MAT257 Tutorial Worksheet 5

Adriano Pacifico – Invert that frown upside down.

Problem 1. Is $f(x,y,z) = \big(ye^x + \sin(\pi y)\cos(z), \cos(yz), z^2\big)$ invertible near $(0, 1, \pi/2)$?

Problem 2. Let $M_2(\mathbb{R})$ denote the set of 2×2 matrices. Define a map $g : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ by $g(A) = A^2$. Determine whether g is invertible in a neighbourhood of the identity matrix I.

Problem 3. Let $f(x,y) = \big(\sinh(x) + y, \sinh(y) - x\big)$. Show that if $U \subset \mathbb{R}^2$ is open, then $f^{2019}(U)$ is also open.

Problem 4. Construct a function $f : \mathbb{R} \to \mathbb{R}$ with $f'(0) = 0$ but nevertheless is invertible in every neighbourhood of 0. What can you say about the regularity of the inverse?

Problem 5.

(a) If $f : \mathbb{R} \to \mathbb{R}$ satisfies $f'(a) \neq 0$ for all $a \in \mathbb{R}$, show that f is injective.

(b) Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x,y) = (e^x \cos y, e^x \sin y)$. Show that $\det Df(x,y) \neq 0$ for all $(x,y) \in \mathbb{R}^2$ but f is not injective.

An alternate proof of the Inverse Function Theorem

A very useful idea in mathematics is hat often times an existence theorem can be rephrased as the existence of a fixed point of some map. This is the basic idea in the following proof of IFT. Many of these "fixed point existence theorems" rely on the following theorem:

Problem 6. (Contraction mapping principle/Banach's fixed point theorem). If $f : (X, d \to (X, d)$ is a constraction of a complete metric space, then f has a unique fixed point.

The basic idea is to pick any point an iterate f. To understand this result, we recall some

Definitions:

(a) A *metric space*, (X, d), is a set X equipped with a function $d : X \times X \to \mathbb{R}_{\geq 0}$ sastisfying:

(i) $\forall x \in X$, $d(x,x) = 0$ (positive definite)

(ii) $\forall x, y \in X$, $d(x,y) = d(y,x)$ (symmetry)

(iii) $\forall x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality)

(b) A subset $U \subset X$ is called *open* if for each $x \in U$, there is an $\varepsilon > 0$ such that the epsilon ball around x is constained in U. **Remark.** Similarly, notions like sequence convergence and Cauchy sequences generalizes to metric spaces by replacing $|x - y|$ with $d(x, y)$ in the corresponsing definitions.

(c) The metric space (X, d) is *complete* if every Cauchy seuqence converges.

(d) A map $f : (X, d) \to (X, d)$ is a *contraction* if there exists $c < 1$ such that for all $x, y \in X$ we have $d\big(f(x), f(y)\big) \leq cd(x, y)$.

The other lemma on which our proof of IFT will rely upon is the following exercise:

Problem 7. If U is convex and $f : U \to \mathbb{R}^n$ is differentiable with derivative satisfying $\sup_{x \in U} |Df(x)| \leq M$, then $|f(a) - f(b)| \leq M|a - b|$.

We now tackle the main theorem. Recall the statement

Theorem. (Inverse function theorem). Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$ mapping with $\det Df(p) \neq 0$. There there exists open sets U, V around p and $f(p$ on which f is a bijection and its inverse is $\mathcal{C}^1$.

Problem 8. (A symplifying assumeption). Show we can assume WLOG that $f(p) = 0 = p$ and that $Df(0) = I$.

Problem 9. We want to build and inverse to f. The key is to consider

$$\varphi_{y_0}(x) = x + \big(y_0 - f(x)\big).$$

Notice that a fixed point of $varphi_{y_0}$ is equivalent to $y + 0 = f(x)$. In particular, if we can show each φ_{y_0} has a unique fixed point as y_0 varies in some open set, then we have successfully defined an inverse to f. Show, using the contraction mapping principle, that this is the case. Specifically, show problem 7 and continuous differentiability imply there is some $r > 0$ such that φ_{y_0} constracts B_r (the ball of radius r centered at 0) by a factor of 10. Use this to show f maps B_r onto $B_{r/2}$. Find open sets U, V as in the statement of the theorem on which f is invertible.

The next part is to show $g = f|_U^{-1}$ is $\mathcal{C}^1$.

Problem 10. Show $\dfrac{|y|}{2} \leq |g(y)| \leq 2|y|$. Using this and the fact $y = g(f(y))$ show that $g(y) = y + h(y)$ where h is some function satisfying $\displaystyle\lim_{y \to \infty} \frac{h(y)}{|y|} = 0$. Use this to conclude $Dg(0) = I$.

Problem 11. Show $U = \{x \mid \det Df(x) \neq 0\}$ is open. Use this and the fact that our starting point (which we did take to be 0) was arbitrary to conclude g is differentiable for $y \in V$. Use chain rule to show g is $\mathcal{C}^1$.