Regularity of elliptic partial differential equations

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1 Introduction

1.1 Notation

We borrow notation from [3]. In particular:

- Open subsets of $\mathbb{R}^n$ are denoted by U, V and W; we write $V \subset\subset U$ to mean that $\overline{V} \subset U$.
- We let $L^p_{\text{loc}}(U) = \{u\colon U \to \mathbb{R} \mid u|_V \in L^p(V) \text{ for all } V \subset\subset U\}$, and similarly for Sobolev space $W^{k,p}_{\text{loc}}$ and Hölder space $C^{k,\alpha}_{\text{loc}}$.
- A matrix function $A\colon U \to \mathbb{R}^{n\times n}$ is uniformly elliptic if there exist constants $0 < \lambda \le \Lambda < \infty$ such that for all $x \in U$ and $\xi \in \mathbb{R}^n$,

$$\lambda|\xi|^2 \le \sum_{i,j} A_{ij}\xi_i\xi_j \le \Lambda|\xi|^2.$$

  When this is the case, we write $0 < \lambda I \le A \le \Lambda I$.

1.2 Motivation

In 1900, Hilbert presented a list of twenty-three unsolved (at the time) problems in mathematics [5], which influenced much of the development of mathematics throughout the 20th century. The nineteenth problem asked, roughly speaking, if solutions of regular problems in the calculus of variations are always analytic. More specifically, fix a convex C^1 function $F\colon \mathbb{R}^n \to \mathbb{R}$, a bounded C^1 domain $U \subset \mathbb{R}^n$, $f \in L^2(U)$, and $g \in C^0(\partial U)$. The archetypal problem in the calculus of variations is to find a minimizer for the functional

$$E[v] = \int_U (F(\nabla v) + fv) \tag{1.1}$$

among all $v \in W^{1,2}(U)$ such that $v = g$ on ∂U.

Remark 1.1. Strictly speaking, it does not make sense to ask that $v|_{\partial U} = g$, as Sobolev functions are only defined up to measure zero sets. This can be rectified with the use of trace operators, which give a meaningful notion of "boundary values" for Sobolev functions—but we will not concern ourselves with this technicality.

In fact, if F is strictly convex, C^1, and satisfies certain decay hypotheses, then a minimizer u exists and is unique. Moreover, u is a weak solution of the Euler–Lagrange equation

$$\int_U (DF(\nabla u)\cdot\nabla w + fw) = 0, \quad \forall w \in W^{1,2}_0(U). \tag{1.2}$$

(Roughly speaking, $W^{1,2}_0(U)$ is the set of functions in $W^{1,2}(U)$ that vanish on the boundary; see Remark 1.1.)

THEOREM 1.2. *Suppose $F\colon \mathbb{R}^n \to \mathbb{R}$ is C^2, and that there exist constants $\lambda, \Lambda > 0$ such that $|D^2F(x)| \leq \Lambda$ and $\langle \xi, D^2F(x) \cdot \xi \rangle \geq \lambda |\zeta|^2$ for all $x, \xi \in \mathbb{R}^n$. If $u \in W^{1,2}(U)$ solves the Euler–Lagrange equation (1.2), then $u \in W^{2,2}_{\mathrm{loc}}(U)$.*

Proofs of these facts can be found in Chapter 8 of [3].

From this point on, we will always assume that $f \equiv 0$, as this case captures the key ideas. Suppose that $u \in W^{1,2}(U)$ is a minimizer of (1.1) and fix $V \subset\subset U$. Let $\varphi \in C_c^\infty(V)$, extending φ to U by $\varphi \equiv 0$ outside V. For $k \in \{1, \dots, n\}$, we have $\partial_k \varphi \in W_0^{1,2}(U)$, so

$$0 = \sum_i \int_V \partial_i F(\nabla u) \partial_i \partial_k \varphi = -\sum_i \int_V \partial_k (\partial_i F(\nabla u)) \partial_i \varphi = -\sum_{i,j} \int_V \partial_{ij} F(\nabla u) \partial_{jk} u \partial_i \varphi.$$

Thus, defining $U_k := \partial_k u$, we have that

$$\sum_{i,j} \int_V \partial_{ij} F(\nabla u) \partial_j U_k \partial_i \varphi = 0.$$

This also holds for $\varphi \in W_0^{1,2}(V)$ by approximating φ with a sequence of functions $\varphi_m \in C_c^\infty(V)$ that converges strongly to φ in $W^{1,2}(V)$. Thus, U_k is a weak solution to the equation

$$\sum_{i,j} \partial_i (\partial_{ij} F(\nabla u) \partial_j U_k) = 0 \quad \text{in } V.$$

If we define the matrix function $A = (a_{ij})_{ij}$ by $a_{ij}(x) = \partial_{ij} F(\nabla u(x))$, then we can write this as

$$\sum_{i,j} \partial_i (a_{ij} \partial_j U_k) = \nabla \cdot (A \nabla U_k) = 0,$$

a linear PDE for U_k in divergence form. The hypotheses on F guarantee that A is elliptic; this is where regularity theory of elliptic PDEs comes into play, and where we find an answer to the question that Hilbert posed.

A priori, even if F is smooth, we do not even know that the a_{ij} are continuous. However, we have a result, proved independently by De Giorgi and Nash in 1957, that provides us with an answer: we will see (Theorem 2.1) that each U_k is, in fact, α-Hölder continuous for some $\alpha > 0$. Thus, if F is smooth, then each $a_{ij} = \partial_{ij} F \circ \nabla u$ is the composition of a smooth function and an α-Hölder continuous function, and hence, is also α-Hölder continuous. In this case, a theorem of Schauder (Theorem 3.12) will imply that U_k is $C^{1,\alpha}$. But now we can apply a bootstrapping argument: it follows that $a_{ij} \in C^{1,\alpha}$, so $U_k \in C^{2,\alpha}$, so $a_{ij} \in C^{2,\alpha}$, and so on. Thus U_k is smooth, and by extension, so is u.

Such a strong result requires a fair bit of set up to reach. For brevity's sake, we will not concern ourselves with overly technical details.

2 Di Giorgi's result

In this section and the following, we study the second order equation

$$\sum_{i,j} \partial_i (a_{ij} \partial_j u) = 0 \quad \text{in } U \subset \mathbb{R}^n, \tag{2.1}$$

where U is a bounded domain, each a_{ij} is bounded and measurable, and $0 < \lambda I \leq (a_{ij})_{ij} \leq \Lambda I$. The key result follows:

THEOREM 2.1 (DE GIORGI–NASH). *Suppose $u \in W^{1,2}_{\text{loc}}(U)$ is a weak solution of (2.1). Then $u \in C^{0,\alpha}_{\text{loc}}(U)$ for some $\alpha = \alpha(n, \lambda, \Lambda) > 0$. Furthermore, for $W \subset\subset V \subset\subset U$, we have the estimate*

$$\|u\|_{C^{0,\alpha}(W)} \leq C\|u\|_{W^{1,2}(V)}.$$

Proving Theorem 2.1 is a complicated affair; hence, we will only sketch an outline. (Otherwise, this paper would be dozens of pages long.)

Proof sketch. Fix a ball $B \subset\subset U$. Since ellipticity of the PDE is preserved by translation and dilation, we may assume $B = B_1$ is the unit ball centred at 0 without loss of generality. We proceed in three steps.

Step 1: L^2 to L^∞. We want to show that

$$\|u_+\|_{L^\infty(B_{1/2})} \leq C\|u_+\|_{L^2(B_1)},$$

and similarly for u_-. Up to replacing u with $\varepsilon u/\|u_+\|_{L^2(B_1)}$ for ε sufficiently small, it suffices to prove that, if $\|u_+\|_{L^2(B_1)}$ is small enough, then $\|u_+\|_{L^\infty(B_{1/2})} \leq 1$. To do so, we will construct a sequence of functions $(\zeta_k u_k)_{k=0}^\infty$, where the ζ_k are cutoff functions converging to $\chi_{B_{1/2}}$ and the u_k are "truncations" of u converging to $(u-1)_-$.

More precisely, for each $k = 0, 1, 2, \ldots$, let $u_k = (u - (1 - 2^{-k}))_+$, and choose a cutoff function $\zeta_k \in C^\infty_c(B_1)$ such that $\zeta_k \equiv 1$ for $|x| \leq \frac{1}{2} + 2^{-(k+1)}$, $\zeta_k \equiv 0$ for $|x| \geq \frac{1}{2} + 2^{-k}$, and $|\nabla \zeta_k| \leq C \cdot 2^k$. By construction, $\zeta_k \equiv 1$ on $\operatorname{supp}(\zeta_{k+1})$, and if $\zeta_{k+1}u_{k+1} > 0$, then $\zeta_k u_k = u_k > 2^{-(k+1)}$. The sequence

$$a_k = \int_{B_1} (u_k \zeta_k)^2$$

satisfies $a_0 = \|u_+\|^2_{L^2(B_1)}$ and $a_{k+1} \leq A^{k+1} (a_k)^{1+\delta}$ for some constant A. One shows that if a_0 is sufficiently small, then this goes to 0 as $k \to \infty$. Thus

$$0 = \int_{B_{1/2}} [(u-1)_+]^2,$$

so $u \leq 1$ almost everywhere in $B_{1/2}$. The proof for u_- is similar, and gives the corresponding lower bound.

Step 2: L^∞ to oscillation decay. In this step, it is shown that there exists $\sigma = \sigma(n, \lambda, \Lambda) < 1$ such that

$$\operatorname*{osc}_{B_{1/4}} u \leq \sigma \operatorname*{osc}_{B_1} u.$$

Since σ is independent from the radius of oscillation decay, we can iterate to obtain

$$\operatorname*{osc}_{B_{1/4^k}} \leq \sigma^k \operatorname*{osc}_{B_1} u = 4^{-k\alpha} \operatorname*{osc}_{B_1} u,$$

where $\alpha = -\log_4(\sigma)$ will be our desired Hölder coefficient for u.

Step 3: oscillation decay to $C^{0,\alpha}$. Given $x \neq y$ in $B_{1/2}$, choose k such that $4^{-(k+1)} \leq |x - y| < 4^{-k}$. Then

$$|u(x) - u(y)| \leq \operatorname*{osc}_{B_{4^{-k}}(x)} u \leq 4^{-k\alpha} \operatorname*{osc}_{B_1(x)} u \leq 4^\alpha |x-y|^\alpha \operatorname*{osc}_{B_1(x)} u \leq 4^\alpha |x-y|^\alpha C\|u\|_{L^2(B_2(x))}.$$

■

Remark 2.2. While De Giorgi proved Theorem 2.1, Nash proved the same for the parabolic equation

$$\partial_t u = \sum_{i,j} \partial_i (a_{ij} \partial_j u),$$

where $u = u(t, x)$—hence the "Nash" in "De Giorgi–Nash".

3 Schauder estimates

First, we show that solutions of constant coefficient equations are smooth. Then, the general principle for Hölder continuous coefficient equations is that the coefficients are "nearly constant" in sufficiently small balls, and that the solution to a Hölder continuous coefficient PDE is "close" to solutions of its local constant coefficient approximations. Details can be found in [1].

3.1 Constant coefficient equations

LEMMA 3.1 (CACCIOPPOLI'S INEQUALITY). *If $u \in W^{1,2}(U)$ is a weak solution of (2.1), then*

$$\int_{B_r} |\nabla u|^2 \leq \frac{C}{(R-r)^2} \int_{B_R} u^2$$

for all balls $B_r \subset\subset B_R \subset\subset U$ centred at the same point.

Caccioppoli's inequality allows us to control the L^2 norm of ∇u in a small ball with the L^2 norm of u in a bigger ball. As a result, we obtain regularity for constant coefficient equations.

THEOREM 3.2. *Suppose that $u \in W^{1,2}(U)$ is a weak solution of (2.1), where each $a_{ij}(x) \equiv a_{ij}$ is constant. Then for all $\kappa \geq 2$, we have $u \in W^{\kappa,2}_{\text{loc}}(U)$, with the estimate*

$$\int_{B_r} |D^\kappa u|^2 \leq \frac{C}{(R-r)^{2\kappa}} \int_{B_R} u^2 \tag{3.1}$$

for all balls $B_r \subset\subset B_R \subset\subset U$ centred at the same point. In particular, since $W^{\kappa',2}_{\text{loc}}(U)$ embeds into $C^0(U)$ for some $\kappa' = \kappa'(n)$ and $D^{\kappa-\kappa'}u \in W^{\kappa',2}_{\text{loc}}$ for all $\kappa > \kappa'$, we have $u \in C^\infty(U)$.

Proof sketch. The idea is that, if u is a solution to the constant coefficient equation, then so is every partial derivative of u, as differentiating both sides of (2.1) with respect to the kth coordinate yields

$$\sum_{i,j} \partial_i(a_{ij}\partial_j(\partial_k u)) = 0.$$

A priori, u is not smooth enough to carry out this argument. Thus, to be more precise, we use the finite difference quotient

$$D^h_k u(x) = \frac{u(x+he_k) - u(x)}{h}, \quad |h| > 0,$$

in place of $\partial_k u$ (see [3] for more on difference quotients). If $B_r \subset\subset B_\rho \subset\subset B_{\rho'} \subset\subset B_R$ and $|h| < \rho' - \rho$, then by Caccioppoli's inequality,

$$\int_{B_r} |\nabla D^h_k u|^2 \leq \frac{C}{(\rho-r)^2} \int_{B_\rho} |D^h_k u|^2 \leq \frac{C}{(\rho-r)^2} \int_{B'_\rho} |\partial_k u|^2 \leq \frac{C}{(\rho-r)^2(R-\rho')^2} \int_{B_R} u^2.$$

Taking $\rho' \to \rho$ and $h \to 0$ yields

$$\int_{B_r} |\nabla \partial_k u|^2 \leq \frac{C}{(\rho-r)^2(R-\rho)^2} \int_{B_R} u^2,$$

and taking $\rho = \frac{r+R}{2}$ yields the result for $\kappa = 2$. Inductively, we obtain (3.1) for all $\kappa \geq 2$. ■

As a consequence of Theorem 3.2, we obtain decay estimates for solutions of constant coefficient equations.

THEOREM 3.3 (CONSTANT COEFFICIENT DECAY ESTIMATES). *Suppose that $u \in W^{1,2}(U)$ is a weak solution of (2.1), where each $a_{ij}(x) \equiv a_{ij}$ is constant. Fix $\kappa \geq 1$. Then for all balls $B_r \subset\subset B_R \subset U$ centred at the same point, we have*

$$\int_{B_r} |D^\kappa u|^2 \leq C \left(\frac{r}{R}\right)^n \int_{B_R} |D^\kappa u|^2 \tag{3.2}$$

and

$$\int_{B_r} \left| D^\kappa u - \fint_{B_r} D^\kappa u \right|^2 \leq C \left(\frac{r}{R}\right)^{n+2} \int_{B_R} \left| D^\kappa u - \fint_{B_R} D^\kappa u \right|^2, \tag{3.3}$$

where $\fint_U f = \int_U f/|U|$ denotes the average of f over U.

3.2 Hölder continuous coefficient equations

Given $u \in C^{0,\alpha}(U)$ (not necessarily a solution to (2.1)), consider the function

$$\omega_p(x, r; u) = \left(\fint_{B_r(x)} \left| u(y) - \fint_{B_r(x)} u \right|^p dy \right)^{\frac{1}{p}}.$$

Since u is continuous, there exists some $\overline{x} \in B_r(x)$ such that $u(\overline{x}) = \fint_{B_r(x)} u$. Since u is also α-Hölder continuous and $|y - \overline{x}| \leq 2r$, we thus have

$$\omega_p(x, r; u) \leq C \left(\fint_{B_r(x)} |y - \overline{x}|^{\alpha p} dy \right)^{\frac{1}{p}} \leq C \left(\fint_{B_r(x)} (2r)^{\alpha p} dy \right)^{\frac{1}{p}} \leq C r^\alpha.$$

In fact, we have a partial converse to this for L^p functions.

THEOREM 3.4 (CAMPANATO). *Fix $p \in [1, \infty)$, $u \in L^p_{\text{loc}}(U)$, and $V \subset\subset U$. Let $r_0 < d(V, \partial U)$. If $\alpha \in (0, 1]$ is such that $\omega_p(x, r; u) \leq K r^\alpha$ for all $x \in V$ and $r < r_0$, then $u \in C^{0,\alpha}(V)$, with the estimate*

$$[u]_{C^{0,\alpha}(V)} \leq CK.$$

COROLLARY 3.5 (MORREY). *Fix $p \in [1, \infty)$, $u \in W^{1,p}_{\text{loc}}(U)$, and $V \subset\subset U$. Let $r_0 < d(V, \partial U)$. If $\alpha \in (0, 1]$ is such that*

$$\left(\frac{1}{r^{n-p}} \int_{B_r(x)} |\nabla u|^p \right)^{\frac{1}{p}} \leq K r^\alpha$$

for all $x \in V$ and $r \leq r_0$, then $u \in C^{0,\alpha}(V)$, and $[u]_{C^{0,\alpha}(V)} \leq CK$.

Proof. By the Poincaré inequality,

$$\int_{B_r(x)} \left| u - \fint_{B_r(x)} u \right|^p \leq C r^p \int_{B_r(x)} |\nabla u|^p \leq C K^p r^{\alpha p} r^{n-p} r^p = C K^p r^{n+\alpha p}.$$

The result follows from Theorem 3.4. ∎

THEOREM 3.6 (HÖLDER DECAY ESTIMATE 1). *Fix $V \subset\subset U$ and $\alpha \in (0, 1)$. Suppose that $u \in W^{1,2}(U)$ is a weak solution of (2.1), where each a_{ij} is α-Hölder continuous. Then for all balls $B_r \subset\subset B_R \subset\subset V$ centred at the same point, we have*

$$\int_{B_r} |\nabla u|^2 \leq C \left[\left(\frac{r}{R}\right)^n + R^{2\alpha} \right] \int_{B_R} |\nabla u|^2 \tag{3.4}$$

and

$$\int_{B_r}\left|\nabla u - \fint_{B_r}\nabla u\right|^2 \le C\left[\left(\frac{r}{R}\right)^{n+2}\int_{B_R}\left|\nabla u - \fint_{B_R}\nabla u\right|^2 + R^{2\alpha}\int_{B_R}|\nabla u|^2\right]. \tag{3.5}$$

(cf. the estimates (3.2) *and* (3.3) *for constant coefficients.)*

Remark 3.7. The interplay between calculus of variations and elliptic regularity is highlighted by the proof of Theorem 3.6: the proof involves minimizing a functional depending on the coefficients $a_{ij}^0 = a_{ij}(x_0)$, where x_0 is the common centre of the balls, and using Theorem 3.3 (hence the similarity between the two estimates).

Estimate (3.4) gives us some regularity for a weak solution $u \in W^{1,2}(U)$ of (2.1). Fix $V \subset\subset U$. Given $x_0 \in V$, define

$$f(r) = \int_{B_r(x_0)}|\nabla u|^2, \quad \text{for } B_r(x_0) \subset\subset V.$$

Then f is increasing, $f(0) = 0$, and Theorem 3.6 shows that

$$f(r) \le C\left(\left(\frac{r}{R}\right)^n + R^{2\alpha}\right)f(R), \quad \forall 0 < r \le R \le R_0,$$

where R_0 is some constant. Thus, up to an error term, f decays to 0 like r^n as $r \to 0$. We claim that, for all $m < n$, there exists C_m such that

$$f(r) \le C_m\left(\frac{r}{R}\right)^m f(R).$$

In other words, the error term can be killed by exchanging the $O(r^n)$ decay for $O(r^m)$ decay, for any $m < n$. More precisely:

LEMMA 3.8. *Suppose that $f\colon [0,d] \to [0,\infty)$ is increasing and*

$$f(r) \le C_1\left(\left(\frac{r}{R}\right)^a + R^{2\alpha}\right)f(R), \quad \forall 0 < r \le R \le d.$$

Then for all $b \in [0,a)$, there exists $d_0 \in [0,d]$ and $C_2 = C_2(a,b,d,C_1) > 0$ such that

$$f(R) \le C_2\left(\frac{r}{R}\right)^b f(R), \quad \forall 0 < r \le R \le d_0.$$

We can use this to obtain

$$\int_{B_r}|\nabla u|^2 \le C_m\left(\frac{r}{R}\right)^m\int_{B_R}|\nabla u|^2 \le C_{R,m}r^m\int_V|\nabla u|^2, \quad \forall m < n. \tag{3.6}$$

Dividing both sides by r^{n-2} and taking square roots, we have

$$\left(\frac{1}{r^{n-2}}\int_{B_r}|\nabla u|^2\right)^{\frac{1}{2}} \le Cr^{\frac{m-(n-2)}{2}}\|\nabla u\|_{L^2(V)}.$$

By Corollary 3.5, we have $u \in C^{0,\gamma}(V)$, where $\gamma := \frac{m-(n-2)}{2} = 1 - \frac{n-m}{2}$ can be chosen arbitrarily close to 1 by taking m sufficiently close to n. Thus, we have proven:

THEOREM 3.9. *If $u \in W^{1,2}(U)$ is a weak solution of* (2.1)*, then $u \in C^{0,\gamma}_{loc}(U)$ for all $\gamma < 1$.*

In fact, we can obtain even higher regularity using a variant of estimate (3.5).

PROPOSITION 3.10 (HÖLDER DECAY ESTIMATE 2). *Fix $V \subset\subset U$ and $\alpha \in (0,1)$. Suppose that $u \in W^{1,2}(U)$ is a weak solution of (2.1), where each a_{ij} is α-Hölder continuous. Then for all balls $B_r \subset\subset B_R \subset\subset V$ centred at the same point and all $m < n$, we have*

$$\int_{B_r} \left| \nabla u - \fint_{B_r} \nabla u \right|^2 \leq C \left[\left(\frac{r}{R} \right)^{n+2} \int_{B_R} \left| \nabla u - \fint_{B_R} \nabla u \right|^2 + R^{m+2\alpha} \int_V |\nabla u|^2 \right].$$

Proof. Apply (3.6) to the last term on the right hand side of estimate (3.5). (Note that we use R instead of r in (3.6).) ∎

Following a similar procedure to before, we fix a centre point $x_0 \in V \subset\subset U$ and consider

$$f(r) = \int_{B_r(x_0)} \left| \nabla u - \fint_{B_r(x_0)} \nabla u \right|^2, \quad B_r(x_0) \subset\subset V. \tag{3.7}$$

Proposition 3.10 tells us that

$$f(r) \leq C \left[\left(\frac{r}{R} \right)^{n+2} f(R) + R^{m+2\alpha} \int_V |\nabla u|^2 \right], \quad \forall m < n.$$

LEMMA 3.11. *Suppose that $f\colon [0,d] \to [0,\infty)$ is increasing and*

$$f(r) \leq C \left[\left(\frac{r}{R} \right)^a f(R) + R^b \right],$$

where $0 < b < a$. Then

$$f(r) \leq C_1 \left(\frac{f(R)}{R^b} + C_2 \right) r^b.$$

Of course, to use Lemma 3.11, we need to know that f is increasing. If $r < r'$, then $B_r(x_0) \subset B_{r'}(x_0)$, hence

$$\int_{B'_r(x_0)} \left| \nabla u - \fint_{B'_r(x_0)} \nabla u \right|^2 \geq \int_{B_r(x_0)} \left| \nabla u - \fint_{B'_r(x_0)} \nabla u \right|^2 \geq \inf_{v \in \mathbb{R}^n} \int_{B_r(x_0)} |\nabla u - v|^2 \, dx.$$

We claim that this infimum is achieved precisely when

$$v = \fint_{B_r(x_0)} \nabla u. \tag{3.8}$$

This is because the function $v \mapsto \int_{B_r(x_0)} |\nabla u - v|^2 \, dx$ is a quadratic parabola with a unique minimum. The minimum can be calculated by setting the derivative equal to 0, from which we obtain (3.8). Thus, we have $f(r) \leq f(r')$.

With this, we are prepared to prove Hölder continuity for ∇u.

THEOREM 3.12 (SCHAUDER). *Suppose that $u \in W^{1,2}_{\text{loc}}(U)$ is a weak solution of (2.1), where $0 < \lambda I \leq (a_{ij})_{ij} \leq \Lambda I$, and $[a_{ij}]_{C^{0,\alpha}(U)} < \infty$ for some fixed $\alpha \in (0,1)$. Then $u \in C^{1,\alpha}_{\text{loc}}(U)$. Furthermore, for $W \subset\subset V \subset\subset U$, we have the estimate*

$$\|\nabla u\|_{C^{0,\alpha}(V)} \leq C \|\nabla u\|_{L^2(W)}.$$

Proof sketch. Fix $x_0 \in V \subset\subset U$ and $m < n$. By Lemma 3.11 with f defined by (3.7), we have

$$\int_{B_r(x_0)} \left| \nabla u - \fint_{B_r(x_0)} \nabla u \right|^2 \leq C r^{m+2\alpha} \int_V |\nabla u|^2.$$

Thus,

$$\omega(x_0, r; \nabla u) = \left(\fint_{B_r(x_0)} \left| u - \fint_{B_r(x_0)} u \right|^2 \right)^{\frac{1}{2}} \leq C r^{\frac{m+2\alpha-n}{2}} \|\nabla u\|_{L^2(V)}.$$

Applying Theorem 3.4, we obtain $[\nabla u]_{C^{0,\beta}(V)} \leq C\|\nabla u\|_{L^2(V)}$ with $\beta = \frac{2\alpha+m-n}{2} = \alpha - \frac{n-m}{2}$.

We have shown that $\nabla u \in C^{0,\beta}_{\text{loc}}(U)$ for any $\beta < \alpha$, but we seek α-Hölder continuity. Given $W \subset\subset V$ and $x, y \in W$, we have

$$|\nabla u(x) - \nabla u(y)| \leq [\nabla u]_{C^{0,\beta}(W)} |x-y|^\beta \leq C\|\nabla u\|_{L^2(V)} |x-y|^\beta.$$

Thus, by the triangle inequality,

$$|\nabla u(x)| \leq |\nabla u(y)| + C\|\nabla u\|_{L^2(V)} |x-y|^\beta.$$

Regarding x as fixed and integrating y over $W \cap B_1(x)$, we use the β-Hölder continuity of ∇u to show that

$$|\nabla u(x)||W \cap B_1(x)| \leq C\|\nabla u\|_{L^2(V)}.$$

If W is sufficiently "nice", then $\|\nabla u\|_{L^\infty(W)} \leq C\|\nabla u\|_{L^2(V)}$. Since ∇u is bounded on W, we have

$$\int_{B_R(x_0)} |\nabla u| \leq C\|\nabla u\|_{L^2(V)} R^n, \quad \forall B_R(x_0) \subset W.$$

Now use Lemma 3.11 with $b = 2\alpha + n$ and repeat the proof to obtain $[\nabla u]_{C^{0,\beta}} \leq C\|\nabla u\|_{L^2(V)}$, where $\beta = \frac{2\alpha+n-n}{2} = \alpha$. ∎

4 Fully nonlinear second order elliptic equations

We briefly sketch some results in the theory of fully nonlinear second order elliptic PDEs; for more details, we refer the reader to [2] and [4]. These equations can be written in the form

$$F(D^2u, Du, u, x) = 0 \quad \text{in } U \subset \mathbb{R}^n, \tag{4.1}$$

where F is (uniformly) elliptic, in the sense that its first-order Taylor expansion with respect to u and its derivatives centred at any point is (uniformly) elliptic. Such equations appear, for example, in probability and geometry as the Bellman and Monge–Ampère equations, respectively. Roughly speaking, the nonlinearity of these equations reflects the fact that they do not arise from minimizing an energy functional.

In fact, using Schauder-type estimates and mild assumptions on the dependence of F on Du, u, and x, regularity results for (4.1) can be reduced to the study of solutions to

$$F(D^2u) = 0 \quad \text{in } U \subset \mathbb{R}^n. \tag{4.2}$$

Thus, we focus our attention on this special case.

The question of regularity has a satisfactory answer in two dimensions thanks to Nirenberg [7], who proved that if $F\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ is uniformly elliptic and $u \in C^2(B_1)$ solves (4.2) in the usual sense, then

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)}$$

for some $\alpha \in (0,1)$. One proof is modelled off the idea that, if we had $u \in C^4(B_1)$, then the second derivatives of u satisfy a uniformly elliptic linear equation in divergence form, and thus, by Theorem 2.1, are $C^{0,\alpha}(B_{1/2})$, from which it follows that $u \in C^{2,\alpha}(B_{1/2})$; the lower regularity of general C^2 solutions is bypassed by using difference quotients. Once α-Hölder continuity is known, a similar bootstrapping technique to the one used in Section 1 can be used to prove smoothness.

In general, we lack the $C^{2,\alpha}$ estimates required to prove regularity in dimensions higher than 2. However, if we weaken the meaning of solution to that of *viscosity solutions* (see [8] for an introduction), then we obtain existence and uniqueness for the boundary value problem

$$\begin{cases} F(D^2u) = 0 & \text{in } U, \\ u = g & \text{on } \partial U, \end{cases}$$

provided that F is uniformly elliptic, $U \subset\subset \mathbb{R}^n$ is a Lipschitz domain, and $g \in C^0(\partial U)$. Furthermore, for viscosity solutions, we have a nonlinear analogue to De Giorgi–Nash:

THEOREM 4.1 (KRYLOV-SAFONOV). *If $u \in C^0(\overline{B}_1)$ is a viscosity solution of (4.2), then $u \in C^{1,\alpha}(B_1)$.*

Thus, a uniformly continuous viscosity solution is $C^{1,\alpha}$, and a C^2 solution in the usual sense is C^∞. Unfortunately, the jump from $C^{1,\alpha}$ to C^2 is not always possible in higher dimensions: for $n \geq 5$, one can construct nonlinear equations with viscosity solutions that are $C^{1,\tau}$ but not $C^{1,\tau+\varepsilon}$ (see [6] for an example with $n = 24$), and regularity remains an open problem for $n = 3$ and $n = 4$.

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