Euler-like vector fields

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April 28, 2021

A motivating example

- Recall: for a vector space V and $\xi \in V$, there is a canonical linear isomorphism $V \cong T_\xi V$ given by $u \mapsto \frac{d}{dt}\big|_{t=0}(\xi + tu)$.
- When $V = T_x M$, this is called the **vertical lift** $\mathrm{vl}_\xi : T_x M \to T_\xi(T_x M)$.
- Identifying $T_\xi(T_x M)$ with a subspace of $T_{(x,\xi)}(TM)$, we obtain the **Euler vector field** $\mathcal{E} \in \mathfrak{X}(TM)$:

$$\mathcal{E}(x,\xi) = \mathrm{vl}_\xi(\xi) \in T_{(x,\xi)}(TM).$$

- In coordinates $(x^i, v^i = dx^i)$, $\mathcal{E} = \sum_{i=1}^n v^i \frac{\partial}{\partial v^i}$.

(c.f. tautological one-form $\alpha = \underbrace{\sum_{i=1}^n p_i \, dx^i}_{\text{in coordinates } (x^i,\, p_i = \frac{\partial}{\partial x^i})} \in \Omega^1(T^*M)$)

Same game, arbitrary vector bundles

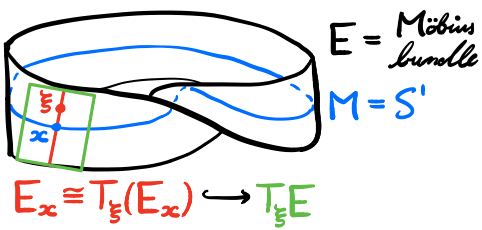
- For a vector bundle $\pi\colon E \to M$, the **Euler vector field** $\mathcal{E} \in \mathfrak{X}(E)$ is defined for $x \in M$, $\xi \in E_x := \pi^{-1}(x)$ by

  a vector space

$$\mathcal{E}(\xi) = \mathrm{vl}_\xi(\xi) \in T_\xi E,$$

  $= V_\xi E$, "vertical space at ξ"

  where $\mathrm{vl}_\xi\colon E_x \xrightarrow{\sim} T_\xi(E_x) \hookrightarrow T_\xi E$.



- In bundle coordinates (x^i, v^i), $\mathcal{E} = \sum_{i=1}^n v^i \frac{\partial}{\partial v^i}$.

Euler-like vector fields on $\mathbb{R}^n$

- Take $M = \{*\}$, $E = \mathbb{R}^n$, $\mathcal{E} = \sum_{i=1}^n x^i \frac{\partial}{\partial x^i}$ (relabel $v^i \to x^i$).

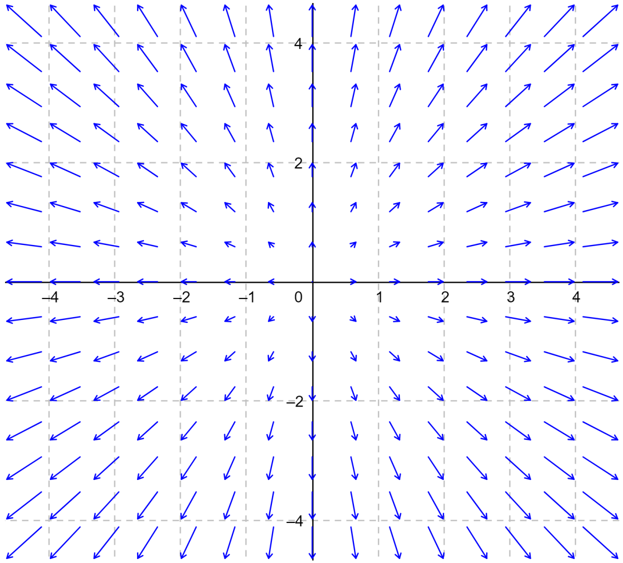


Figure: The Euler vector field $\mathcal{E} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ on $\mathbb{R}^2$.

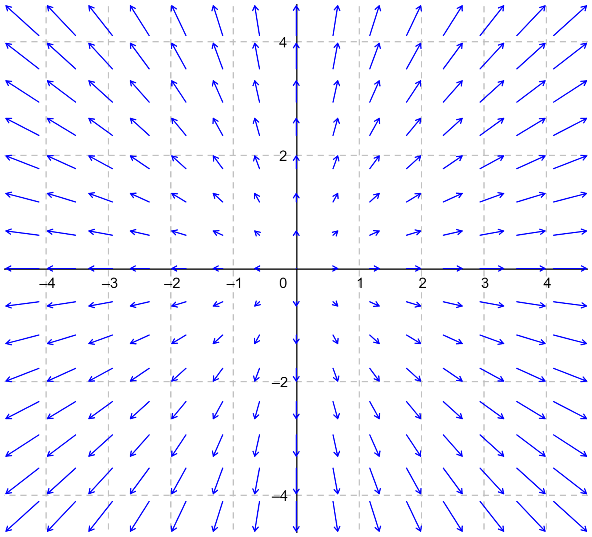
- Idea: a vector field $X \in \mathfrak{X}(\mathbb{R}^n)$ (or in a neighbourhood of 0) is Euler-like if "$X = \mathcal{E}$ + higher order terms".

Euler-like vector fields on $\mathbb{R}^n$

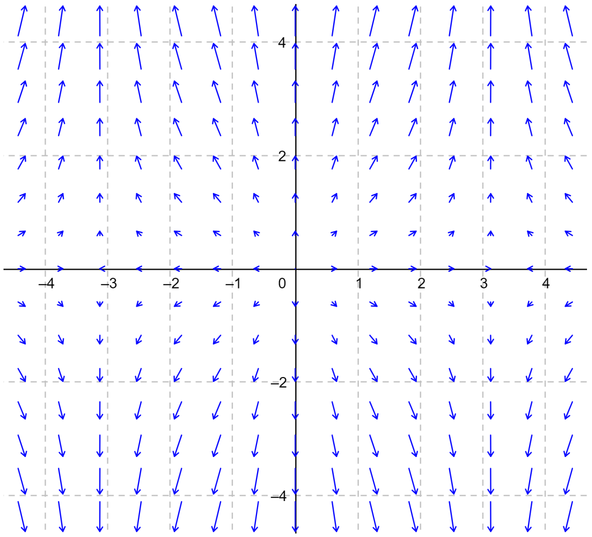
- If $X(0) = 0$, the **linear approximation** of $X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$ is the vector field $X_{[0]} \in \mathfrak{X}(\mathbb{R}^n)$ obtained by replacing each X^i with its first-order Taylor expansion at 0:

$$X_{[0]} = \sum_{i,j=1}^n a^i_j x^j \frac{\partial}{\partial x^i}, \qquad a^i_j = \frac{\partial X^i}{\partial x^j}(0).$$

- X is **Euler-like** if $X_{[0]} = \mathcal{E}$ ($\Leftrightarrow a^i_j = \delta^i_j$).



(a) The Euler vector field $\mathcal{E} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.



(b) The Euler-like vector field $X = \sin(x)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.

Euler-like vector fields on $\mathbb{R}^n$ are linearizable

Lemma (Linearization)

If $X \in \mathfrak{X}(\mathbb{R}^n)$ is Euler-like, then there exists a germ at 0 of a diffeomorphism φ of $\mathbb{R}^n$ such that $\varphi(0) = 0$, $D\varphi(0) = \mathrm{id}$, and $\varphi^* X = \mathcal{E}$.

$$\varphi^* \underset{X_{[0]}}{\underbrace{X = \mathcal{E}}}$$

Proof (Moser-type argument).

Write $X = \sum_i X^i \frac{\partial}{\partial x^i}$. The TDVF $X_t := \sum_i \frac{1}{t} X^i(tx) \frac{\partial}{\partial x^i}$ $(t \neq 0)$ extends smoothly to $t=0$ by $X_0 = X_{[0]} = \mathcal{E}$. Computation $\Longrightarrow \frac{dX_t}{dt} = \frac{1}{t}[\mathcal{E}, X_t]$ $(\ddagger)$

$X_t = \mathcal{E} + o(t)$ as $t \to 0 \Rightarrow \omega_t := \frac{1}{t}(X_t - \mathcal{E})$ also extends smoothly to $t=0$.

Let $t \mapsto \Phi_t$ be the flow of the TDVF (ω_t).

Moser: $\frac{d}{dt} \Phi_t^* X_t = \Phi_t^* \Big(\underset{(\ddagger)}{\underset{\shortparallel}{\frac{dX_t}{dt}}} + \underset{[\omega_t, X_t]}{\underset{\shortparallel}{\mathcal{L}_{\omega_t} X_t}} \Big) = \Phi_t^* (t^{-1} [X_t, X_t]) = 0 \Longrightarrow \Phi_t^* X_t$ const.

Set $\varphi := \Phi_1 \Rightarrow \varphi^* X = \Phi_1^* X_1 = \Phi_0^* X_0 = \mathcal{E}$.

$\forall t$, ω_t vanishes to 2^{nd} order at $x=0 \Rightarrow \varphi(0) = 0$, $D\varphi(0) = \mathrm{id}$.

Hadamard's lemma:
$f \in C^\infty(\mathbb{R}^n) \Rightarrow \exists g_1, \dots, g_n \in C^\infty(\mathbb{R}^n)$ w/
$g_i(0) = \frac{\partial f}{\partial x^i}(0)$, $f(x) = f(0) + \sum_{i=1}^n x^i g_i(x)$.

$\square$

Euler-like vector fields on $\mathbb{R}^n$: applications

Lemma (Morse)

Let $f \in C^\infty(\mathbb{R}^n)$ be a smooth function with $f(0) = 0$. If f has a non-degenerate critical point at 0, then there exists a diffeomorphism φ of two neighbourhoods of 0 such that $\varphi(0) = 0$ and $\varphi^* f$ is a homogeneous quadratic polynomial.

Theorem (Darboux)

Let $\omega \in \Omega^2(\mathbb{R}^n)$ be a closed 2-form. If ω is non-degenerate at 0, then there exists a diffeomorphism φ of two neighbourhoods of 0 such that $\varphi(0) = 0$ and $\varphi^* \omega$ is constant.

Facts

We will use two facts about the Euler vector field $\mathcal{E} \in \mathfrak{X}(\mathbb{R}^n)$:

1. A smooth function $f \in C^\infty(\mathbb{R}^n)$ (or in a neighbourhood of 0) satisfies $\mathcal{L}_\mathcal{E} f = kf$ if and only if f is a homogeneous polynomial of degree k.
2. A smooth k-form $\omega \in \Omega^k(\mathbb{R}^n)$ (or in a neighbourhood of 0) satisfies $\mathcal{L}_\mathcal{E}\omega = k\omega$ if and only if ω has constant coefficients.

Application: Morse's lemma

Lemma (Morse)

Let $f \in C^\infty(\mathbb{R}^n)$ be a smooth function with $f(0) = 0$. If f has a non-degenerate critical point at 0, then there exists a diffeomorphism φ of two neighbourhoods of 0 such that $\varphi(0) = 0$ and $\varphi^* f$ is a homogeneous quadratic polynomial.

Fact: $f \in C^\infty(\mathbb{R}^n)$ (or in a neighbourhood of 0) satisfies $\mathcal{L}_\mathcal{E} f = kf$ if and only if f is a homogeneous polynomial of degree k. (Relevant: $k = 2$.)

Proof.

Taylor expand $f(x) = \frac{1}{2}\sum_{i,j} A_{ij}(x)x^i x^j$ w/ $x \mapsto A(x) = (A_{ij}(x))$ smooth, symmetric, $A(0) = \mathrm{Hess}\, f(0)$.

Computation $\Rightarrow \frac{\partial f}{\partial x^j} = \sum_k B_{jk}(x)x^k$, where $B_{jk} = A_{jk} + \frac{1}{2}\sum_\ell \frac{\partial A_{k\ell}}{\partial x^j} x^\ell$.

$B(0) = A(0)$ non-degenerate $\Rightarrow B(x)$ non-degenerate for x near 0.

Thus $X := \sum_{i,j} \underbrace{(A(x)B(x)^{-1})_{ij}}_{= I + \text{h.o.t.}} x^i \frac{\partial}{\partial x^j}$ is well-def'd, E-L near 0.

Linearization lemma $\Rightarrow \exists \varphi$ w/ $\varphi^* X = \mathcal{E}$.

Computation $\Rightarrow \mathcal{L}_X f = 2f \Longrightarrow \mathcal{L}_\mathcal{E} \varphi^* f = \varphi^* \mathcal{L}_X f = 2\varphi^* f$.

$\square$

Application: Darboux's theorem

Theorem (Darboux)

Let $\omega \in \Omega^2(\mathbb{R}^n)$ be a closed 2-form. If ω is non-degenerate at 0, then there exists a diffeomorphism φ of two neighbourhoods of 0 such that $\varphi(0) = 0$ and $\varphi^*\omega$ is constant.

Fact: $\omega \in \Omega^k(\mathbb{R}^n)$ (or in a neighbourhood of 0) satisfies $\mathcal{L}_\mathcal{E}\omega = k\omega$ if and only if ω has constant coefficients. (Relevant: $k = 2$.)

Proof.

Taylor expand $\omega = \sum_{i<j} (c_{ij} + O(|x|))\, dx^i \wedge dx^j$. Poincaré lemma $\Rightarrow \exists \alpha \in \Omega^1(\mathbb{R}^n)$ s.t. $d\alpha = \omega$.

Coordinate expression $\Rightarrow \alpha$ can be chosen of the form $\alpha = \sum_{i,j} (c_{ij} x^i + O(|x|^2))\, dx^j$.

ω non-degenerate at $0 \Rightarrow$ non-degen near $0 \Rightarrow$ can solve $\iota_X \omega = 2\alpha$ for X near 0.

Coordinate expressions $\Rightarrow X$ is E-L $\Rightarrow \exists \varphi$.

Cartan $\Rightarrow \mathcal{L}_X \omega = d\iota_X\omega + \iota_X d\omega = 2d\alpha = 2\omega$

$\Rightarrow \mathcal{L}_\mathcal{E} \varphi^*\omega = \varphi^* \mathcal{L}_X \omega = 2\varphi^*\omega.$

$\square$

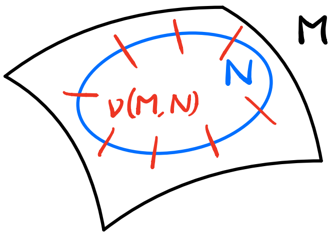
Euler-like vector fields for submanifolds: setting up

- The category Man:
  - objects: smooth manifolds
  - morphisms from M to M': smooth maps $\varphi\colon M \to M'$
- The category Man^2:
  - objects: pairs (M, N) with M a smooth manifold and $N \subseteq M$ a closed submanifold
  - morphisms from (M, N) to (M', N'): smooth maps $\varphi\colon M \to M'$ with $\varphi(N) \subseteq N'$

Euler-like vector fields for submanifolds: setting up

- The *tangent bundle functor* $T\colon \mathsf{Man} \to \mathsf{Man}$:
  - objects M sent to TM
  - morphisms $\varphi\colon M \to M'$ sent to $D\varphi\colon TM \to TM'$

- The *normal bundle functor* $\nu\colon \mathsf{Man}^2 \to \mathsf{Man}$:
  - objects (M,N) sent to $\nu(M,N) := TM|_N/TN$ (vector bundle over N)
  - morphisms $\varphi\colon (M,N) \to (M',N')$ sent to $\nu(\varphi)\colon \nu(M,N) \to \nu(M',N')$



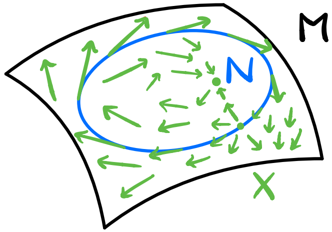
$\varphi(N) \subseteq N' \Rightarrow D\varphi\colon TM|_N \to TM'|_{N'}$

$\forall x \in N,\ D\varphi(x)[T_xN] \subseteq T_{\varphi(x)}N'$ (chain rule)

$$\nu(\varphi)(x)\colon \frac{T_xM}{T_xN} \to \frac{T_{\varphi(x)}M'}{T_{\varphi(x)}N'}$$

- Compatibility: $\nu(TM, TN) \cong T\nu(M,N)$

Euler-like vector fields for submanifolds: definitions



Vector field $X \in \mathfrak{X}(M)$ tangent to submanifold N

$(\forall x \in N, X(x) \in T_x N)$

$\Downarrow$

Morphism $X\colon (M, N) \to (TM, TN)$ in Man^2

$\downarrow \nu$

$\nu(X)\colon \nu(M, N) \to \nu(TM, TN) \cong T\nu(M, N)$

$\downarrow$

Linear approximation $X_{[0]} := \nu(X) \in \mathfrak{X}(\nu(M, N))$

$\downarrow$

Euler-like if $X_{[0]} = \mathcal{E}$ (of $\nu(M, N)$).

- Previous definition: when $(M, N) = (\text{open neighbourhood of } 0 \in \mathbb{R}^n, \{0\})$.

Euler-like vector fields for submanifolds: applications

Theorem (Bursztyn, Lima, Meinrenken)

An Euler-like vector field X for (M, N) determines a unique maximal tubular neighbourhood embedding $\varphi\colon O \to M$ of a star-shaped open neighbourhood $O \subseteq \nu(M, N)$ of the zero section of $\nu(M, N)$ such that $\varphi^* X = \mathcal{E}$. + Functoriality!

Corollaries:

- Weinstein's Lagrangian neighbourhood theorem
- Morse–Bott lemma
- Grabowski–Rodkievicz theorem
- Linearization of proper Lie groupoids
- Linearization of proper symplectic groupoids

+ G-equivariant versions!