SIMPLE HOMOTOPY EQUIVALENCE AND WHITEHEAD GROUPS

ARTHUR LEI QIU

1. Introduction

If you ask a student taking an introductory topology course what the right notion of isomorphism between topological spaces is, they might respond with "homeomorphism" if they've paid attention in class. If you ask the same question to an algebraic topologist, they might instead choose homotopy equivalence. Within the world of finite CW complexes, there is a third notion of isomorphism intermediate to these two which is often useful: that of *simple homotopy equivalence*.

Roughly speaking, a simple homotopy equivalence of two CW complexes is a map between them which is given by a finite sequence of "expansions" and "collapses" of cells (to be defined more rigorously in a moment). Since the operations of "expanding" and "collapsing" undo each other, it is easy to describe a homotopy inverse for such a map: one simply[1] reverses the sequence of expansions and collapses. Thus, simple homotopy equivalences preserve some of the combinatorial data of CW complexes. In contrast, a homotopy inverse for an arbitrary homotopy equivalence need not be so easily describable—even worse, it might only be known to exist by some abstract nonsense[2]. Simple homotopy equivalences are just that: simple.

Let us put these ideas on more formal footing. We begin with some notation: denote by $B^n \subset \mathbb{R}^n$ the closed n-dimensional unit ball and $S^{n-1} = \partial B^n$ its boundary. Let S^{n-1}_+ and S^{n-1}_- denote the closed upper and lower hemispheres of S^{n-1}, which meet at the "equator" $S^{n-1}_+ \cap S^{n-1}_- \cong S^{n-2}$. (When $n = 1$, we have $S^0_+ = \{1\}$ and $S^0_- = \{-1\}$, so the "equator" is $S^{-1} = \varnothing$.)

Given a finite CW complex X with skeletal filtration $\varnothing = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq \cdots \subseteq X^d = X$ and a map of pairs $f\colon (S^{n-1}_-, S^{n-2}) \to (X^{n-1}, X^{n-2})$ where $n \geq 1$, we can build a CW complex structure on the pushout $Y := X \sqcup_{S^{n-1}_-} B^n$ by attaching cells in a two-step process:

Step 1: Attach the $(n-1)$-cell $e^{n-1} = S^{n-1}_+ \setminus S^{n-2}$ corresponding to the open upper hemisphere of S^{n-1} to X along the restriction $f|_{S^{n-2}}\colon S^{n-2} \to X^{n-2}$ of f to the equator.

Step 2: Attach the n-cell $e^n = B^n \setminus S^{n-1}$ to the CW complex $X \sqcup_{S^{n-2}} S^{n-1}_+$ resulting from Step 1 via the map $f \sqcup \mathrm{id}\colon S^{n-1} = S^{n-1}_- \sqcup_{S^{n-2}} S^{n-1}_+ \to X^{n-1} \sqcup_{S^{n-2}} S^{n-1}_+$.

The resulting CW complex structure on Y is called an *elementary expansion* of X; we also refer to the inclusion map $X \hookrightarrow Y$ as such. For instance, the elementary expansion of the one-point space $X = \{*\}$ along the constant map $f\colon S^{n-1}_- \to \{*\}$ is just $Y = B^n$.

The inverse operation—that of *elementary collapse*—can be thought of as undoing this process. More formally, S^{n-1}_- is a (deformation) retract of B^n, and any choice of retraction $B^n \to S^{n-1}_-$ induces a map $Y \to X$ by the universal property of pushouts. The homotopy class of this map does not depend on the choice of retraction, and is such that elementary collapse is homotopy inverse to elementary expansion.

To illustrate what elementary expansions look like for small values of n, consider the CW complex structure X on B^2 whose 0-skeleton consists of four vertices as depicted in Figure 1.1.

[1] Pun intended.

[2] The author means this as a term of endearment for category theory.

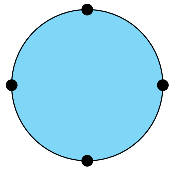

FIGURE 1.1. A CW complex structure X on B^2.

To perform an elementary expansion of X by a 1-cell, we need a map of pairs $f\colon (S^0_-, S^{-1}) \to (X^0, X^{-1})$. Since S^0_- is a single point and $S^{-1} = X^{-1} = \varnothing$, this is equivalent to a choice of a vertex in X^0. Picking, say, the left vertex, we obtain the CW complex Y depicted in Figure 1.2.

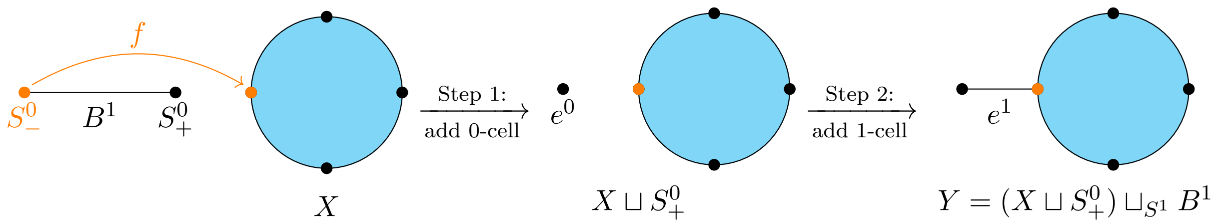

FIGURE 1.2. The outcome Y of performing an elementary expansion on X by a 1-cell.

To further perform an elementary expansion of Y by a 2-cell, we need a map of pairs $g\colon (S^1_-, S^0) \to (Y^1, Y^0)$, i.e., a path in Y^1 with endpoints lying in Y^0. Choosing, for example, a path joining the vertex added in the previous step and the bottom vertex of the disk, we obtain the CW complex Z depicted in Figure 1.3.

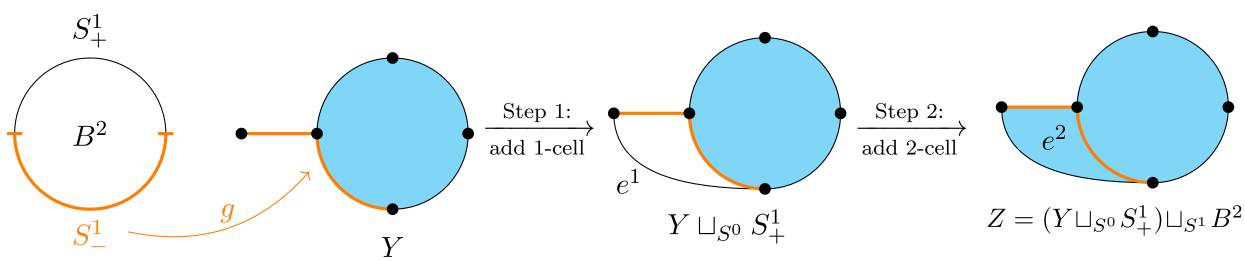

FIGURE 1.3. The outcome Z of performing an elementary expansion on Y by a 2-cell.

Finally, to perform an elementary expansion of Z by a 3-cell, we choose $h\colon (S^2_-, S^1) \to (Z^2, Z^1)$ to be the inclusion of $S^2_- \cong B^2$ as our original CW complex $X \subset Z$. The resulting CW complex W is shown in Figure 1.4.

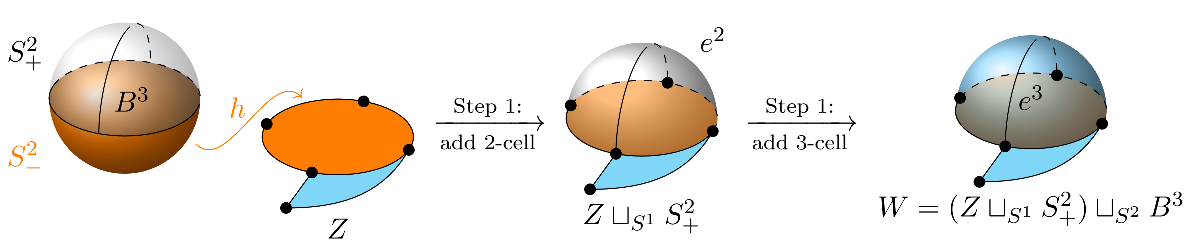

FIGURE 1.4. The outcome W of performing an elementary expansion on Z by a 3-cell.

In this way, we have constructed a sequence of elementary expansions $X \to Y \to Z \to W$. Elementary collapses $W \to Z \to Y \to X$ can be visualized by "smooshing" back down the two cells added by each elementary expansion. All of these maps are homotopy equivalences, and this idea is the foundation for our first definition:

Definition 1.1. *A map $f\colon X \to Y$ of finite CW complexes is called a* ***simple homotopy equivalence*** *if it is homotopic to a finite sequence of compositions*

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots \xrightarrow{f_n} X_n = Y,$$

where each f_i is either an elementary expansion or an elementary collapse. We say that X and Y are ***simple homotopy equivalent*** *if there exists a simple homotopy equivalence between them.*

As alluded to before, homeomorphisms (of finite CW complexes) are simple homotopy equivalences[3], and simple homotopy equivalences are homotopy equivalences. This raises the question: when are the converses true? That is,

- Does every simple homotopy equivalence yield a homeomorphism?
- Does every homotopy equivalence yield a simple homotopy equivalence?

Even if we restrict our attention to the subcategory of compact topological manifolds, the answer to both questions is "sometimes, yes; in general, no":

- In dimensions $d \leq 3$, simple homotopy equivalent d-manifolds are homeomorphic. However, for every $d \geq 4$, there exist d-dimensional manifolds that are simple homotopy equivalent but not homeomorphic.
- In dimensions $d \leq 2$, homotopy equivalent d-manifolds are simple homotopy equivalent. However, for every odd $d \geq 3$, there exist d-dimensional manifolds that are homotopy equivalent but not simple homotopy equivalent.

One can ask if there is a way to quantify the difference between homeomorphism and simple homotopy equivalence, or between simple homotopy equivalence and homotopy equivalence. The latter question has a relatively satisfactory answer—this is the subject of this paper. It goes as follows: given a (path-connected, say) topological space X with fundamental group $G = \pi_1(X)$, one associates to G an abelian group $\mathrm{Wh}(G)$ called the *Whitehead group* of G. Every homotopy equivalence $f\colon X \to Y$ of finite CW complexes yields an element $\tau(f)$ of $\mathrm{Wh}(G)$, the *Whitehead torsion* of f, and it turns out that f is a simple homotopy equivalence if and only if $\tau(f) = 0$. Thus, the obstruction for a homotopy equivalence to be simple can be characterized in terms of a purely algebraic description.

In Section 2, we formally introduce the definition of a Whitehead group. This is most succinctly accomplished by detouring through algebraic K-theory, so we begin by discussing the first K-group of a ring. In Section 3, we briefly mention some known examples of Whitehead groups before expositing a general result: the Whitehead group of a finite group is finitely generated. Finally, in Section 4, we return to the world of CW complexes and show how to define the Whitehead torsion of a homotopy equivalence. We conclude by compiling some properties of Whitehead torsion and showcasing several applications to problems in geometry and topology.

2. K-groups and Whitehead groups

Let R be a ring (henceforth always assumed to be associative and unital). For each $n \in \mathbb{N}$, we denote by $\mathrm{GL}_n(R)$ the group of $n \times n$ invertible matrices with entries in R. Given $m \leq n$, there is an inclusion homomorphism

$$\mathrm{GL}_m(R) \hookrightarrow \mathrm{GL}_n(R), \qquad A \mapsto \begin{pmatrix} A & O \\ O & I \end{pmatrix},$$

[3]This fact is highly nontrivial; see the discussion around Theorem 4.6.

where I and O denote identity and zero matrices of appropriate sizes. We set

$$\mathrm{GL}(R) := \varinjlim \mathrm{GL}_n(R).$$

Elements of $\mathrm{GL}(R)$ can thus be thought of as "infinite invertible matrices which differ from the infinite identity matrix by finitely entries".

Definition 2.1. *The* ***first algebraic K-group*** $\mathrm{K}_1(R)$ *of a ring* R *is the abelianisation of* $\mathrm{GL}(R)$.

Remark 2.2. As the name suggests, $\mathrm{K}_1(R)$ is but one of a sequence of "K-groups" associated to a ring R. In fact, one can define $\mathrm{K}_n(R)$ for any $n \in \mathbb{Z}$ in a way which gives rise to a long exact sequence of K-groups, but we will not make use of these beyond the briefest of mentions.

While Definition 2.1 makes it clear that K_1 defines a functor from the category of rings to the category of abelian groups, it is somewhat opaque. To explicate this, let us determine the commutator subgroup of $\mathrm{GL}(R)$. For this, we need the following definition: a *shear matrix*[4] is a square matrix that coincides with the identity matrix except for one off-diagonal element. Given non-zero $r \in R$ and positive integers $i \neq j$, we denote by $e_{ij}(r)$ the shear matrix with r in the (i,j) entry (of unspecified size). The subgroups $\mathrm{E}_n(R) := \langle e_{ij}(r) \mid r \in R, i \neq j\rangle$ of $\mathrm{GL}_n(R)$ generated by these matrices form a direct system, and we set

$$\mathrm{E}(R) := \varinjlim \mathrm{E}_n(R).$$

It turns out that $\mathrm{E}(R)$ is a normal subgroup of $\mathrm{GL}(R)$. In fact, more is true:

Lemma 2.3 (Whitehead [10, §1]). *The commutator subgroup of* $\mathrm{GL}(R)$ *is* $\mathrm{E}(R)$.

Thus, given $A \in \mathrm{GL}_n(R) \subset \mathrm{GL}(R)$, its image in $\mathrm{K}_1(R) = \mathrm{GL}(R)/\mathrm{E}(R)$ measures the obstruction to reducing A to the identity matrix via shear transformations.

We now define the Whitehead group of a group G. For this, recall that for a ring R, the *group ring* $R[G]$ is the R-module of finitely supported functions $G \to R$; it can also be thought of as the set of all formal finite sums of the form $\sum_i r_i g_i$ where $r_i \in R$ and $g_i \in G$. As the name suggests, $R[G]$ is also a ring with the product

$$\Big(\sum_i r_i g_i\Big)\Big(\sum_j s_j h_j\Big) = \sum_{i,j} (r_i s_j)(g_i h_j).$$

(This is well-defined by the assumption of finite support.) In general, the group of units $R[G]^\times$ is quite mysterious. However, it always contains the subgroup of *trivial units*, which are elements of the form rg where $r \in R^\times$ and $g \in G$. There is a homomorphism $R^\times \times G \to \mathrm{K}_1(R[G])$ given by mapping $(r,g) \in R^\times \times G$ to the trivial unit $rg \in R[G]^\times = \mathrm{GL}_1(R[G])$ thought of as a 1×1 invertible matrix, then taking its image in $\mathrm{K}_1(R[G])$. The Whitehead group of G is the cokernel of this map in the case where $R = \mathbb{Z}$:

Definition 2.4. *The* ***Whitehead group*** $\mathrm{Wh}(G)$ *of a group* G *is the quotient group*

$$\mathrm{Wh}(G) := \mathrm{K}_1(\mathbb{Z}[G])/\langle \pm g | g \in G\rangle.$$

Remark 2.5. In some references (notably [10] and [14]), $\mathrm{K}_1(R)$ of a ring R is sometimes called the "Whitehead group of R". This is unfortunate terminology, as $\mathrm{K}_1(R)$ need not coincide with $\mathrm{Wh}(R)$ (where we regard R as a group); for example, if one considers $\mathbb{Z}$ as a ring, then $\mathrm{K}_1(\mathbb{Z}) = \{\pm 1\}$, but thinking of $\mathbb{Z}$ only as a group, we have $\mathrm{Wh}(\mathbb{Z}) = 0$. Thus, calling K_1 the "Whitehead group" leads to seemingly contradictory statements like "the Whitehead group of $\mathbb{Z}$ is nontrivial and the Whitehead group of $\mathbb{Z}$ is trivial".

[4]These are often called *elementary matrices* in the literature, but I prefer to use the term "elementary matrix" to mean a matrix obtained from the identity matrix by applying one of *any* of the elementary row or column operations (of which shear matrices are a special case).

3. Whitehead groups of finite groups

Determining the Whitehead group of a group G is, in general, quite a difficult task. Oftentimes, it is unclear if $\mathrm{Wh}(G)$ is even non-zero. For example:

- The Whitehead group of a free abelian group (including the trivial group) is trivial, though this fact is not easy to prove [4]; the special case where $G = \mathbb{Z}$ is considerably easier [7].
- The Whitehead groups of the cyclic groups C_n of order $n \in \{2, 3, 4, 6\}$ are trivial; however, $\mathrm{Wh}(C_5) \cong \mathbb{Z}$ [10, Example 6.6]. In fact, $\mathrm{Wh}(C_p)$ is infinite for any prime $p \geq 5$.
- It is not known if there exists a torsion-free group with non-trivial Whitehead group.

The main goal of this section is to sketch a proof of the following result. Recall that the *rank* of an abelian group A is the dimension of $A \otimes_{\mathbb{Z}} \mathbb{Q}$ as a $\mathbb{Q}$-vector space.

Theorem 3.1 (Bass [3, §6]). *If G is a finite group, then $\mathrm{K}_1(\mathbb{Z}[G])$ is finitely generated (hence so too is $\mathrm{Wh}(G)$). Moreover, the ranks of $\mathrm{K}_1(\mathbb{Z}[G])$ and $\mathrm{Wh}(G)$ are both equal to the number of irreducible real representations of G minus the number of irreducible rational representations of G.*

Before proving Theorem 3.1, we recall some necessary algebraic definitions.

3.1. **Algebras and orders.** By an *algebra* over a field F (or *F-algebra*), we mean an F-vector space A equipped with a bilinear product $A \times A \to A$ (hence in particular a ring structure). For the remainder of Section 3, we assume all algebras are associative, unital, and finite-dimensional over their base field. Thus in particular, F canonically embeds into the centre $Z(A)$ of A.

Definition 3.2. *An F-algebra A is called*

- ***central*** *if $Z(A) = F$,*
- ***simple*** *if A is non-zero and has no two-sided ideals besides $\{0\}$ and A, and*
- ***semisimple*** *if A is (isomorphic to) a direct product of simple algebras.*

A semisimple F-algebra A is simple if and only if $Z(A)$ is a field. When this is the case, A is central when regarded as an algebra over $Z(A)$.

Definition 3.3. *Let A be a $\mathbb{Q}$-algebra. A finitely generated $\mathbb{Z}$-submodule $\Lambda \subset A$ is called a **full $\mathbb{Z}$-lattice** if it spans A over $\mathbb{Q}$ (i.e., every element of A can be written as a finite sum of the form $\sum q_i u_i$, where $q_i \in \mathbb{Q}$ and $u_i \in \Lambda$). If, in addition, Λ is a subring of A, then Λ is called a **$\mathbb{Z}$-order**. A **maximal $\mathbb{Z}$-order** is a $\mathbb{Z}$-order which is maximal with respect to set inclusion.*

If G is a finite group, then $A = \mathbb{Q}[G]$ is a semisimple $\mathbb{Q}$-algebra by Maschke's theorem [9, Ch. XVIII, Theorem 1.2], and $\Lambda = \mathbb{Z}[G]$ is a $\mathbb{Z}$-order in A. Usually Λ is not maximal, but it is contained in a maximal $\mathbb{Z}$-order. More generally:

Theorem 3.4. *Let A be a semisimple $\mathbb{Q}$-algebra.*

(i) A contains at least one maximal $\mathbb{Z}$-order, and every $\mathbb{Z}$-order in A is contained in a maximal $\mathbb{Z}$-order [12, Corollary 10.4].

(ii) Suppose $A = \prod_{i=1}^n A_i$ where each A_i is simple. A subset $\Lambda \subset A$ is a $\mathbb{Z}$-order if and only if it splits into a product $\Lambda = \prod_{i=1}^n \Lambda_i$, where each $\Lambda_i \subset A_i$ is a $\mathbb{Z}$-order; moreover, Λ is maximal if and only if each Λ_i is maximal [12, Theorem 10.5].

Remark 3.5. One can analogously define the notion of an R-order inside an F-algebra, where R is any Dedekind domain and F is its field of fractions. In fact, to study $\mathrm{K}_1(\Lambda)$ where Λ is a $\mathbb{Z}$-order in a $\mathbb{Q}$-algebra, one often examines the K-groups of $R = \mathbb{Z}_p$-orders over the p-adic numbers $F = \mathbb{Q}_p$, then appeals to localization sequences. We will only need this for one step of the proof of Theorem 3.1 (namely in Corollary 3.8(iii)), which we will take for granted.

Lemma 3.6. *If $\Lambda \subseteq \Gamma$ is any pair of $\mathbb{Z}$-orders in a $\mathbb{Q}$-algebra A, then there exists an integer $m > 0$ such that $m\Gamma \subseteq \Lambda$, where $m\Gamma = \{m\gamma \mid \gamma \in \Gamma\}$ denotes the two-sided principal ideal of Γ generated by $\underbrace{1 + \cdots + 1}_{m \text{ times}}$.*

Proof. Choose $\mathbb{Q}$-bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ of A such that $\Lambda = \operatorname{span}_{\mathbb{Z}}\{v_i\}$ and $\Gamma = \operatorname{span}_{\mathbb{Z}}\{w_i\}$. Then for each i, we can write $w_i = \sum_{j=1}^n q_{ij} v_j$ where $q_{ij} \in \mathbb{Q}$. Writing $q_{ij} = a_{ij}/b_{ij}$ in lowest terms with $a_{ij} \in \mathbb{Z}$ and $b_{ij} \in \mathbb{Z}_{>0}$, let m be the lowest common multiple of all the b_{ij}'s.

We claim that $m\Gamma \subseteq \Lambda$. Indeed, for all $x = \sum_i k_i w_i \in \Gamma$ where $k_i \in \mathbb{Z}$, we have $mx = \sum_i k_i(mw_i)$. But $mw_i = \sum_j (m\frac{a_{ij}}{b_{ij}}) v_j$, and each coefficient ma_{ij}/b_{ij} is an integer since m is a multiple of b_{ij}; thus, $mx \in \Lambda$. ∎

3.2. **Reduced norms.** Given a field F and a central simple F-algebra A, there exists a function $A \to F$, called the *reduced norm*, which coincides with the determinant map when $A = \operatorname{Mat}_n(F)$ is the algebra of $n \times n$ matrices with entries in F. Its precise definition is not of import to us, so we refer the reader to [12, §9a] for details on its construction. What *is* important is the following:

- Reduced norm is a multiplicative function, and the reduced norm of $a \in A$ is non-zero if and only if a is invertible. Thus, the reduced norm restricts to a homomorphism $A^\times \to F^\times$ between the groups of units.
- For any integer $n > 1$, $\operatorname{Mat}_n(A)$ is also a central simple F-algebra, and the reduced norm $\operatorname{Mat}_n(A)^\times = \operatorname{GL}_n(A) \to F^\times$ extends the reduced norm $\operatorname{GL}_{n-1}(A) \to F^\times$.

By the universal property of colimits, we obtain a corresponding homomorphism $\operatorname{GL}(A) \to F^\times$. Since $F^\times$ is abelian, this factors through the abelianisation to define a map $\operatorname{nr}_A \colon \mathrm{K}_1(A) \to F^\times$; henceforth, "reduced norm" shall refer to this map.

We can extend the notion of reduced norms to semisimple F-algebras: suppose $A = \prod_{i=1}^n A_i$, where each A_i is simple. Then $\mathrm{K}_1(A) \cong \bigoplus_{i=1}^n \mathrm{K}_1(A_i)$ [14, III.1.1.3], and each A_i is a central simple algebra over the field $Z(A_i)$. The reduced norms $\operatorname{nr}_{A_i} \colon \mathrm{K}_1(A_i) \to Z(A_i)^\times$ can be combined to define

$$\operatorname{nr}_A \colon \mathrm{K}_1(A) \to \prod_{i=1}^n Z(A_i)^\times \left(= Z(A)^\times\right), \qquad \operatorname{nr}_A(x_1, \ldots, x_n) = \left(\operatorname{nr}_{A_1}(x_1), \ldots, \operatorname{nr}_{A_n}(x_n)\right).$$

If $F = \mathbb{Q}$ and $\Lambda \subset A$ is a $\mathbb{Z}$-order, we denote by $\mathrm{SK}_1(\Lambda)$ the kernel of the map $\mathrm{K}_1(\Lambda) \to \mathrm{K}_1(A)$, and we define $\operatorname{nr}_\Lambda$ as the composition

$$\mathrm{K}_1(\Lambda) \to \mathrm{K}_1(A) \xrightarrow{\operatorname{nr}_A} Z(A)^\times.$$

Let us now list some useful properties of reduced norms for semisimple $\mathbb{Q}$-algebras. Recall that an *algebraic number field* is a finite field extension of $\mathbb{Q}$; that is, a field F containing $\mathbb{Q}$ such that $\dim_{\mathbb{Q}}(F) < \infty$. The *ring of integers* $\mathcal{O}_F$ of F is the integral closure of $\mathbb{Z}$ in F, i.e., the set of elements in F which are roots of a monic polynomial with coefficients in $\mathbb{Z}$. These concepts apply, in particular, to the centre of a simple $\mathbb{Q}$-algebra.

Theorem 3.7. *Let A be a simple $\mathbb{Q}$-algebra and $\Lambda \subset A$ a maximal $\mathbb{Z}$-order.*

(i) The reduced norm $\operatorname{nr}_A \colon \mathrm{K}_1(A) \to Z(A)^\times$ is injective, and $\operatorname{coker}(\operatorname{nr}_A)$ is finite.
(ii) The image of $\operatorname{nr}_\Lambda \colon \mathrm{K}_1(\Lambda) \to Z(A)^\times$ is contained in $(\mathcal{O}_{Z(A)})^\times$, and $(\mathcal{O}_{Z(A)})^\times / \operatorname{im}(\operatorname{nr}_\Lambda)$ is finite.

The proof of Theorem 3.7 is rather lengthy, so we refer the reader to [11, Theorem 2.3] for details. Analogous statements for semisimple $\mathbb{Q}$-algebras follow immediately; we make them explicit for the record, along with an additional statement about the kernel of $\operatorname{nr}_\Lambda$.

Corollary 3.8. *Let A be a semisimple $\mathbb{Q}$-algebra and $\Lambda \subset A$ a maximal $\mathbb{Z}$-order. Write $A = \prod_{i=1}^n A_i$ where each A_i is simple, and set $\mathcal{O} = \prod_{i=1}^n \mathcal{O}_{Z(A_i)}$.*

(i) The reduced norm $\mathrm{nr}_A\colon \mathrm{K}_1(A) \to Z(A)^\times$ *is injective, and* $\mathrm{coker}(\mathrm{nr}_A)$ *is finite.*
(ii) The image of $\mathrm{nr}_\Lambda\colon \mathrm{K}_1(\Lambda) \to Z(A)^\times$ *is contained in* $\mathcal{O}^\times$*, and* $\mathcal{O}^\times / \mathrm{im}(\mathrm{nr}_\Lambda)$ *is finite.*
(iii) The kernel of nr_Λ *is finite.*

Proof. We have $\ker(\mathrm{nr}_A) = \prod_{i=1}^n \ker(\mathrm{nr}_{A_i}) = 0$, and $\mathrm{coker}(\mathrm{nr}_A) \cong \prod_{i=1}^n \mathrm{coker}(\mathrm{nr}_{A_i})$ is finite since each factor is finite. Item (ii) follows from Theorem 3.4(ii). The fact that $\ker(\mathrm{nr}_\Lambda)$ is finite requires several inputs:

- Since nr_A is injective by (i), we have $\mathrm{SK}_1(\Lambda) = \ker(\mathrm{nr}_\Lambda)$.
- By a theorem of Bass [2, Proposition 11.2], the map $\mathrm{GL}_2(\Lambda) \to \mathrm{K}_1(\Lambda)$ is surjective.
- A result of Siegel [13] implies that $\mathrm{GL}_2(\Lambda)$ is finitely generated, hence so too is $\mathrm{SK}_1(\Lambda)$.
- Quillen's localization sequence (see Remark 3.5 and [11, Theorem 1.17]) implies that $\mathrm{SK}_1(\Lambda)$ is torsion. (This step uses the fact that Λ is maximal.)

Combining these, we see that $\ker(\mathrm{nr}_\Lambda)$ is a finitely generated abelian group which is torsion; thus it is finite. ∎

Generalizations of Corollary 3.8(ii) and (iii) holds for non-maximal $\mathbb{Z}$-orders (Proposition 3.10 below); to prove these, we will need to be able to compare $\mathrm{K}_1(\Lambda)$ and $\mathrm{K}_1(\Gamma)$ where $\Lambda \subseteq \Gamma$ are $\mathbb{Z}$-orders in the same $\mathbb{Q}$-algebra.

Lemma 3.9. *Let R be a subring of S. If there exists a two-sided ideal I of S such that $I \subseteq R$ and $|S/I^2| < \infty$, then the map $K_1(R) \to K_1(S)$ has finite kernel and cokernel.*

See [11, Lemma 2.4] for a proof (this is where one must discuss the *second* K-*group* $\mathrm{K}_2(R)$).

Proposition 3.10. *Let A be a semisimple $\mathbb{Q}$-algebra, and let $\Lambda \subset A$ be any $\mathbb{Z}$-order. Write $A = \prod_{i=1}^n A_i$ where each A_i is simple, and set $\mathcal{O} = \prod_{i=1}^n \mathcal{O}_{Z(A_i)}$. Then the map $\mathrm{nr}_\Lambda\colon \mathrm{K}_1(\Lambda) \to \mathcal{O}^\times$ has finite kernel and cokernel.*

Proof. By Theorem 3.4(i), there exists a maximal $\mathbb{Z}$-order $\Gamma \subset A$ containing Λ. By Lemma 3.6, there exists an integer $m > 0$ such that $m\Gamma \subseteq \Lambda$. Then the product ideal $(m\Gamma)^2 = m^2\Gamma$ is also contained in Γ, and $|\Gamma/(m\Gamma)^2| = |\Gamma/m^2\Gamma| < \infty$. It follows from Lemma 3.9 that the map $\mathrm{K}_1(\Lambda) \to \mathrm{K}_1(\Gamma)$ has finite kernel and cokernel. Since nr_Λ is equal to the composition

$$\mathrm{K}_1(\Lambda) \to \mathrm{K}_1(\Gamma) \xrightarrow{\mathrm{nr}_\Gamma} \mathcal{O}^\times$$

and $\mathrm{nr}_\Gamma\colon \mathrm{K}_1(\Gamma) \to \mathcal{O}^\times$ has finite kernel and cokernel by Corollary 3.8(ii) and (iii), the result follows. ∎

3.3. **Towards the theorem of Bass.** We are almost in a position to prove Theorem 3.1; the last few inputs we shall need come from number theory and representation theory.

Theorem 3.11 (Dirichlet unit theorem [8, Theorem 13.12]). *Let F be an algebraic number field, and $\mathcal{O}_F$ its ring of integers. The group of units $(\mathcal{O}_F)^\times$ is finitely generated; moreover, if r denotes the number of field summands of $\mathbb{R} \otimes_{\mathbb{Q}} F$, then* $\mathrm{rank}((\mathcal{O}_F)^\times) = r - 1$.

For the next result, we make a provisional definition: given a finite group G and $g, h \in G$, we say that g and h are $\mathbb{Q}$-*conjugate* if the cyclic subgroups generated by g and h are conjugate; we say that g and h are $\mathbb{R}$-*conjugate* if g is conjugate to h or h^{-1}.

Theorem 3.12 (Berman–Witt [6, Theorem 42.8]). *Let F be a field of characteristic 0. For any finite group G, the number of simple $F[G]$-modules (i.e., the number of simple summands of $F[G]$) equals the number of F-conjugacy classes of elements in G.*

Since simple $F[G]$-modules are equivalent to irreducible F-representations of G, the Berman–Witt theorem provides the link to the rank formula in the statement of Theorem 3.1 when combined with the following:

Lemma 3.13. *Let $f\colon A \to B$ be a homomorphism of abelian groups.*
(i) If $\ker(f)$ is torsion, then $\operatorname{rank}(A) \leq \operatorname{rank}(B)$.
(ii) If $\operatorname{coker}(f)$ is torsion, then $\operatorname{rank}(A) \geq \operatorname{rank}(B)$.

Proof. Consider the following exact sequence of $\mathbb{Z}$-modules:

$$0 \longrightarrow \ker(f) \hookrightarrow A \xrightarrow{f} B \twoheadrightarrow \operatorname{coker}(f) \longrightarrow 0$$

Tensoring with $\mathbb{Q}$ is an exact functor, so the sequence

$$0 \longrightarrow \ker(f) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow A \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{f\otimes 1} B \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \operatorname{coker}(f) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow 0$$

is also exact. If $\ker(f)$ is torsion, then $\ker(f) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. Thus $f \otimes 1$ is an injective linear map of $\mathbb{Q}$-vector spaces, whence it follows that

$$\operatorname{rank}(A) = \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q}) \leq \dim_{\mathbb{Q}}(B \otimes_{\mathbb{Z}} \mathbb{Q}) = \operatorname{rank}(B).$$

Similarly, if $\operatorname{coker}(f)$ is torsion, then $\operatorname{coker}(f) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. Thus $f \otimes 1$ is surjective, so $\operatorname{rank}(A) \geq \operatorname{rank}(B)$. ∎

Proof of Theorem 3.1. Let $A = \mathbb{Q}[G]$ and $\Lambda = \mathbb{Z}[G]$. If q denotes the number of simple summands of A, then we can write $Z(A) = \prod_{i=1}^{q} F_i$ where each F_i is a field with ring of integers $\mathcal{O}_{F_i}$. Denote by r and r_i the number of field summands of $\mathbb{R}\otimes_{\mathbb{Q}} Z(A)$ and $\mathbb{R}\otimes_{\mathbb{Q}} F_i$, respectively, so that $r = \sum_i r_i$. By the Dirichlet unit theorem, each $(\mathcal{O}_{F_i})^\times$ is finitely generated, hence so too is $\prod_{i=1}^{q}(\mathcal{O}_{F_i})^\times$. By Proposition 3.10, the same is true of $\mathrm{K}_1(\Lambda)$; moreover, Lemma 3.13 implies that

$$\operatorname{rank}(\mathrm{K}_1(\Lambda)) = \operatorname{rank}\left(\prod_{i=1}^{q}(\mathcal{O}_{F_i})^\times\right) = \sum_{i=1}^{q}\operatorname{rank}((\mathcal{O}_{F_i})^\times) = \sum_{i=1}^{q}(r_i - 1) = r - q,$$

which yields the desired rank formula when combined with the Berman–Witt theorem. ∎

4. Whitehead torsion

Henceforth, we assume all rings R satisfy the *invariant basis number* property: if R^m is isomorphic to R^n (as left or right R-modules), then $m = n$. This is the case, for example, if R admits a non-zero ring homomorphism to a division ring. The integral group rings $R = \mathbb{Z}[G]$ considered previously are examples of such rings, as they admit a non-zero homomorphism to $\mathbb{Q}$ given by sending all elements of $G \subset \mathbb{Z}[G]$ to 1.

To build up to the definition of Whitehead torsion, we must first discuss torsion for chain complexes and maps between them. We mostly follow the exposition of Jacob Lurie's notes for the remainder of this section.

4.1. **Torsion for chain complexes.** Let R be a ring. By a *based chain complex* over R, we mean a bounded chain complex $C_\bullet = (C_\bullet, \partial)$ of finitely generated free R-modules, together with a preferred choice of unordered bases for each C_n. Suppose, in addition, that $C_\bullet$ is *acyclic* (i.e., the homology groups $H_\bullet(C)$ all vanish). Then the identity map $C_\bullet \to C_\bullet$ is chain homotopic to zero; that is, there exist module homomorphisms $P_n\colon C_n \to C_{n+1}$ such that $\partial P + P\partial = \mathrm{id}_{C_\bullet}$. If we set

$$C_{\text{even}} = \bigoplus_{n\in\mathbb{Z}} C_{2n}, \qquad C_{\text{odd}} = \bigoplus_{n\in\mathbb{Z}} C_{2n+1},$$

then $\partial + P$ defines homomorphisms $C_{\text{even}} \to C_{\text{odd}}$ and $C_{\text{odd}} \to C_{\text{even}}$. In fact, these are isomorphisms: since $(\partial + P)^2 = \partial^2 + \partial P + P\partial + P^2 = \mathrm{id} + P^2$, the geometric series formula suggests that an inverse for $(\partial + P)^2$ should be given, at least formally, by

$$(\mathrm{id} + P^2)^{-1} = \mathrm{id} - P^2 + P^4 - P^6 + \cdots.$$

But this sum is finite since $C_\bullet$ is bounded and P increases degree, and hence the right-hand side defines an actual inverse for $(\partial + P)^2$.

We can build bases for the free R-modules C_{even} and C_{odd} from the preferred bases of the C_n's, and therefore identify $C_{\text{even}} \cong R^m$ and $C_{\text{odd}} \cong R^n$. Since R has the invariant basis number property and $C_{\text{even}} \cong C_{\text{odd}}$, we have $m = n$. Expressing $\partial + P \colon C_{\text{even}} \to C_{\text{odd}}$ as a matrix with respect to these bases, we may regard $\partial + P \in \mathrm{GL}_n(R)$. Of course, this matrix will depend on how we chose to order the bases of C_{even} and C_{odd}; however, any two orderings will yield matrices which are equivalent up to multiplication by permutation matrices. It turns out that the image of any permutation matrix under the map $\mathrm{GL}_n(R) \to \mathrm{GL}(R) \to \mathrm{K}_1(R)$ is given by $[\varepsilon]$, where $\varepsilon = \pm 1$ is the sign of the permutation. Thus, if we work with the *reduced* K_1*-group*

$$\overline{\mathrm{K}}_1(R) := \mathrm{K}_1(R)/\langle \pm 1 \rangle,$$

then we can make the following definition:

Definition 4.1. *The **torsion** $\tau(C_\bullet) \in \overline{\mathrm{K}}_1(R)$ of the based acyclic chain complex $C_\bullet$ is the image of $\partial + P \in \mathrm{GL}_n(R)$ under the map $\mathrm{GL}_n(R) \to \mathrm{GL}(R) \to \mathrm{K}_1(R) \to \overline{\mathrm{K}}_1(R)$.*

It turns out that $\tau(C_\bullet)$ is independent of the choice of the chain null-homotopy P.

To define the Whitehead torsion of a homotopy equivalence of CW complexes, we will also need to be able to make sense of torsion for certain maps of chain complexes $f\colon (C_\bullet, \partial^C) \to (D_\bullet, \partial^D)$. This can be defined as follows: let $C[-1]_\bullet := (C_{\bullet-1}, -\partial^C)$ denote the chain complex obtained from $C_\bullet$ by shifting each degree down by 1 and changing the sign of the boundary operator. The *mapping cone* of f is the chain complex $\mathrm{Cone}(f)_\bullet = C[-1]_\bullet \oplus D_\bullet$, with the boundary operator

$$\partial(x, y) := \begin{pmatrix} -\partial^C & 0 \\ f & \partial^D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (-\partial^C x, f(x) + \partial^D y).$$

The mapping cone of f is acyclic if and only if f is a quasi-isomorphism (i.e., induces isomorphisms on all homology groups). If $C_\bullet$ and $D_\bullet$ are based, then assembling the preferred bases of $C_\bullet$ and $D_\bullet$ yields a basis for $\mathrm{Cone}(f)_\bullet$; again, the ordering is unimportant for our purposes.

Definition 4.2. *The **torsion** $\tau(f) \in \overline{\mathrm{K}}_1(R)$ of a quasi-isomorphism $f\colon C_\bullet \to D_\bullet$ of based chain complexes is the torsion of $\mathrm{Cone}(f)_\bullet$ in the sense of Definition 4.1.*

4.2. **Torsion for CW complexes.** Let $f\colon X \to Y$ be a cellular homotopy equivalence of finite connected CW complexes.[5] Fixing a basepoint $x \in X$, we set $G = \pi_1(X, x) \cong \pi_1(Y, f(x))$. Choose a universal cover $\widetilde{Y} \to Y$ and let $\widetilde{X} = X \times_Y \widetilde{Y} \to X$ be the corresponding universal cover of X, both equipped with the natural CW complex structures obtained by lifting the cells of X and Y along the respective covering maps. Then the lift $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ induces a chain homotopy equivalence $\widetilde{f}_*\colon C_\bullet(\widetilde{X}, \mathbb{Z}) \to C_\bullet(\widetilde{Y}, \mathbb{Z})$. By choosing a lift for each cell in X and Y, we may regard $C_\bullet(\widetilde{X}, \mathbb{Z})$ and $C_\bullet(\widetilde{Y}, \mathbb{Z})$ as chain complexes of free $\mathbb{Z}[G]$-modules and apply the previously detailed constructions to obtain an element of $\overline{K}_1(\mathbb{Z}[G])$. This element depends on the choice of lifts for each cell, and therefore is only determined up to the action of G by deck transformations of $\widetilde{X}$ and $\widetilde{Y}$. Notice, however, that

$$\overline{\mathrm{K}}_1(\mathbb{Z}[G])/G = \Big(\mathrm{K}_1(\mathbb{Z}[G]) \big/ \langle \pm 1 \rangle\Big) \big/ G = \mathrm{Wh}(G).$$

[5]If X and Y are disconnected, then the following discussion can be applied to each connected component.

Definition 4.3. *The **Whitehead torsion** $\tau(f) \in \mathrm{Wh}(G)$ is the image of the chain homotopy equivalence $\widetilde{f}_*\colon C_\bullet(\widetilde{X}, \mathbb{Z}) \to C_\bullet(\widetilde{Y}, \mathbb{Z})$ in $\mathrm{Wh}(G)$.*

Let us make some remarks about Whitehead torsion and its properties. First, note that changing the basepoint $x \in X$ induces isomorphisms of $\mathbb{Z}[G]$ which come from conjugation by a path in X. This does not affect the Whitehead torsion, as conjugation acts trivially on the Whitehead group:

$$(gxg^{-1}) \sim \begin{pmatrix} gxg^{-1} & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} xg^{-1} & 0 \\ 0 & g \end{pmatrix} \sim \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \sim (x).$$

Thus $\tau(f)$ is also independent of the basepoint. Second, Whitehead torsion is homotopy invariant:

Proposition 4.4. *Let $f\colon X \to Y$ and $g\colon X \to Y$ be cellular homotopy equivalences of finite connected CW complexes. If f and g are homotopic, then $\tau(f) = \tau(g)$.*

In particular, using the cellular approximation theorem, we can extend Whitehead torsion to homotopy equivalences which are not necessarily cellular. As promised, this provides a way of distinguishing arbitrary homotopy equivalences from those which are simple:

Theorem 4.5. *A homotopy equivalence $f\colon X \to Y$ of finite connected CW complexes is simple if and only if $\tau(f) = 0$.*

As mentioned at the beginning of Section 3, the Whitehead group of the trivial group is trivial; thus, Theorem 4.5 implies that there is no difference between homotopy equivalence and simple homotopy equivalence for simply connected CW complexes.

In the earliest papers discussing simple homotopy theory, it was conjectured that simple homotopy equivalence is a topological invariant, i.e., does not depend on the cellular structure. It would take decades for this to be proven:

Theorem 4.6 (Chapman [5], Theorem 1). *Every homeomorphism of finite connected CW complexes is a simple homotopy equivalence.*

4.3. **Epilogue.** We have developed a most impressive algebraic apparatus for detecting when a homotopy equivalence of finite CW complexes can be seen from within the world of finite CW complexes. Let us conclude by mentioning a few applications of simple homotopy theory to questions in differential topology. Throughout, all manifolds are smooth.

Recall that a *cobordism* between two compact d-dimensional manifolds M and N is a compact $(d+1)$-dimensional manifold with boundary W such that $\partial W = M \sqcup N$. If the inclusions $M \hookrightarrow W$ and $N \hookrightarrow W$ are both homotopy equivalences, then W is called an *h-cobordism*.

Theorem 4.7 (s-cobordism[6] theorem). *Let M and N be compact manifolds of dimension $d \geq 5$. A path-connected h-cobordism W between M and N is diffeomorphic to $M \times [0,1]$ relative to M if and only if the inclusion $M \hookrightarrow W$ is a simple homotopy equivalence.*

The s-cobordism theorem provides a tool for constructing diffeomorphisms $M \to N$ in higher dimensions. For instance, in view of the fact that $\mathrm{Wh}(\mathbb{Z}^d) = 0$ for all integers $d \geq 0$, the s-cobordism theorem implies that any manifold of dimension $d \geq 5$ which is h-cobordant to the d-torus is also diffeomorphic to it.

For a second application of simple homotopy theory, recall that a *lens space* is a quotient of the unit sphere $S^{2n-1} \subset \mathbb{C}^{2n}$ by a finite cyclic subgroup of the unitary group $\mathrm{U}(n)$.

[6] If the h in "h-cobordism" stands for "homotopy", then we propose that the s in "s-cobordism" should stand for "shmotopy". Homotopy shmotopy.

Theorem 4.8 (Franz–Reidemeister [10, §12]). *Two lens spaces are diffeomorphic if and only if they are simple homotopy equivalent.*

In fact, lens spaces produce examples of odd-dimensional manifolds that are homotopy equivalent but not simple homotopy equivalent as mentioned in Section 1.

Finally, the author cannot resist mentioning an application within symplectic topology. For any closed smooth manifold M, the cotangent bundle $\pi\colon T^*M \to M$ admits a canonical exact symplectic form on its total space. The *nearby Lagrangian conjecture* states that every closed exact Lagrangian submanifold of T^*M is Hamiltonian isotopic to the zero section. While this conjecture is wide open in general, progress has been made recently:

Theorem 4.9 (Abouzaid–Kragh [1]). *If $L \subset T^*M$ is a closed exact Lagrangian, then the projection $\pi|_L\colon L \to M$ is a simple homotopy equivalence.*

As a consequence, we have the following:

Corollary 4.10. *Two lens spaces are diffeomorphic if and only if their cotangent bundles are symplectomorphic.*

Indeed, if L_1 and L_2 are lens spaces such that there exists a symplectomorphism $T^*L_1 \to T^*L_2$, then the image of L_1 in T^*L_2 under this symplectomorphism is a closed exact Lagrangian; thus L_1 and L_2 are simple homotopy equivalent, and the corollary follows from Theorem 4.8.

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