EULER-LIKE VECTOR FIELDS

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Abstract. Euler-like vector fields provide a convenient framework for obtaining normal form results for various geometric structures, as the proofs of such results can often be reduced to constructing an Euler-like vector field that is compatible with a given structure and exploiting homogeneity. We introduce Euler-like vector fields on $\mathbb{R}^n$ and demonstrate this heuristic principle with proofs of Morse's lemma and Darboux's theorem. We then discuss how this framework generalizes to Euler-like vector fields for submanifolds.

1. Introduction: Euler vector fields

Given a vector space V and $\xi \in V$, there is a natural linear isomorphism $V \cong T_\xi V$ mapping $u \in V$ to $\frac{d}{dt}\Big|_{t=0}(\xi + tu) \in T_\xi V$. When $V = T_xM$ is the tangent space of a smooth manifold M at a point $x \in M$, this isomorphism is called the *vertical lift* $\mathrm{vl}_\xi \colon T_xM \to T_\xi(T_xM)$. This terminology comes from the fact that $T_\xi(T_xM)$ can be naturally identified with the *vertical space* $\ker(D\pi(x,\xi)) \subseteq T_{(x,\xi)}(TM)$, where $\pi\colon TM \to M$ is the footpoint map $\pi(x,\xi) = x$. Using this identification, we define a section of the double tangent bundle

$$\mathcal{E}\colon TM \to T(TM), \qquad \mathcal{E}(x,\xi) = \mathrm{vl}_\xi(\xi).$$

If (x^i) are local coordinates on M and $v^i = dx^i$, then $\mathcal{E} = \sum_i v^i \frac{\partial}{\partial v^i}$ with respect to the local coordinates (x^i, v^i) on TM. Thus $\mathcal{E}$ defines a smooth vector field, i.e. an element of $\mathfrak{X}(TM)$. The existence of such a vector field on tangent bundles is analogous to the existence of the tautological one-form on cotangent bundles.

The above construction generalizes to any vector bundle $\pi\colon E \to M$. Namely, the **Euler vector field** $\mathcal{E} \in \mathfrak{X}(E)$ is defined as follows: given $x \in M$, the fibre $E_x := \pi^{-1}(x)$ is a vector space, so for $\xi \in E_x$, the vertical lift $\mathrm{vl}_\xi\colon E_x \to T_\xi(E_x)$ identifies E_x with the vertical space $\ker(D\pi(\xi)) \subseteq T_\xi E$. We then set $\mathcal{E}(\xi) = \mathrm{vl}_\xi(\xi)$. As is the case when $E = TM$, the coordinate expression of $\mathcal{E}$ in bundle coordinates (x^i, v^i) is given by $\mathcal{E} = \sum_i v^i \frac{\partial}{\partial v^i}$, hence $\mathcal{E}$ is smooth.

2. Euler-like vector fields on $\mathbb{R}^n$

The Euler vector field of $\mathbb{R}^n$ is obtained by regarding $\mathbb{R}^n$ as a vector bundle over a one-point space. In this case, we relabel the fibre coordinates from v^i to x^i, so that $\mathcal{E} = \sum_i x^i \frac{\partial}{\partial x^i}$. The notion of Euler-like vector fields in this setting is given by those vector fields which are equal to $\mathcal{E}$ up to first order near the origin.

Definition 2.1. *Let X be a vector field on a neighbourhood of $0 \in \mathbb{R}^n$ such that $X(0) = 0$. The* ***linear approximation*** *of X (at 0) is the vector field $X_{[0]} \in \mathfrak{X}(\mathbb{R}^n)$ obtained by writing $X = \sum_i X^i \frac{\partial}{\partial x^i}$ and replacing each X^i with its first-order Taylor expansion at 0. We say X is* ***Euler-like*** *if $X_{[0]} = \mathcal{E}$.*

In the proofs to come of Morse's lemma and Darboux's theorem, we will use the following crucial lemma, which asserts that Euler-like vector fields on $\mathbb{R}^n$ are smoothly linearizable.

Lemma 2.2 ([2]). *If $X \in \mathfrak{X}(\mathbb{R}^n)$ is Euler-like, then there exists a diffeomorphism φ of two neighbourhoods of 0 such that $\varphi(0) = 0$, $D\varphi(0) = \mathrm{id}$, and $\varphi^* X = \mathcal{E}$.*

The proof we present of Lemma 2.2, from [2], follows a variation of Moser's method.

Proof. Write $X = \sum_i X^i \frac{\partial}{\partial x^i}$ and consider the time-dependent vector field

$$X_t := \sum_i t^{-1} X^i(tx) \frac{\partial}{\partial x^i}, \qquad t \neq 0.$$

This extends smoothly to $t = 0$ by $X_0 = X_{[0]} = \mathcal{E}$, and a straightforward computation shows that

$$\frac{dX_t}{dt} = t^{-1}[\mathcal{E}, X_t].$$

Since $X_t = \mathcal{E} + o(t)$ as $t \to 0$ (in little-o notation), the time-dependent vector field

$$W_t := t^{-1}(X_t - \mathcal{E})$$

also extends smoothly to $t = 0$. Letting $t \mapsto \Phi_t$ denote the flow generated by (W_t), we have[1]

$$\frac{d}{dt}\Phi_t^* X_t = \Phi_t^* \left(\frac{dX_t}{dt} + \mathcal{L}_{W_t} X_t \right) = \Phi_t^*([t^{-1}\mathcal{E} + W_t, X_t]) = \Phi_t^*(t^{-1}[X_t, X_t]) = 0.$$

Thus, $\Phi_t^* X_t$ is constant, so $\varphi := \Phi_1$ satisfies

$$\varphi^* X = \Phi_1^* X_1 = \Phi_0^* X_0 = (\mathrm{id})^* \mathcal{E} = \mathcal{E}.$$

For any fixed t, the vector field W_t vanishes to second order at $x = 0$, hence $\Phi_t(0) = 0$ and $D\Phi_t(0) = \mathrm{id}$; in particular, φ satisfies $\varphi(0) = 0$ and $D\varphi(0) = \mathrm{id}$. ∎

Although Lemma 2.2 was stated for Euler-like vector fields on $\mathbb{R}^n$, a similar result holds for Euler-like vector fields X defined on a neighbourhood of 0. (For example, choose a smooth function η which equals 1 in a neighbourhood of 0 and has compact support contained in the domain of X, and apply the lemma to ηX.)

We will use the following two facts, which demonstrate how differentiation along the Euler vector field $\mathcal{E}$ can detect homogeneity:

(1) A smooth function $f \in C^\infty(\mathbb{R}^n)$ (or in a neighbourhood of 0) satisfies $\mathcal{L}_{\mathcal{E}} f = kf$ if and only if f is a homogeneous polynomial of degree k.
(2) A smooth k-form $\omega \in \Omega^k(\mathbb{R}^n)$ (or in a neighbourhood of 0) satisfies $\mathcal{L}_{\mathcal{E}}\omega = k\omega$ if and only if ω has constant coefficients.

Neither facts are terribly difficult to prove; for example, fact (1) follows from Euler's homogeneous function theorem and Taylor expanding at 0.

Lemma 2.3 (Morse). *Let $f \in C^\infty(\mathbb{R}^n)$ be a smooth function with $f(0) = 0$. If f has a non-degenerate critical point at 0, then there exists a diffeomorphism φ of two neighbourhoods of 0 such that $\varphi(0) = 0$ and $\varphi^* f$ is a homogeneous quadratic polynomial.*

Proof. By a second-order version of Hadamard's lemma, we have

$$f(x) = \frac{1}{2} \sum_{i,j} A_{ij}(x) x^i x^j,$$

[1] To keep in line with the rest of the course, our sign convention for the flow differs from that used in [2].

where the matrix-valued function $x \mapsto A(x) = (A_{ij}(x))$ is smooth, takes values in symmetric matrices, and satisfies $A(0) = \operatorname{Hess} f(0)$. It follows that

$$\frac{\partial f}{\partial x^j}(x) = \sum_k B_{jk}(x) x^k,$$

where $B_{jk} = A_{jk} + \frac{1}{2}\sum_\ell \frac{\partial A_{k\ell}}{\partial x^j} x^\ell$. The smooth matrix-valued function $x \mapsto B(x) = (B_{jk}(x))$ satisfies $B(0) = A(0)$, hence $B(x)$ is invertible for x in a neighbourhood of 0. Thus, the expression

$$X = \sum_{i,j} (A(x)B(x)^{-1})_{ij} x^i \frac{\partial}{\partial x^j}$$

gives a well-defined vector field near 0, which is Euler-like because $A(x)B(x)^{-1}$ equals the identity matrix up to higher order terms. By Lemma 2.2, we obtain a diffeomorphism φ with $\varphi(0) = 0$ and $\varphi^* X = \mathcal{E}$. More computation yields $\mathcal{L}_X f = 2f$, and pulling this equality back along φ yields $\mathcal{L}_{\mathcal{E}} \varphi^* f = \varphi^*(\mathcal{L}_X f) = 2\varphi^* f$. Thus $\varphi^* f$ is a homogeneous polynomial of degree 2. ∎

THEOREM 2.4 (DARBOUX). *Let $\omega \in \Omega^2(\mathbb{R}^n)$ be a closed 2-form. If ω is non-degenerate at 0, then there exists a diffeomorphism φ of two neighbourhoods of 0 such that $\varphi(0) = 0$ and $\varphi^*\omega$ is constant.*

Proof. By the Poincaré lemma, ω has a primitive $\alpha \in \Omega^1(\mathbb{R}^n)$. By Taylor expanding the coordinate expression for ω at 0, we have

$$\omega = \sum_{i<j} (c_{ij} + O(|x|))\, dx^i \wedge dx^j,$$

so we may choose α to be of the form

$$\alpha = \sum_{i<j} (c_{ij} x^i + O(|x|^2))\, dx^j.$$

Since ω is non-degenerate at 0, hence near 0 by continuity, the equation $\iota_X \omega = 2\alpha$ can be solved for a vector field X in a neighbourhood of 0, which we see is Euler-like in view of the coordinate expressions for ω and α. By Lemma 2.2, we obtain a diffeomorphism φ with $\varphi(0) = 0$ and $\varphi^* X = \mathcal{E}$. Since $d\omega = 0$, Cartan's magic formula yields

$$\mathcal{L}_X \omega = d\iota_X \omega + \iota_X d\omega = 2d\alpha = 2\omega,$$

and by pulling this equality back along φ, we obtain $\mathcal{L}_{\mathcal{E}} \varphi^*\omega = 2\varphi^*\omega$. ∎

3. EULER-LIKE VECTOR FIELDS FOR SUBMANIFOLDS

We have seen how Euler-like vector fields on $\mathbb{R}^n$ can be used to simplify proofs of normal form results such as Morse's lemma and Darboux's theorem. These results are pointwise, in the sense that they describe how geometric objects behave *only* in a neighbourhood of a point satisfying certain non-degeneracy conditions. This should be unsurprising: Euler-like vector fields on $\mathbb{R}^n$ need only behave like the Euler vector field near the submanifold $\{0\} \subset \mathbb{R}^n$. To generalize these proofs to their analogues along more general submanifolds, we will also need to generalize the notion of Euler-like vector fields. We do so in this section, contenting ourselves with introducing the main definitions and statements; we direct the reader to [1] and [2] for more detailed discussion.

We denote by Man the category of smooth manifolds and smooth maps between them. We will also make use of the category Man^2, whose objects are pairs (M, N) with M a smooth manifold and $N \subseteq M$ a closed submanifold, and whose morphisms $\varphi\colon (M, N) \to (M', N')$ are smooth maps $\varphi\colon M \to M'$ with $\varphi(N) \subseteq N'$. The *normal bundle functor* $\nu\colon \mathsf{Man}^2 \to \mathsf{Man}$ assigns to each object (M, N) the normal bundle

$$\nu(M, N) := TM|_N/TN,$$

a vector bundle over N, and to each morphism $\varphi\colon (M, N) \to (M', N')$ the vector bundle morphism

$$\nu(\varphi)\colon \nu(M, N) \to \nu(M', N')$$

along $\varphi|_N\colon N \to N'$ induced by $D\varphi\colon TM|_N \to TM'|_{N'}$.

The normal bundle functor is compatible with the tangent bundle functor $T\colon \mathsf{Man} \to \mathsf{Man}$ in the sense that there is a natural isomorphism $\nu(TM, TN) \cong T\nu(M, N)$; see [1, Appendix A] for details. Thus, if $X \in \mathfrak{X}(M)$ is tangent to N (meaning $X(x) \in T_xN$ for all $x \in N$), then X can be regarded as a morphism $(M, N) \to (TM, TN)$ in Man^2. By applying the normal bundle functor, we obtain

$$\nu(X)\colon \nu(M, N) \to \nu(TM, TN) \cong T\nu(M, N).$$

We define the **linear approximation** of X to be $X_{[0]} := \nu(X) \in \mathfrak{X}(\nu(M, N))$, and we say that X is **Euler-like** for (M, N) if $X_{[0]}$ equals the Euler vector field of $\nu(M, N)$. Definition 2.1 is recovered when M is an open neighbourhood of $0 \in \mathbb{R}^n$ and $N = \{0\}$.

The following theorem is a generalization of Lemma 2.2; its proof is analogous, and can be found in [1].

Theorem 3.1 ([1]). *An Euler-like vector field X for (M, N) determines a unique maximal tubular neighbourhood embedding $\varphi\colon O \to M$ of a star-shaped open neighbourhood $O \subseteq \nu(M, N)$ of the zero section of $\nu(M, N)$ such that $\varphi^*X = \mathcal{E}$.*

Here, *star-shaped* means invariant under multiplication by scalars in $[0, 1]$, and a *tubular neighbourhood embedding* means a morphism $\varphi\colon (O, N) \to (M, N)$ which is an embedding $\varphi\colon O \to M$ such that $\nu(\varphi)$ is the identity map on $\nu(M, N)$ after making the canonical identification $\nu(O, N) \cong \nu(M, N)$.

As one might expect, Theorem 3.1 plays the same role in proving normal form results that Lemma 2.2 played in our proofs of Morse's lemma and Darboux's theorem. More can be said about the tubular neighbourhood embedding φ obtained from the theorem. For example,

(1) there is a more-or-less explicit description of the image $\varphi(O)$;
(2) this construction is functorial;
(3) if X is complete, then O can be taken to be all of $\nu(M, N)$.

Examples of normal form results that can be proven using Theorem 3.1 include the Morse–Bott lemma and Weinstein's Lagrangian neighbourhood theorem. Euler-like vector fields also find applications in linearization results for proper Lie groupoids and symplectic groupoids. We defer the reader to [2] for proofs of these and more.

References

1. H. Bursztyn, H. Lima, and E. Meinrenken, *Splitting theorems for Poisson and related structures*, J. Reine Angew. Math. **754** (2019), 281–312.
2. E. Meinrenken, *Euler-like vector fields, normal forms, and isotropic embeddings*, Indag. Math. **32** (2021), 224–245.