MAT257 Tutorial Worksheet 9

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The definition of a manifold comes in essentially 3 different flavours:

1. **(Smooth level sets)** Given $f : \mathbb{R}^{n+k} \to \mathbb{R}^n$ a $\mathcal{C}^r$–function, let $S = f^{-1}(y)$ for some $y \in \mathbb{R}^k$, e.g. S is a level set of f. If $Df(x)$ has rank k for all $x \in S$ (smoothness), then S is an n dimensional $\mathcal{C}^r$–manifold.

2. **(Locally a graph)** We say $S \subset \mathbb{R}^{n+k}$ is an n dimensional $\mathcal{C}^r$–manifold if for all $(x_0, y_0) \in S$, there is an neighbourhood U of (x_0, y_0) in $\mathbb{R}^{n+k}$ for which there exists a $\mathcal{C}^r$ function $g : \mathbb{R}^n \to \mathbb{R}^k$ such that $S = \{(x, y) \in U : y = g(x)\}$.

3. **(Collection of charts)** A set $S \subset \mathbb{R}^m$ is an n dimensional $\mathcal{C}^r$–manifold if for each $x \in S$, there is are open sets $U \subset \mathbb{R}^m$ containing x and $V \subset \mathbb{R}^n$ and a continuous invertible map $\varphi : U \cap S \to V$ for which $\varphi^{-1} : V \to U \cap S$ is continuous (the pair (V, φ^{-1}) is called a chart), and given any other chart, (V', ψ^{-1}), the map $\phi \circ \psi^{-1}$ is $\mathcal{C}^r$ (provided $\psi^{-1}(V') \cap \varphi^{-1}(V)$ is nonempty) .

And for each definition, we get 3 flavours of tangent spaces.

Problem 1. Consider the ellipse E defined by

$$4x^2 + 9y^2 = 36.$$

(a) Show using the level set definition of a manifold that the ellipse is indeed a manifold. That is show $\nabla f(x, y) \neq (0, 0)$ at points on the ellipse.

(b) Use the implicit function theorem to conclude that near each point $p \in E$, E can be expressed as the graph of a smooth function.

(c) For any $p \in E$, construct a chart near p. (*Hint:* Think about the projection map $\pi : \mathbb{R}^2 \to \mathbb{R}$ given by $\pi(x, y) = x$.)

Problem 2. Let E be the ellipse as in problem 1.

(a) Describe the tangent space of E at $p = (3/2, \sqrt{3})$. What is it at $(5/2, 3/2)$?

(b) Using your description of E as a graph near p, describe the tangent space of E at p.

(c) Using charts around E describe the tangent space of E at p.

Problem 3. Repeat problem 1 and 2 with the ellipsoid

$$x^2 + 4y^2 + 9z^2 = 37,$$

and the point $p = (1, 3/2, \sqrt{3})$.

Problem 4. Consider the orthogonal group of $\mathbb{R}^2$ given by

$$O_2(\mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) \ : \ AA^t = I\}.$$

Show that $O_2(\mathbb{R})$ is a manifold and compute its dimension.

Problem 5. Is the figure eight a manifold?

For concreteness, one possible description of the firgure eight is the following: A figure eight is the image of the map

$$\begin{aligned} \gamma : (-\pi/2, 3\pi/2) &\to \mathbb{R}^2 \\ t &\mapsto \big(\sin(2t), \cos(t)\big) \end{aligned}$$

Problem 6. (Hard) An alternative characterization of tangent spaces is based on the following problem (which is described for $\mathbb{R}^n$ at 0 for simplicity).

Let $v : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathbb{R}$ be a linear map such that for every $c \in \mathbb{R}$, and every $f, g \in \mathcal{C}^\infty(\mathbb{R}^n)$ we have

- $v(c) = 0$;
- $v(fg) = v(f)g(0) + v(g)f(0)$.

Show there exists $w \in \mathbb{R}^n$ such that $v(f) = Df(0) \cdot w$.