraction for $q=1$. For $q=2$ the contraction follows from (6.5). By the standard interpolation theorem it follows that for each $q\in[1,2]$,

$$\|P_t f\|_{2,q} \le \|f\|_{2,q}\,. \tag{6.7}$$

The well known Nash inequality on $\mathbb{Z}^d$ states that there exists a finite, positive constant C_q depending only on q, the jump law $p(\cdot)$ and the dimension d such that for all $g:\mathbb{Z}^d\to\mathbb{R}$,

$$\sum_x g_x^2 \le C_q\left(\sum_{x,y} p(y-x)\left(g_y-g_x\right)^2\right)^{\gamma}\left(\sum_x g_x^q\right)^{2(1-\gamma)/q}$$

where

$$\gamma = \left(\frac{1}{q} - \frac{1}{2}\right)\left(\frac{1}{d} + \frac{1}{q} - \frac{1}{2}\right)^{-1}.$$

Translating into an expression for f we obtain

$$|||f|||_{2,2}^2 \le C_q \mathbb{D}(f)^\gamma |||f|||_{q,2}^{2(1-\gamma)}. \tag{6.8}$$

Integrating (6.8), with (6.5), (6.7) and (6.2) we obtain the decay estimate.

Remarks. 1. If one is willing to use the norm $|||\cdot|||_{\infty,q}$ in the estimate (6.1) there is an even easier proof. Rewrite (6.3) as

$$\frac{\partial[P_t f]}{\partial \eta_x} = \sum_y \mathbb{E}_{(x,\eta)}\left[\frac{\partial f}{\partial \eta_y}(\eta(t)) \;\Big|\; x(t) = y\right] \mathbb{P}_{(x,\eta)}\left(x(t) = y\right).$$

For the symmetric simple exclusion the marginal distribution of the second class particle is a simple random walk, so

$$\mathbb{P}_{(x,\eta)}\left(x(t) = y\right) = p_t(y-x)$$

where $p_t(y-x)$ is the solution of the (discrete) heat equation $\partial_t p = \Delta_d p$ with the discrete Laplacian $\Delta_d f(x) = \sum_y p(y-x)(f(y) - f(x))$ and $p_0(y) = 1_{\{y=0\}}$. By the previous formula we have

$$\left\|\frac{\partial[P_t f]}{\partial \eta_x}\right\|_\infty \le \sum_y \left\|\frac{\partial f}{\partial \eta_y}\right\|_\infty p_t(x,y).$$

The result follows immediately from standard estimates for the heat kernel on $\mathbb{Z}^d$ which state that for $q \in [1,2)$ there exists a constant $C_q(d)$ so that for any function $a \in \ell^q(\mathbb{Z}^d)$,

$$\|a * p_t\|_{\ell^2}^2 \le t^{-d(\frac{1}{q}-\frac{1}{2})}\|a\|_{\ell^q}^2.$$

2. The coupling method descibed above for the symmetric simple exclusion model can also be applied, for example, to the zero range model, in the case that the model is attractive [$g(k)$ increasing in k in (1.1)]. However, the rates of the resulting second class particle depend in a nontrivial way on the process and therefore the method described in this section does not seem to apply to this setting.

Acknowledgements. J. Quastel and H. T. Yau would like to thank the very kind hospitality at IMPA, Rio de Janeiro where part of this work was done. C. Landim was partially supported by CNPq grant 300358/93-8 and PRONEX 41.96.0923.00 "Fenomenos Criticos em Probabilidade e Processos Estocasticos". H.T. Yau is partially supported by U.S. National Science Foundation grant 9403462 and David and Lucile Packard Foundation Fellowship.

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