Existence and Uniqueness Theorem

\( \sigma : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}^{d \times d}, \ b : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}^d \) be Borel measurable, \( \exists A < \infty, \) 

\[ \| \sigma(x, t) \| + |b(x, t)| \leq A(1 + |x|) \quad x \in \mathbb{R}^d, \ 0 \leq t \leq T \]

and Lipschitz;

\[ \| \sigma(x, t) - \sigma(y, t) \| + |b(x, t) - b(y, t)| \leq A|x - y|. \]

\( x_0 \in \mathbb{R}^d \) indep of \( B_t, \ E[|x_0|^2] < \infty. \)

Then there exists a unique solution \( X_t \) on \([0, T]\) to

\[ dX_t = b(X_t, t)dt + \sigma(X_t, t)dB_t, \quad X_0 = x_0 \]

and \( E[\int_0^T |X_t|^2dt] < \infty. \)

Uniqueness means that if \( X^1_t \) and \( X^2_t \) are two solutions then

\[ P(X^1_t = X^2_t, \ 0 \leq t \leq T) = 1 \]
Proof of Uniqueness

Suppose $X^1_t$ and $X^2_t$ are solutions

$$X^1_t - X^2_t = \int_0^t (b(X^1_s, s) - b(X^2_s, s))ds + \int_0^t (\sigma(X^1_s, s) - \sigma(X^2_s, s))dB_s$$

$$+ x^1_0 - x^2_0$$

$$E[|X^1_t - X^2_t|^2] \leq 4E[|\int_0^t (b(X^1_s, s) - b(X^2_s, s))ds|^2]$$

$$+ 4E[|\int_0^t (\sigma(X^1_s, s) - \sigma(X^2_s, s))dB_s|^2] + 4E[|x^1_0 - x^2_0|^2]$$

$$E[|\int_0^t (b(X^1_s, s) - b(X^2_s, s))ds|^2] \leq A^2 \int_0^t E[|X^1_s - X^2_s|^2]ds$$

$$E[\int_0^t |\sigma(X^1_s, s) - \sigma(X^2_s, s)|^2ds] = E[|\int_0^t (\sigma(X^1_s, s) - \sigma(X^2_s, s))ds|^2]$$

$$\leq A^2 \int_0^t E[|X^1_s - X^2_s|^2]ds$$
Call $\phi(t) = E[|X^1_t - X^2_t|^2]$

$$\phi(t) \leq 8A^2 \int_0^t \phi(s)ds + 4\phi(0)$$

$$\Phi(t) = \int_0^t \phi(s)ds$$

$$(e^{-8A^2t}\Phi(t))' = (\Phi'(t) - 8A^2\Phi(t))e^{-t} \leq 4\phi(0)e^{-t}$$

$$e^{-8A^2t}\Phi(t) \leq 4\phi(0)$$

$$\phi(t) \leq 8A^2\Phi(t) \leq 4e^{8A^2t}\phi(0)$$

$$E[|X^1_t - X^2_t|^2] \leq 4e^{8A^2t}E[|x_0^1 - x_0^2|^2]$$

For each $0 \leq t \leq T$, $X^1_t = X^2_t$ a.s. so $X^1_t = X^2_t$ for all rational $t \in [0, T]$ a.s. By continuity this implies that $X^1_t = X^2_t$ for all $t \in [0, T]$ a.s.
Proof of Existence

\[ X_0(t) \equiv x_0 \]

\[ X_n(t) = x_0 + \int_0^t \sigma(s, X_{n-1}(s)) dB(s) + \int_0^t b(s, X_{n-1}(s)) ds \]

Doob's inequality

\[ E[ \sup_{0 \leq t \leq T} |X_n(t) - X_{n-1}(t)|^2 ] \]

\[ \leq 4 E[ \int_0^T \| \sigma(s, X_{n-1}(s)) - \sigma(s, X_{n-2}(s)) \|^2 ds ] \]

\[ + TE[ \int_0^T |b(s, X_{n-1}(s)) - b(s, X_{n-2}(s)) |^2 ds ] \]

\[ \leq C \int_0^T E[|X_{n-1}(s) - X_{n-2}(s)|^2] ds \]

\[ \leq CE[ \sup_{0 \leq t \leq T} |X_{n-1}(t) - X_{n-2}(t)|^2 ] \]
Proof.

\[
E\left[ \sup_{0 \leq t \leq T} |X_n(t) - X_{n-1}(t)|^2 \right] \leq CE\left[ \sup_{0 \leq t \leq T} |X_{n-1}(t) - X_{n-2}(t)|^2 \right]
\]

\[
E\left[ \sup_{0 \leq t \leq T} |X_n(t) - X_{n-1}(t)|^2 \right] \leq \frac{(CT)^n}{n!}
\]

\[
P\left( \sup_{0 \leq t \leq T} |X_n(t) - X_{n-1}(t)| > \frac{1}{2^n} \right) \leq 2^{2n} E\left[ \sup_{0 \leq t \leq T} |X_n(t) - X_{n-1}(t)|^2 \right]
\]

summable

Borel – Cantelli \quad \Rightarrow \quad P\left( \sup_{0 \leq t \leq T} |X_n(t) - X_{n-1}(t)| > \frac{1}{2^n} \text{ i.o.} \right) = 0.

Hence for almost every \( \omega \), \( X_n(t) = X_0(t) + \sum_{j=0}^{n-1} (X_{j+1}(t) - X_j(t)) \) converges uniformly on \([0, T]\) to a limit \( X(t) \) which solves the required stochastic integral equation.
Lipschitz condition is \textit{not} necessary

\textbf{Theorem}

Let \( d = 1 \) and

\[
|b(t, x) - b(t, y)| \leq C|x - y| \\
|\sigma(t, x) - \sigma(t, y)| \leq C|x - y|^{\alpha}, \quad \alpha \geq 1/2
\]

Then there exists a solution of \( dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \) and it is unique

\textbf{Example}

\( \sigma(x) = \text{sgn}(x) \) and \( dX = \sigma(B)dB \) \textit{Not a stochastic differential equation}

But \( X \) is a Brownian motion \( dB = \sigma(B)dX \textit{ is a stochastic differential equation} \)

But also \( d(-B) = \sigma(-B)dX \) so no uniqueness
Markov property

$X_t$ can be obtained by solving the stochastic differential equation up to time $s < t$ and then solving in $[s, t]$ with initial condition $X_s$
By uniqueness this gives the same answer
Define the transition probability

$$p(s, x, t, A) = P(X_t^{s,x} \in A)$$

where $X_t^{s,x}$ is the solution starting at $x$ at time $s$
From the construction we have

$$P(X_t^{0,x} \in A \mid \mathcal{F}_s) = p(s, X_s^{0,x}, t, A)$$

which is the Markov property
A diffusion is a Markov process with transition probabilities \( p(s, x, t, dy) \) satisfying, for each \( \delta > 0 \) as \( h \to 0 \),

\[
\begin{align*}
  i. & \quad \frac{1}{h} \int_{|y-x| \geq \delta} p(t, x, t+h, dy) \to 0 \quad \Rightarrow \text{continuous paths} \\
  ii. & \quad \frac{1}{h} \int_{|y-x| < \delta} (y-x)p(t, x, t+h, dy) \to b(t, x) \\
  iii. & \quad \frac{1}{h} \int_{|y-x| < \delta} (y_i-x_i)(y_j-x_j)p(t, x, t+h, dy) \to a_{ij}(t, x)
\end{align*}
\]
Formal derivation of the backward equation

\[ p(s, x, t, A) = \int p(s, x, s + h, dy)p(s + h, y, t, A) \]

\[ 0 = \int p(s, x, s + h, dy) \left\{ p(s + h, y, t, A) - p(s, x, t, A) \right\} \]

\[ 0 = \int p(s, x, s + h, dy) \left\{ h \frac{\partial p(s, x, t, A)}{\partial s} + \sum_{i=1}^{d} (y_i - x_i) \frac{\partial p(s, x, t, A)}{\partial x_i} \right. \]

\[ + \frac{1}{2} \sum_{i,j=1}^{d} (y_i - x_i)(y_j - x_j) \frac{\partial^2 p(s, x, t, A)}{\partial x_i \partial x_j} + \cdots \right\} \]

\[ - \frac{\partial p(s, x, t, A)}{\partial s} = \sum_{i=1}^{d} b_i(t, x) \frac{\partial p(s, x, t, A)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(t, x) \frac{\partial^2 p(s, x, t, A)}{\partial x_i \partial x_j} \]
Proof.

$f(x)$ smooth

\[-\frac{\partial}{\partial s} u = L_s u \quad 0 \leq s < t \quad u(t, x) = f(x)\]

Ito’s formula: $u(s, X(s))$ martingale up to time $t$

\[u(s, x) = E_{s,x}[u(s, X(s))] = E_{s,x}[u(t, X(t))] = \int f(z)p(s, x, t, dz)\]

Let $f_n(z)$ smooth functions tending to $\delta(y - z)$

\[u(s, x) = p(s, x, t, y) \quad \text{if} \quad -\frac{\partial}{\partial s} u = L_s u \quad 0 \leq s < t \quad u(t, x) = \delta(x - y)\]
Existence result from PDE

Suppose that $a(t, x)$ and $b(t, x)$ are bounded and that there are $\alpha > 0$, $\gamma \in (0, 1]$, $C < \infty$ such that for all $s, t \geq 0, x, y \in \mathbb{R}^d$,

i. $\xi^T a(t, x)\xi \geq \alpha |\xi|^2, \quad \xi \in \mathbb{R}^d$,

ii. $\|a(s, x) - a(t, y)\| + |b(s, x) - b(t, y)| \leq C(|x - y|^\gamma + |t - s|^\gamma)$.

Then the backward equation has a solution and furthermore

$$p(s, x, t, A) = \int_A p(s, x, t, y)dy$$

with $p(s, x, t, y) \geq 0$ jointly continuous in $s, x, t, y$. Furthermore, $p(s, x, t, y)$ is the unique weak solution of the forward equation, i.e.

$$\int f(t, y)p(s, x, t, y)dy - f(s, x) = \int_s^t \int \{\partial_u + \mathcal{L}\} f(u, y)p(s, x, u, y)dydu$$
The solution $X_t$, $t \geq 0$ of $dX_t = b(X_t)dt + \sigma(X_t)dB_t$ with $X_0 = x$ is a Markov process with infinitesimal generator

$$L = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i}, \quad a = \sigma \sigma^*.$$ 

$$f(t, X_t) = f(0, X_0) + \int_0^t \left\{ \partial_s f(s, X_s) + Lf(s, X_s) \right\} ds$$

$$+ \int_0^t \sum_{i,j=1}^{d} \sigma_{ij}(s, X_s) \frac{\partial}{\partial x_i} f(s, X_s) dB^i_s$$
Example. Brownian motion $d = 1$

$$\mathcal{L} = \frac{1}{2} \frac{\partial^2}{\partial x^2}$$

**Forward**

$$\frac{\partial p(s, x, t, y)}{\partial t} = \frac{1}{2} \frac{\partial^2 p(s, x, t, y)}{\partial y^2}, \quad t > s$$

$$p(s, x, s, y) = \delta(y - x)$$

**Backward**

$$- \frac{\partial p(s, x, t, y)}{\partial s} = \frac{1}{2} \frac{\partial^2 p(s, x, t, y)}{\partial x^2}, \quad s < t,$$

$$p(t, x, t, y) = \delta(y - x)$$
Example. Ornstein-Uhlenbeck Process

\[ \mathcal{L} = \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} - \alpha x \frac{\partial}{\partial x} \]

**Forward**

\[
\frac{\partial p(s, x, t, y)}{\partial t} = \frac{1}{2} \frac{\partial^2 p(s, x, t, y)}{\partial y^2} + \frac{\partial}{\partial y} (\alpha y p(s, x, t, y), \quad t > s,
\]

\[ p(s, x, s, y) = \delta(y - x) \]

**Backward**

\[
-\frac{\partial p(s, x, t, y)}{\partial s} = \frac{1}{2} \frac{\partial^2 p(s, x, t, y)}{\partial x^2} - \alpha x \frac{\partial p(s, x, t, y)}{\partial x}, \quad s < t,
\]

\[ p(t, x, t, y) = \delta(y - x) \]