The classical no-hair theorems

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1 Introduction

The no-hair conjecture of general relativity states that all black hole solutions to Einstein's field equations are characterized by their mass, angular momentum, and electric charge. That is, in some sense, black holes are very simple objects: outside of their event horizons, the geometry of spacetime is determined by only three quantities. The conjecture is known to hold in several astrophysically relevant situations[1] (see [10] for a physical test of the conjecture using data from the binary black hole merger GW150914) but lacks a complete rigorous mathematical proof, with resolution only in some special cases. For example, there are proofs in the classical setting of stationary electrovacuum spacetimes with some additional assumptions, but the conjecture is known to be false in more general settings (which we will not discuss).

In this paper, we examine some of these classical cases. Specifically, in Section 3, we outline the proof of the no-hair theorem in the case of non-rotating black holes with no electric charge. We extend this proof to the case of non-zero electric charge in Section 4. Finally, in Section 5, we briefly discuss the no-hair theorem in the context of rotating black holes both with and without electric charge.

We adopt the following conventions throughout:

- (M, g) denotes a four-dimensional Lorentzian manifold with metric signature $(-,+,+,+)$. The metric induces a natural inner product on each tensor bundle, which we denote by $\langle \cdot, \cdot \rangle$.
- We work in units where $G = c = 1$. Greek indices denote both space and time components, while Latin indices are reserved for spatial components only.

2 Preliminaries

For details on the statements included in Section 2.1, we refer the reader to [15]; for the proof of Theorem 2.1 specifically, see [5]. In Section 2.2, we refer heavily to [7] and [8].

2.1 Conformal geometry of 3-manifolds

Recall that two Riemannian metrics h and $\hat{h}$ on the same underlying 3-dimensional manifold Σ are conformally equivalent if there exists a strictly positive function $\Omega \in C^\infty(\Sigma)$ such that

$$\hat{h} = \Omega^2 h.$$

The scalar curvatures R and $\hat{R}$ corresponding respectively to h and $\hat{h}$ are related by

$$\frac{\Omega^4}{2}\hat{R} = \frac{\Omega^2}{2}R - 2\Omega\Delta\Omega + \langle \operatorname{grad}\Omega, \operatorname{grad}\Omega \rangle, \tag{2.1}$$

where the Laplacian Δ and the inner product $\langle \cdot, \cdot \rangle$ are taken with respect to h.

We say that (Σ, h) is conformally flat if h is conformally equivalent to a flat metric on Σ. In three dimensions, we have a characterization of conformal flatness in terms of a particular tensor.

[1] Indeed, many physicists refer to the general statement as the no-hair *theorem*.

Theorem 2.1. *A 3-dimensional Riemannian manifold* (Σ, h) *is conformally flat if and only if the Cotton tensor*

$$C_{kij} := 2\nabla_{[i}R_{j]k} + \frac{1}{2}h_{k[i}\nabla_{j]}R = \nabla_i R_{jk} - \nabla_j R_{ik} + \frac{1}{4}\left(h_{ki}\nabla_j R - h_{kj}\nabla_i R\right) \tag{2.2}$$

vanishes identically. (Here and throughout the paper, square brackets denote antisymmetrization.)

2.2 Static spacetimes

We say a spacetime (M, g) is stationary if it admits a global timelike Killing vector field k. Such a stationary Killing field gives rise to the 3-dimensional quotient manifold $\Sigma = M/G$, where G is the one-parameter group of isometries generated by the flow of k. If, in addition, k is hypersurface-orthogonal (i.e. (M, g) is static), then Σ can be realized as a spacelike hypersurface of M orthogonal to the trajectories of k, whose induced Riemannian metric we denote by $\tilde{g}$. In this case, the metric on M can be written as

$$g = -S^2\,dt^2 + \tilde{g}, \tag{2.3}$$

where $S^2 = -\langle k, k\rangle$ is positive on Σ. One can then use Cartan's structure equations to express the Ricci tensor and scalar curvature of the static metric (2.3) in terms of S and the Levi-Civita connection $\tilde{\nabla}$ on $(\Sigma, \tilde{g})$. Specifically, letting tildes denote quantities associated to $(\Sigma, \tilde{g})$, we have

$$R_{tt} = S\tilde{\Delta}S, \qquad R_{ij} = \tilde{R}_{ij} - S^{-1}\tilde{\nabla}_j\tilde{\nabla}_i S, \qquad R = \tilde{R} - 2S^{-1}\tilde{\Delta}S. \tag{2.4}$$

In particular, when we are interested in the Einstein vacuum equations $R_{\mu\nu} = 0$, we obtain

$$\tilde{\Delta}S = 0, \qquad \tilde{R}_{ij} = S^{-1}\tilde{\nabla}_j\tilde{\nabla}_i S, \qquad \tilde{R} = 0. \tag{2.5}$$

These equations characterize the geometry of $(\Sigma, \tilde{g})$, a spacelike hypersurface embedded in (M, g) which physically represents the "$t = 0$ slice" exterior to a black hole[2]. In general, Σ has non-empty boundary $\mathcal{H} = \partial\Sigma$, representing the event horizon; we assume that $S = 0$ on $\mathcal{H}$, so that $\mathcal{H}$ is a *Killing horizon*. Then the surface gravity κ of $\mathcal{H}$ is defined by

$$\mathrm{grad}(S^2) = 2\kappa k \quad \text{on } \mathcal{H}. \tag{2.6}$$

The assumption that the horizon is regular (i.e. non-degenerate) means that $\kappa \neq 0$. Lastly, we assume that $\Sigma \cup \mathcal{H}$ is orientable.

We can compute the Cotton tensor (2.2) of $(\Sigma, \tilde{g})$ in a vacuum spacetime by using the vacuum equations (2.5) and $R_{\mu\nu} = 0$. Letting $S_{|k} := \tilde{\nabla}_k S$, we find that

$$C_{kij} = \frac{2}{S^2}S_{|[j}S_{|i]|k} + \frac{1}{S}\tilde{R}_{jink}S^{|n}. \tag{2.7}$$

In three dimensions, the Riemann tensor satisfies

$$\tilde{R}_{jink} = 2\tilde{g}_{j[n}\tilde{R}_{k]i} - 2\tilde{g}_{i[n}\tilde{R}_{k]j} + \tilde{R}\tilde{g}_{j[k}\tilde{g}_{n]i}. \tag{2.8}$$

Substituting this into (2.7) and using (2.5) results in

$$C_{kij} = \frac{4}{S^2}S_{|[j}S_{|i]|k} + \frac{2}{S^2}g_{k[i}S_{|j]|n}S^{|n}, \tag{2.9}$$

[2] Or possibly several, a priori—we will actually not need to assume that $\mathcal{H}$ is connected.

an expression which will be important in proving Israel's theorem.

Notice that the vacuum equations (2.5) can be made sense of independently from the ambient spacetime by specifying an arbitrary positive function $S \in C^\infty(\Sigma)$ (the positivity of S corresponding to stationarity). Before proceeding, we need an appropriate notion of asymptotic flatness for spacelike hypersurfaces without reference to an embedding into an ambient spacetime.

Definition 2.2. *We say a 3-dimensional Riemannian manifold $(\Sigma, \tilde{g})$ satisfying the vacuum equations (2.5) is* ***asymptotically flat with mass*** $M \geq 0$ *if there exists a compact set $K \subset \Sigma$ such that*

1. *$\Sigma \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus \overline{D}$, where $\overline{D}$ denotes the closed unit ball in $\mathbb{R}^3$ centred at 0.*
2. *With respect to the standard coordinates $\{y^1, y^2, y^3\}$ on $\mathbb{R}^3$ and the standard Euclidean metric $\delta = \operatorname{diag}(1,1,1)$, we have $\tilde{g} = \delta + O(|y|^{-1})$ and $S = 1 - M|y|^{-1} + O(|y|^{-2})$ in $\Sigma \setminus K$.*

If Σ is an embedded hypersurface of a spacetime (M, g), we also refer to (M, g) as asymptotically flat.

In this case, equations (2.5) allow us to choose coordinates $\{x^1, x^2, x^3\}$ on Σ such that (denoting $r = |x|$)

$$\tilde{g} = \left(1 + \frac{2M}{r}\right)\delta + O(r^{-2}), \qquad S = 1 - \frac{M}{r} + O(r^{-2}), \tag{2.10}$$

where the first derivatives of the $O(r^{-2})$ terms are themselves $O(r^{-3})$.

The last ingredient we need is a corollary of the positive mass theorem; for details, see [14].

Theorem 2.3. *Let (N, h) be a complete, orientable 3-dimensional Riemannian manifold with non-negative scalar curvature. If (N, h) is asymptotically flat with vanishing mass in the sense of Definition 2.2, then (N, h) is isometric to $\mathbb{R}^3$ with the standard Euclidean metric.*

3 Uniqueness of the Schwarzschild black hole

The first "no-hair" result was proven in [11]. Its original formulation can be paraphrased as:

Theorem 3.1 (Israel 1967). *The Schwarzschild metric is the unique static, asymptotically flat vacuum spacetime with closed, simply connected equipotential surfaces and regular event horizon.*

Israel's original proof consisted of the construction of two integral identities used to determine the geometric properties of the 2-dimensional equipotential hypersurfaces $V = -\langle k, k\rangle = \text{constant}$, where k is the static Killing vector field. By assuming that these surfaces are regular and homeomorphic to S^2, Israel was able to deduce spherical symmetry, and hence the uniqueness of the Schwarzschild metric.

Over the decades, numerous simplifications of the arguments used in Israel's seminal paper have arisen. The strategy we present, first published by Bunting and Masood-ul-Alam [3] in 1987, is to look for conformal transformations of the 3-dimensional hypersurface $(\Sigma, \tilde{g})$. By working on a conformally transformed version of $(\Sigma, \tilde{g})$, we may use the positive mass theorem (Theorem 2.3) to deduce conformal flatness. One particular advantage of this proof is that we need not assume that the event horizon is connected; instead, we will deduce its connectedness as a consequence (and hence, multiple black holes cannot exist in an asymptotically flat static vacuum spacetime).

We restate Theorem 3.1 with our relaxed assumption.

Theorem 3.2 (Bunting and Masood-ul-Alam 1987). *The Schwarzschild metric is the unique static, asymptotically flat vacuum spacetime with regular event horizon.*

Our goal is to turn $(\Sigma, \tilde{g})$ into an asymptotically flat, complete Riemannian manifold with vanishing scalar curvature and mass by making a suitably chosen conformal transformation and gluing along the boundary (i.e. forming the double). The following is adapted from [3], with simplifications provided by [8].

Lemma 3.3. *Suppose $(\Sigma, \tilde{g})$ is a 3-dimensional Riemannian manifold with boundary $\mathcal{H}$ which satisfies the vacuum equations (2.5) and the asymptotic condition (2.10) with $S > 0$ in Σ and $S = 0$ on $\mathcal{H}$. Let h_+ and h_- be the metrics on Σ defined by*

$$h_\pm = \Omega_\pm^2 \tilde{g}, \quad \text{where } \Omega_\pm = \frac{(1 \pm S)^2}{4}. \tag{3.1}$$

Then

1. *$\Omega_\pm$ vanishes nowhere in Σ;*
2. *(Σ, h_+) and (Σ, h_-) both have zero scalar curvature;*
3. *(Σ, h_+) is asymptotically flat with vanishing mass, in the sense of Definition 2.2;*
4. *h_- "compactifies the infinity", in the sense that h_- extends to $\Sigma \cup \{\infty\}$, where ∞ denotes the point at infinity.*

Proof sketch. **1.** Under the assumption of stationarity ($S > 0$ on Σ), we immediately obtain $\Omega_+ > \frac{1}{4}$ in Σ. To show that $\Omega_- \neq 0$ in Σ, one can show the surface gravity (2.6) satisfies $\kappa^2 = -\langle \operatorname{grad} S, \operatorname{grad} S \rangle$, from which it follows that the outward unit normal of $\mathcal{H}$ is given by

$$n = -\kappa^{-1} \operatorname{grad} S. \tag{3.2}$$

Regularity of the horizon ($\kappa \neq 0$) then implies that

$$\frac{\partial (S-1)}{\partial n} < 0 \quad \text{on } \mathcal{H}.$$

On the other hand, we have from (2.10) that

$$S - 1 = -\frac{M}{r} + O(r^{-2}) \quad \text{as } r \to \infty,$$

but $S - 1$ is harmonic (2.5) and hence can only attain its maximum of 0 at ∞, so $\Omega_- < 0$ on Σ.

2. Let $\hat{R}$ denote the scalar curvatures of both h_+ and h_-. From (2.1), we obtain

$$\frac{\Omega_\pm^4}{2} \hat{R} = \frac{\Omega_\pm^2}{2} R - \Omega_\pm (S \pm 1) \Delta S,$$

which vanishes by (2.5).

3. From the expansion of S given by (2.10), we have $\Omega_+ = 1 - M/r + O(r^{-2})$ as $r \to \infty$. Combining this with the expansion of $\tilde{g}$ also given by (2.10), we obtain

$$h_+ = \Omega_+^2 \tilde{g} = \delta + O(r^{-2}) \quad \text{as } r \to \infty.$$

Thus, (Σ, h_+) is asymptotically flat with vanishing mass.

4. Using the asymptotic expansion (2.10) in coordinates $z^i = x^i / |x|^2$, one can show that

$$h_\varepsilon(z) = \begin{cases} h_-(z) & \text{if } 0 < |z| < \varepsilon, \\ \left(\frac{M}{2}\right)^4 \delta_{ij}\, dz^i \otimes dz^j & \text{if } z = \infty \end{cases}$$

defines a metric on the compactification $\Sigma \cup \{\infty\}$. ∎

Unfortunately, it is not enough to pass from $\tilde{g}$ to one of the conformally equivalent metrics $h_\pm$ to use Theorem 2.3, the reason being that the boundary has negative mean curvature relative to the outward unit normal. However, we can bypass this obstruction; recall that the double of a manifold with boundary Σ is the manifold $N = (\Sigma \times \{0,1\})/\sim$ where $(x,0) \sim (x,1)$ for all $x \in \partial\Sigma$.

Since $S = 0$ on $\mathcal{H}$, the metrics h_+ and h_- match continuously on $\mathcal{H}$ and hence define a metric h on N. In fact, the same is true of the second fundamental forms corresponding to $h_\pm$. We compactify one end of N by adjoining a point at infinity; by Lemma 3.3, h extends to a metric on $N^* = N \cup \{\infty\}$. By construction, the manifold (N^*, h) satisfies the hypotheses of Theorem 2.3. We thus obtain:

Corollary 3.4. *The spatial geometry $(\Sigma, \tilde{g})$ of the domain of outer communications of an asymptotically flat, static vacuum spacetime with regular event horizon is conformally flat. In fact, if k denotes the hypersurface-orthogonal Killing field and $S^2 = -\langle k, k\rangle$, then the flat metric conformally equivalent to $\tilde{g}$ is given by $\frac{1}{16}(1+S)^4\tilde{g}$.*

Proof of Theorem 3.2. All that remains is to establish spherical symmetry. For this, we will use the conformal flatness of $(\Sigma, \tilde{g})$ along with the characterization given by Theorem 2.1 to deduce spherical symmetry of the 2-dimensional equipotential hypersurfaces Σ' (i.e. those of constant S). The unit normal to Σ' is given by

$$n = w^{-1/2}\,\mathrm{grad}\, S, \quad \text{where } w = \langle \mathrm{grad}\, S, \mathrm{grad}\, S\rangle,$$

and the induced metric on Σ' is

$$\beta_{ij} = \tilde{g}_{ij} - n_i n_j.$$

The extrinsic curvature H_{ij} is

$$H_{ij} = \beta_i^m \beta_j^k n_{k|m}$$

with trace-free part

$$\mathring{H}_{ij} = H_{ij} - \frac{1}{2}H\beta_{ij}.$$

One can show the (purely geometric) identity

$$\mathring{H}_{ij}\mathring{H}^{ij} + \frac{1}{8w^2}\beta^{ij}w_i w_j = \frac{S^4}{8w^2}\left(\frac{4}{S^2}S_{[j}S_{i]k} + \frac{2}{S^2}g_{k[i}S_{j]}S^n\right)^2 - \frac{1}{2w}(\tilde{\Delta}S)^2, \tag{3.3}$$

where $w_i := w_{|i}$ and $S_{ij} := S_{|i|j} := \tilde{\nabla}_j\tilde{\nabla}_i S$. Combined with expression (2.9) for the Cotton tensor in vacuum spacetime and the fact that $\tilde{\Delta}S = 0$, we obtain

$$\frac{S^4}{8w^2}C_{kij}C^{kij} = \mathring{H}_{ij}\mathring{H}^{ij} + \frac{1}{8w^2}\beta^{ij}w_i w_j. \tag{3.4}$$

Using conformal flatness of $(\Sigma, \tilde{g})$, i.e. $C_{kij} = 0$, we deduce that

$$H_{ij} - \frac{1}{2}H\beta_{ij} = 0, \qquad \beta^{ij}w_{;j} = 0,$$

yielding spherical symmetry. It follows that Σ is diffeomorphic to the product $S^2 \times \mathbb{R}$ with metric

$$\tilde{g} = f^2\, dS^2 + r^2\, d\Omega^2, \tag{3.5}$$

where f and r are functions of the radial coordinate S only and $d\Omega^2 = d\theta^2 + \sin^2(\theta)\, d\varphi^2$ is the metric on S^2. Laplace's equation and the $\tilde{R}_{SS}$ equation (2.5) become

$$\left(r^2 f^{-1}\right)' = 0, \qquad \left(\frac{r'}{r}\right)' - \frac{f'r'}{fr} + \left(\frac{r'}{r}\right)^2 = \frac{1}{2S}\frac{f'}{f}.$$

Applying the asymptotic expansion (2.10) for S yields the solution $f = \frac{r^2}{M}$ and $r(S) = \frac{2M}{1-S^2}$. Thus, combining (2.3) with (3.5) and $f^2\, dS^2 = S^{-2}\, dr^2$ yields

$$g = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\, d\Omega^2,$$

which concludes the proof. ■

4 Uniqueness of the Reissner–Nördstrom black hole

We now generalize the results of Section 3 to the case of non-vanishing electric charge. Specifically, we will sketch a proof of the following:

THEOREM 4.1 (MASOOD-UL-ALAM 1992). *The Reissner–Nordström metric is the unique static, asymptotically flat electrovacuum spacetime with regular event horizon.*

Again, we emphasize that connectedness of the horizon is not assumed. The proof of Theorem 4.1 is similar to that of Theorem 3.2: we apply the positive mass theorem to show that $(\Sigma, \tilde{g})$ is conformally flat, and then use the vanishing of the Cotton tensor (2.2) to deduce spherical symmetry. For details, see [12].

Once more, we use the decomposition (2.3) and assume that $(\Sigma, \tilde{g})$ is asymptotically flat with asymptotic expansion (2.10). The electric potential ϕ is defined by $d\phi = E$, the gauge chosen so that ϕ vanishes at spacelike infinity. One can show that, in the same coordinates used in (2.10), ϕ has asymptotic expansion

$$\phi = \frac{Q}{r} + O(r^{-2}), \tag{4.1}$$

where Q denotes the total electric charge. Similarly to before, we assume that the first derivatives of the $O(r^{-2})$ terms are themselves $O(r^{-3})$.

Let $\phi_i := \tilde{\nabla}_i \phi$ and $\langle \cdot, \cdot \rangle$ denote the inner product with respect to $\tilde{g}$. Then the stress-energy tensor on Σ is given by

$$T_{tt} = \frac{1}{8\pi}\langle \operatorname{grad}\phi, \operatorname{grad}\phi\rangle, \qquad T_{ij} = \frac{1}{8\pi S^2}\left(\tilde{g}_{ij}\langle \operatorname{grad}\phi, \operatorname{grad}\phi\rangle - 2\phi_i\phi_j\right).$$

Combined with Einstein's equations $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}$, the analogous equations to (2.4) are

$$\tilde{\Delta} S = S^{-1}\langle \operatorname{grad}\phi, \operatorname{grad}\phi\rangle, \tag{4.2}$$

$$\tilde{R}_{ij} = S^{-1}\tilde{\nabla}_j\tilde{\nabla}_i S + S^{-2}(\tilde{g}_{ij}\langle \operatorname{grad}\phi, \operatorname{grad}\phi\rangle - 2\phi_i\phi_j), \tag{4.3}$$

$$\tilde{R} = 2S^{-1}\tilde{\Delta} S, \tag{4.4}$$

and Maxwell's equation for ϕ is

$$\tilde{\Delta}\phi = S^{-1}\langle \operatorname{grad}\phi, \operatorname{grad} S\rangle. \tag{4.5}$$

We now state a generalization of Lemma 3.3.

LEMMA 4.2. *Suppose* $(\Sigma, \tilde{g})$ *satisfies the electrovacuum equations* (4.2) – (4.5) *and the asymptotic conditions* (2.10) *and* (4.1) *with* $S > 0$ *in* Σ *and* $S = 0$ *on* $\partial\Sigma$. *Let* $h_\pm$ *be the metrics on* Σ *defined by*

$$h_\pm = \Omega_\pm^2 \tilde{g}, \quad \text{where } \Omega_\pm = \frac{(1 \pm S)^2 - \phi^2}{4}.$$

Then, provided that $|Q| < M$*:*

1. $\Omega_\pm$ *vanishes nowhere in* Σ*;*
2. (Σ, h_+) *and* (Σ, h_-) *both have non-negative scalar curvature;*
3. (Σ, h_+) *is asymptotically flat with vanishing mass;*
4. h_- *extends to* $\Sigma \cup \{\infty\}$.

That is, the conclusions of Lemma 3.3 still hold, with the exception that "zero scalar curvature" be replaced with "non-negative scalar curvature".

Proof sketch. **1.** Since S is non-negative, it suffices to show that $(1-S)^2-\phi^2 = (S+\phi-1)(S-\phi-1)$ is positive to prove the result for both Ω_+ and Ω_-. By (4.2) and (4.5), these two factors satisfy

$$\tilde{\Delta}(S \pm \phi - 1) \mp S^{-1}\langle \operatorname{grad} \phi, \operatorname{grad}(S \pm \phi - 1)\rangle = 0.$$

Similarly to before, regularity of the event horizon implies that the outward unit normal of $\partial\Sigma$ is given by (3.2), and we have

$$\frac{\partial(S \pm \phi - 1)}{\partial n} < 0 \quad \text{on } \partial\Sigma.$$

Then using the asymptotic expansion

$$S \pm \phi - 1 = \frac{-M \pm Q}{r} + O(r^{-2}) \quad \text{as } r \to \infty$$

and the maximal principle shows that $S \pm \phi - 1$ is non-positive in Σ and can vanish only at ∞, so both factors are negative in Σ. (This argument fails if $|Q| = M$.)

2. We have

$$d\Omega_\pm = \frac{1}{2}\left((S \pm 1)\, dS - \phi\, d\phi\right)$$

and

$$\tilde{\Delta}\Omega_\pm = \frac{1}{2}\left(\langle \operatorname{grad} S, \operatorname{grad} S\rangle - \langle \operatorname{grad} \phi, \operatorname{grad} \phi\rangle + (S \pm 1)\tilde{\Delta}S - \phi\tilde{\Delta}\phi\right).$$

Combining (4.2), (4.4), and (4.5) with (2.1) and some algebraic manipulation, we see that the conformally transformed scalar curvature $\hat{R}$ satisfies

$$\begin{aligned}
\frac{\Omega_\pm^4}{2}\hat{R} &= \frac{1}{S^2}\left(\Omega_\pm^2 \mp \Omega_\pm S + \frac{1}{4}S^2\phi^2\right)|\operatorname{grad} \phi|^2 + \frac{1}{4}\phi^2|\operatorname{grad} S|^2 + \frac{\phi}{4S}(1 - S^2 - \phi^2)\langle \operatorname{grad} \phi, \operatorname{grad} S\rangle \\
&= \frac{1}{S^2}\left(\frac{1 - S^2 - \phi^2}{4}\right)^2 |\operatorname{grad} \phi|^2 + \frac{1}{4}\phi^2|\operatorname{grad} S|^2 + \frac{\phi}{4S}(1 - S^2 - \phi^2)\langle \operatorname{grad} \phi, \operatorname{grad} S\rangle \\
&= \frac{\left|(1 - S^2 - \phi^2)\operatorname{grad} \phi + 2\phi S \operatorname{grad} S\right|^2}{(4S)^2},
\end{aligned}$$

which is a perfect square and hence non-negative.

The proofs of assertions 3 and 4 are similar to that of Lemma 3.3, so we omit them. ■

Once more, we pass to the double N of Σ and compactify it by adjoining a point at infinity to one end. The Riemannian 3-manifold (N^*, h) obtained in this way satisfies the hypotheses of Theorem 2.3. We thus obtain:

Corollary 4.3. *The spatial geometry $(\Sigma, \tilde{g})$ of the domain of outer communications of an asymptotically flat, static electrovacuum spacetime with regular event horizon is conformally flat. In fact, if k denotes the hypersurface-orthogonal Killing field and $S^2 = -\langle k, k\rangle$, then the flat metric equivalent to $\tilde{g}$ is given by $\frac{1}{16}((1+S)^2 - \phi^2)^2\tilde{g}$. Moreover, the electric and gravitational potentials satisfy*

$$S^2 - \phi^2 + \frac{2M}{Q}\phi = 1. \tag{4.6}$$

We conclude the proof of Theorem 4.1 by establishing spherical symmetry. The technique is similar to the vacuum case but requires more algebra, some of which we will omit.

Proof of Theorem 4.1. First, we eliminate ϕ from the electrovacuum equations: by (4.6), we have $d\phi = \frac{S}{\phi - m}\, dS$, where $m := M/Q$. Thus, (4.2) – (4.4) become

$$\tilde{\Delta} S = \frac{S}{S^2 - 1 + m^2}\langle \operatorname{grad} S, \operatorname{grad} S\rangle, \tag{4.7}$$

$$\tilde{R}_{ij} = \frac{1}{S}S_{ij} + \frac{1}{S^2 - 1 + m^2}\left(\tilde{g}_{ij}\langle \operatorname{grad} S, \operatorname{grad} S\rangle - 2S_iS_j\right), \tag{4.8}$$

$$\tilde{R} = \frac{2}{S^2 - 1 + m^2}\langle \operatorname{grad} S, \operatorname{grad} S\rangle, \tag{4.9}$$

where $S_{ij} = S_{|i|j}$ as before. Then, by using the electrovacuum analogues of (2.7) and (2.8), one can show that the electrovacuum analogue of equation (2.9) for the Cotton tensor is

$$C_{kij} = \left(1 - \frac{S^2}{S^2 - 1 + m^2}\right)\left(\frac{4}{S^2}S_{[j}S_{i]k} + \frac{2}{S^2}\tilde{g}_{k[i}S_{j]n}S^n + \frac{2S^nS_n}{S^2 - 1 + m^2}\frac{1}{S}\tilde{g}_{k[j}S_{i]}\right). \tag{4.10}$$

More (tedious and not terribly enlightening, hence omitted) algebraic manipulation involving (3.3) yields

$$\begin{aligned}\frac{S^4}{8w^2}C_{kij}C^{kij} &= \left(\frac{m^2-1}{S^2-1+m^2}\right)^2\left(\mathring{H}_{ij}\mathring{H}^{ij} + \frac{1}{8w^2}\beta^{ij}w_iw_j + \frac{1}{2}\left(\frac{\tilde{\Delta}S}{w} - \frac{S}{S^2-1+m^2}\right)^2\right)\\ &= \left(\frac{m^2-1}{S^2-1+m^2}\right)^2\left(\mathring{H}_{ij}\mathring{H}^{ij} + \frac{1}{8w^2}\beta^{ij}w_iw_j\right),\end{aligned}$$

where the term in the right-most paranthesis vanishes by (4.7) (recall that $w = \langle \operatorname{grad} S, \operatorname{grad} S\rangle$). This shows that

$$\frac{S^4}{8w^2}C_{kij}C^{kij} = \left(\frac{M^2 - Q^2}{M^2 + Q^2(S^2-1)}\right)\left(\mathring{H}_{ij}\mathring{H}^{ij} + \frac{1}{8w^2}\beta^{ij}w_iw_j\right),$$

which is the same expression as (3.4) up to the first factor. But this first factor cannot vanish as $|Q| < M$, so we deduce that

$$H_{ij} - \frac{1}{2}H\beta_{ij} = 0, \qquad \beta^{ij}w_{;j} = 0,$$

once more yielding spherical symmetry. Thus, as before, the metric on Σ can be written in the form (3.5), where f and r are functions of S. In this case however, the Poisson equation (4.7) becomes

$$\frac{(r^2 f^{-1})'}{r^2 f^{-1}} = \frac{Q^2 S}{Q^2(S^2-1)+M^2},$$

which has solution

$$f(S) = \frac{r^2(S)}{\sqrt{Q^2(S^2-1)+M^2}}. \tag{4.11}$$

The $\tilde{R}_{SS}$ component of (4.8) becomes

$$\left(\frac{r'}{r}\right)' - \frac{f'r'}{fr} + \left(\frac{r'}{r}\right)^2 = \frac{1}{2S}\left(\frac{f'}{f} + \frac{Q^2 S}{Q^2(S^2-1)+M^2}\right)$$

which is satisfied provided that $r'(S) = Sf(S)$ with f given by (4.11). Using asymptotic expansion once more, we obtain

$$r(S) = \frac{Q^2}{M - \sqrt{Q^2(S^2-1)+M^2}},$$

yielding the Reissner–Nördstrom metric

$$g = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2\, d\Omega^2.$$

■

Remark 4.4. We have proven uniqueness of the two-parameter Reissner–Nördstrom metric. Some theories of electromagnetism, however, predict the existence of *magnetic charge*, which we did not account for. Actually, it is relatively straightforward to generalize the techniques we have used to the case of magnetic charge $P \neq 0$, provided that $M^2 > Q^2 + P^2$. The key idea is to perform a "duality rotation" to reduce to the purely electric case. Details can be found in [8].

5 Uniqueness of the Kerr–Newman black hole

The three-parameter Kerr–Newman metric describes the geometry exterior to a black hole of mass M, angular momentum J, and electric charge Q, subject to the inequality

$$M^2 > (J/M)^2 + Q^2.$$

This generalizes both the Schwarzschild and Reissner–Nördstrom metrics as derived in Sections 3 and 4. The special case $Q = 0$, $J \neq 0$ is known as the Kerr metric.

Currently, the no-hair conjecture only has partial resolution in this general setting: even in the case $Q = 0$, combined results from Hawking [6], Carter [4], and Robinson [13] demonstrate that the Kerr family is unique among stationary vacuum solutions with regular event horizons, but require the additional assumption of *real analyticity* of spacetime, which is both restrictive and difficult to justify from a physical standpoint. More recent work by Ionescu and Klainerman [9] has shown that this requirement can be relaxed to that of smoothness and a technical condition relating the Ernst potential and Killing scalar. Shortly thereafter, Alexakis, Ionescu, and Klainerman [1] extended this result to "small perturbations" of Kerr spaces.

One may also consider extremal Kerr(–Newman) black holes, which satisfy

$$M^2 = (J/M)^2 + Q^2.$$

Here also, we have a (much more recent) proof of uniqueness among stationary, rotating, asymptotically flat electrovacuum solutions, but again, only with the additional assumption of analyticity. For details, see [2].

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