”Economics has never been a science - and it is even less now than a few years ago.” Paul Samuelson

”Funeral by funeral, theory advances” Paul Samuelson

”Economics is extremely useful as a form of employment for economists.” J.K. Galbraith

”The only function of economic forecasting is to make astrology look respectable.”J.K. Galbraith

“In economics it is a far, far wiser thing to be right than to be consistent”J.K. Galbraith

Binomial Model

S_n = stock price at time n

$S_{n+1} = uS_n$ with probability p, $S_{n+1} = dS_n$ with probability $q = 1 - p$

$0 < d < u$ fixed

r = interest rate: $\$1 \to \$(1 + r)$ in time 1

$0 < d < 1 + r < u$ or it is not interesting

V = European call option Strike price K, maturity at time N

means $V_N = (S_N - K)_+$

Problem: $V_0 =?$

Arbitrage pricing theory

Start with X_0 dollars

You can have Δ_0 shares of stock (costs $\Delta_0 S_0$) and put the rest in the bank

At time 1 you have $X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0)$ which is either $X_1(u) = \Delta_0 u S_0 + (1+r)(X_0 - \Delta_0 S_0)$ with prob p

or
$X_1(d) = \Delta_0 d S_0 + (1+r)(X_0 - \Delta_0 S_0)$ with prob q

Because it is just two equations in two unknowns, you can choose X_0, Δ_0 so that $X_1 = V_1$

$$\Delta_0 = \frac{V_1(u) - V_1(d)}{S_1(u) - S_1(d)} \qquad X_0 = \frac{1}{1+r}[\tilde{p} V_1(u) + \tilde{q} V_1(d)]$$

$$\tilde{p} = \frac{1+r-d}{u-d} \qquad \tilde{q} = \frac{u-1-r}{u-d}$$

Arbitrage pricing theory says that $X_0 = V_0$

We know $V_N = (S_N - K)_+$

We now have $V_{N-1} = \tilde{E}[\frac{1}{1+r} V_N] = \tilde{E}[\frac{1}{1+r}(S_N - K)_+]$

$V_{N-2} = \tilde{E}[(\frac{1}{1+r})^2 (S_N - K)_+ \mid S_{N-2}]$

$\tilde{E}$ is with respect to random walk S_n with $S_{n+1} = uS_n$ with probability $\tilde{p}$ and $S_{n+1} = dS_n$ with probability $\tilde{q}$

$$V_0 = \tilde{E}[(\frac{1}{1+r})^N (S_N - K)_+ \mid S_0]$$

Brownian motion as limit of random walks

$Y_1, Y_2, \ldots$ independent with $P(Y_i = 1) = P(Y_i = -1) = 1/2$

$$M_n = Y_1 + \cdots + Y_n$$

$$B^{(n)}(t) = \frac{M_{\lfloor nt \rfloor}}{\sqrt{n}}$$

Takes step $\pm 1/\sqrt{n}$ at times $1/n, 2/n, \ldots$
$B^{(n)}(t)$ converges to Brownian motion
$(B^{(n)}(t_1), \ldots, B^{(n)}(t_k)) \stackrel{n\to\infty}{\longrightarrow}$ k dimensional Gaussian
Covariance $E[B^{(n)}(t)B^{(n)}(s)] \to t \wedge s$

Diffusion with variance σ^2 and drift μ as limit of random walks

$X^{(n)}(t)$ jumps at time increments $1/n$ either $\frac{\sigma}{\sqrt{n}} + \frac{\mu}{n}$ or $-\frac{\sigma}{\sqrt{n}} + \frac{\mu}{n}$ with probability $1/2$

$X^{(n)}(t) - \frac{\mu}{n}\lfloor nt \rfloor = \sigma B^{(n)}(t)$

So $X^{(n)}(t) \to X(t) = \sigma B(t) + \mu t$ or $dX = \sigma dB + \mu dt$

Geometric Brownian motion as limit of Binomial model

For each $n = 1, 2, 3, \ldots$ consider a Binomial model $S^{(n)}$ with $u_n = 1 + \frac{\sigma}{\sqrt{n}}$ and $d_n = 1 - \frac{\sigma}{\sqrt{n}}$ and $r_n = 0$ and step size $1/n$

$$\tilde{p} = \frac{1 + r_n - d_n}{u_n - d_n} = \frac{1}{2} = \tilde{q}$$

$X^{(n)}(t) = \log S^{(n)}(t)$ jumps at time increments $1/n$ either $\log u_n$ or $\log d_n$, each with probability $1/2$

$\log u_n = \log(1 + \frac{\sigma}{\sqrt{n}}) = \frac{\sigma}{\sqrt{n}} + \frac{\sigma^2}{2n} + o(\frac{1}{n})$
$\log d_n = \log(1 - \frac{\sigma}{\sqrt{n}}) = -\frac{\sigma}{\sqrt{n}} + \frac{\sigma^2}{2n} + o(\frac{1}{n})$

$X^{(n)}(t) \to \sigma B(t) - \frac{1}{2}\sigma^2 t$

$S^{(n)}(t) \to S(t) = e^{\sigma B(t) - \frac{1}{2}\sigma^2 t}$ Geometric Brownian motion $dS = \sigma S dB$

If $r_n = \frac{r}{n}$ get $dS = \sigma S dB + rSdt$

Option pricing formula

$$V_0 = \tilde{E}[(1+\frac{r}{n})^{-\lfloor nT \rfloor}(S_{\lfloor nT \rfloor} - K)_+]$$

becomes

$$V_0 = E[e^{-rT}(S_T - K)_+] \qquad dS = \sigma S dB + rSdt$$

At time $t \in [0, T]$

$$V(t) = E[e^{-r(T-t)}(S_T - K)_+ \mid \mathcal{F}_t]$$

But we can *compute* transition probabilities for S_t

$$p(t,x,y)dy = P(xe^{\sigma B(t)+(r-\frac{1}{2}\sigma^2)t} \in dy) = \frac{1}{\sigma y\sqrt{2\pi t}} e^{-\frac{(\log\frac{y}{x}-(r-\frac{1}{2}\sigma^2)t)^2}{2t\sigma^2}} dy$$

$$V(t) = \int e^{-r(T-t)}(y-K)_+ p(T-t, S_t, y)dy$$

$$= S(t)\Phi\left(\frac{\log\frac{S_t}{K}+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) - e^{-r(T-t)}K\Phi\left(\frac{\log\frac{S_t}{K}+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \qquad \Phi(x) = \int_{-\infty}^{x} e^{-y^2/2}\frac{dy}{\sqrt{2\pi}}$$

Risk Neutral Valuation

We have a stock and a risk free bond

Stock price $dS(t) = S(t)(\sigma(t)dB(t) + \mu(t)dt)$ Hold $\Delta(t)$ stock at time t

wealth process assumed to satisfy

$$dX(t) = \Delta(t)dS(t) + r(t)(X(t) - \Delta(t)S(t))dt$$

Note that this does *not* follow directly from differentiating equation for the wealth. The extra term is set to be 0 and called *self-financing condition*

The *risk-neutral measure* is a probability measure $\tilde{P}$ on $(C([0,T]), \mathcal{F}_t)$ equivalent to P under which all tradable assets are martingales after discounting by $e^{-\int_0^t r(s)ds}$

If there is a risk neutral measure then the value at time t of claim paying asset $V(T)$ at time T is

$$V(t) = \tilde{E}[e^{-\int_t^T r(s)ds} V(T) \mid \mathcal{F}_t]$$

To find the risk neutral measure we write

$$d(e^{-\int_0^t r(s)ds} S(t)) = e^{-\int_0^t r(s)ds} \sigma(t) S(t)(\theta(t)dt + dB(t)) \quad \theta(t) = \frac{\mu(t) - r(t)}{\sigma(t)}$$

Define $\tilde{B}(t) = \int_0^t \theta(s)ds + B(t)$

$$d(e^{-\int_0^t r(s)ds} S(t)) = e^{-\int_0^t r(s)ds} \sigma(t) S(t) d\tilde{B}(t)$$

define for Borel sets $A \in C([0, T])$,

$$\tilde{P}(A) = \int_A Z(T)dP \qquad Z(T) = e^{-\int_0^T \theta(t)dB(t) - \frac{1}{2}\int_0^T \theta^2(t)dt}$$

By Cameron-Martin-Girsanov $\tilde{B}(t)$ is a Brownian motion under $\tilde{P}$ and all tradable assets in our model are martingales under $\tilde{P}$, so $\tilde{P}$ is the risk neutral measure

European call

$$V(T) = (S(T) - K)_+$$

$$V(0) = u(T, S) = \tilde{E}_S[e^{-\int_0^T r(t)dt}(S(T) - K)_+]$$

$$\frac{\partial u}{\partial T} = \frac{1}{2}\sigma^2 S^2 \frac{\partial u}{\partial S^2} + rS\frac{\partial u}{\partial S} - ru \qquad u(0, S) = (S - K)_+$$

$$V(0) = u(T, S(0))$$

Constructing the hedge

Let

$$M(t) = \tilde{E}[e^{-\int_0^T r(s)ds} V(T) \mid \mathcal{F}_t]$$

$M(t)$ is a martingale with respect to $\tilde{E}$. By the martingale

representation theorem there exists a progressively measurable $\gamma(t)$ so that

$$M(t) = M(0) + \int_0^t \gamma(s) d\tilde{B}(s)$$

$$\Delta(t) = e^{\int_0^t r(s)ds} \frac{\gamma(t)}{\sigma(t)S(t)}$$

$$X(t) = e^{\int_0^t r(s)ds} M(t)$$

$$dX(t) = \Delta(t)dS(t) + r(t)(X(t) - \Delta(t)S(t))dt \qquad X(T) = V(T)$$

So we have constructed a hedge

Interest rate models

An easy model for interest rates is

$$dr(t) = (\theta - \alpha r(t))dt + \sigma dB(t)$$

called mean reverting . This is called *Vasicek model* although it is just Ornstein-Uhlenbeck process $+\theta$. If coefficients are nonrandom functions of time it is called *Hull-White*
Discount function

$$Z_{t,T} = E[e^{-\int_t^T r(s)ds} \mid \mathcal{F}_t] \qquad Z_{t,T} = Z_{t,T}(r(t))$$

$$\frac{\partial Z}{\partial T} = (\theta - \alpha r)\frac{\partial Z}{\partial r} + \sigma^2 \frac{1}{2}\frac{\partial Z}{\partial r^2} - rZ$$

explicit solution in Hull-White case

Example. Cox-Ingersol-Ross model

The interest rate $r(t)$ is assumed to satisfy the equation

$$dr(t) = (\alpha - \beta r(t))dt + \sigma\sqrt{r(t)}dB(t).$$

Note that the Lipschitz condition is not satisfied, but existence/uniqueness holds by the stronger theorem we did not prove