MAT257 Tutorial Worksheet 4

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Warning. Now that you have a tool like differentiation under the integral sign, you might be tempted to look at the world through rose coloured glasses. Don't!

Problem 1. Why can we not apply differentiation under the integral sign to

$$\int_0^\infty \frac{\sin x}{x} dx?$$

At which point in the proof of this result does the argument break down?

Problem 2. It turns out that differentiation under the integral sign does apply to the above integral (but this requires a good deal more sophistication to prove). Define

$$f(y) = \int_0^\infty \frac{\sin x}{x} e^{-xy} dx.$$

Compute $f'(y)$ assuming you may differentiate under the integral sign and use this to compute $f(0)$.

Problem 3. Let χ_A denote the characteristice function of the set A. Let $f_n = n\chi_{[0,\frac{1}{n}]}$. Show $\int_0^1 f_n = 1$ for each $n \in \mathbb{Z}_{>0}$. Use this to conclude that in general

$$\lim_{n\to\infty} \int_0^1 \neq \int_0^1 \lim_{n\to\infty} f_n.$$

What condition from first year calculus do we require to obtain this?

Remark. In general, it is a *very* subtle issue to swap limits with integrals, integrals with integrals with integral, limits with each other and the like. It is often very useful to do so and can be necessary to continue on with a computation. Any time you swap any two limits of sorts, be prepared to defend your manipulation with your life! (Here we are making the identification "gpa=life").

Problem 4. Compute $\displaystyle\int_0^1 \frac{x^{2019} - 1}{\log x} dx$.

Problem 5.

(a) Given a continuous function $g(x)$, find a function y such that $\dfrac{dy}{dx} = g(x)$.

(b) Suppose we are give $\mathcal{C}^1$ functions $P, Q : \mathbb{R}^2 \to \mathbb{R}$ satisfying $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$. Show there exists a $\mathcal{C}^2$ function $f : \mathbb{R}^2 \to \mathbb{R}$ with $P = \dfrac{\partial f}{\partial x}$ and $Q = \dfrac{\partial f}{\partial y}$.

Problem 6. (Hard) Evaluate $\displaystyle\int_0^1 \frac{\log(x+1)}{x^2+1} dx$.