MAT257 Tutorial Worksheet 3

Adriano Pacifico – Chain Mail

Problem 1. Compute the derivative of the following functions:

(a) $f(x,y,z) = x^y$

(b) $f(x,y) = y^x)$

(c) $f(x,y,z) = (x^y, z)$

(d) $f(x,y) = \sin(x \sin y))$

(e) $f(x,y,z) = (\log(x^2+y^2+z^2), xyz)$

(f) $f(x,y) = \sin(xy)$

Problem 2. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x,y) = \big(\sin(xy), e^x, y\log(1+x^2)\big)$ and $g(u,v,w) = uv + w$.

(a) Compute $Df(0,1)$ and $Dg(0,1,0)$.

(b) Compute $g \circ f$ and then $D(g \circ f)(0,1)$

(c) How do your answers compare?

Problem 3. Let (x,y) be coordinates on $\mathbb{R}^2$, and (u,v,w) be coordinates on $\mathbb{R}^3$. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}$ be $\mathcal{C}^1$ functions.

(a) The chain rule asserts $D(g \circ f)$ can be expressed in terms of Dg and Df. Write out these matrices in terms of the partials and do the matrix multiplication to obtain a formula for

$$\frac{\partial(g \circ f)}{\partial x} \quad \text{and} \quad \frac{\partial(g \circ f)}{\partial y}$$

in terms of

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial g}{\partial u}, \frac{\partial g}{\partial v}, \frac{\partial g}{\partial w}.$$

(b) Go back to problem 2, (you should have the partials computed already) and test your formula above. Make sure you see how each piece in the general setup corresponds to each piece in your computation.

Problem 4. (One of the most important exercises at this point in the course. Fix coordinates $(x_1, \dots, x_n)$ for $\mathbb{R}^n$, $(y_1, \dots, y_m)$ for $\mathbb{R}^m$ and $(z_1, \dots, z_\ell)$ for $\mathbb{R}^\ell$. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^\ell$ be $\mathcal{C}^1$ functions.

(a) The chain rule asserts $D(g \circ f) = \big[(Dg) \circ f\big] \cdot \big[Df\big]$, where $\cdot$ denotes matrix multiplication. Write out each of these matrices in terms of the partials

$$\frac{\partial (g \circ f)}{\partial x_i}, \quad \frac{\partial f}{\partial x_j} \quad \text{and} \quad \frac{\partial g}{\partial y_k}.$$

DO THE MATRIX MULTIPLICATION! Use this to write down a formula for

$$\frac{\partial g \circ f}{\partial x_i}$$

in terms of

$$\frac{\partial f}{\partial x_j}, \quad \text{and} \quad \frac{\partial g}{\partial y_k}.$$

(b) Now we get sloppy: In your expression above replace f with y (and then the resulting $g \circ y$ with g since g depends on y as a variable). Now stare at the resulting expression until it feels like the obvious generalizatios of one variable chain rule. (Here it helps to remember more sloppy things like $\frac{df}{dx} = \frac{df}{dy}\frac{dy}{dx}$).

Remark. In practice, we might say something like g depends on the variables $(y_1, \dots, y_m)$ and then say these variables depend on $(x_1, \dots, x_n$, suppresing f altogether. This makes sense because $f : \mathbb{R}^n \to \mathbb{R}^m$ with the coordinates as above effectively means that $y_i = f_i(x)$, so the x dependence of y is specified by f. That is the y_i are the same as the component functions of f so we may as well supress f. Notationally, this seems sensible since we can write $g(y_1, \dots, y_m) = g(y_1(x), \dots, y_m(x)) = (g \circ f)(x_1, \dots, x_n)$, which gives us the freedom to think of y as its own independent variable or as a variable depending on x.

(c) To drill the point home, go back to problem 2, (you should have the partials computed already) and test your formula above. Make sure you see how each piece in the general setup corresponds to each piece in your computation.

Problem 5. (More sloppiness). All dependences that will be described are assumed to be $\mathcal{C}^1$. Suppose that f depends on $x,\ y,\ z,\ t$, that x depends on t, that y depends on $x,\ t, s$ and that z depends on $x,\ y$. Compute $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.

Problem 6. Suppose $G : \mathbb{R}^2 \to \mathbb{R}^2$ is a $\mathcal{C}^1$ functions such that $G(0) = 0$. If $H_n(x) = G^n(x)$ denotes the n–fold composition of G and we are given $DG(0) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then compute $DH_{2019}(0)$.