MAT257 Tutorial Worksheet 7

Adriano Pacifico – It's implicit in the detail

Problem 1.

(a) Show the equation $x^2+y^2 = 25$ defines y as a function of x in a neighbourhood of $(5/2, 5\sqrt{3}/2)$. Compute $\frac{dy}{dx}$ at this point.

(b) Show the system

$$x^2u + vy = 1$$
$$x^2 + y^2 + u^2 + \frac{v^2}{2} = \frac{5}{2}$$

defines u and v as functions of x and y in a neighbourhood around $(x, y, u, v) = (1, 1, 0, 1)$ and compute $\frac{\partial u}{\partial x}$ at this point.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a nonconstant $\mathcal{C}^1$ function such that $f'(0) \neq 0$ and $f(x + y) = f(x)f(y)$. Define $F(x, y) = f(x)f(y)$. Determine what conditions (if any) must be imposed on y to ensure that y can be solved as a $\mathcal{C}^1$ functions of x on the set $\{(x, y) \ : \ F(x, y) = 1\}$ and write down an explicit formula for y in terms of x where possible.

Problem 3. Show that the following system always has a solution for sufficiently small a,

$$x + y + \sin(xy) = a$$
$$\sin(x^2 + y) = 2a$$

Problem 4.

(a) Show there are $\mathcal{C}^1$ functions $u(x, y, z)$ and $v(x, y, z)$ defined in a neighbourhood of $p = (1, 1, 1)$ such that $u(p) = 1 = v(p)$ and

$$x^3 + y + 2xzu + v^3 = 5$$
$$xyv^3 + 3xz - 7uv = -3.$$

(b) Compute $\frac{\partial u}{\partial z}$ at p.

(c) Show that the equations $u(x, y, z) = 1, \ v(x, y, z) = 1$ defines any one of x, y or z as a function of the other two near p.

Problem 5. Let $p(x) = x^n + a_1 x^{n-1} + \ldots + a_n$ be a polynomial with n distinct roots. Show that if we perturb the coefficients slightly, then the resulting polynomial still has n distinct roots. That is, show that there exists a neighborhood U of $(a_1, \ldots, a_n)$ such that if $(b_1, \ldots, b_n) \in U$, then $p(x) = x^n + b_1 x^{n-1} + \ldots + b_n$ also has n distinct roots.

Problem 6. (Very hard)

(a) Let $f(x,y) = (x, S(x))$ be a $\mathcal{C}^1$ mapping on $\mathbb{R}^2$. Show there is a map ψ such that $\psi \circ f(x,y) = (x,0)$.

(b) Let $f(x,y) = \big(u(x,y), v(x,y)\big)$ be a $\mathcal{C}^1$ mapping on $\mathbb{R}^2$. Suppose $Df(x)$ has rank 1 for all $x \in U$, for some open subset U of $\mathbb{R}^2$, and moreover that $\frac{\partial u}{\partial x} \neq 0$ in U. Define $\varphi(x,y) = \big(u(x,y,),y\big)$. Show that φ is (locally) invertible and that there is a $\mathcal{C}^1$ mapping, ψ, for which $\psi \circ f \circ \varphi^{-1}(x,y) = (x,0)$.

(c) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a $\mathcal{C}^1$ map such that for each $y \in \mathbb{R}^2$, the set $f^{-1}(y)$ is finite. Show that $\det Df(x)$ cannot vanish identically on any open subset of $\mathbb{R}^2$.