MAT257 Tutorial Worksheet 2

Adriano Pacifico – Sequentially seeing sequences

Problem 1. Let $\{x_k\}_{k\geq 0} \subset \mathbb{R}^n$ be a sequence with $x_k \to x$ and $x_k \to y$ as $k \to \infty$. Show $x = y$.

Problem 2. Let $K_0 \supset K_1 \supset K_2 \supset \dots$ be a decreasing sequence of compact sets in $\mathbb{R}^n$. Prove that $\bigcap_{n=0}^{\infty} K_n \neq \emptyset$.

Problem 3. Let $\{A_k\}$ denote a sequence of compact subsets of $\mathbb{R}^n$. Which of the following are compact?

(a) $\cup_{k=1}^{2019} A_k$

(b) $\cup_{k=1}^{\infty} A_k$

(c) $\cap_{k=1}^{2019} A_k$

(d) $\cap_{k=1}^{\infty} A_k$

Problem 4. Let $\{a_k\}_{k\geq 0}$ be a sequence in $\mathbb{R}^n$. We say the sequence $\{b_k\}_{k\geq 0}$ is a rearrangement of $\{a_k\}$ if there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $b_k = a_{\sigma(k)}$. Are limits of sequences affected by rearrangement? What if we only ask σ to be an injection? Surjection?

Problem 5. **(Hard)** Let $f : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}$ be defined by $f(x) = \frac{2x+3}{3x+5}$ (where it is understood $f(-5/3) = \infty$ and $f(\infty) = 2/3$). Let f^n denote the n–fold composition of f. Compute

$$\lim_{n\to\infty} f^n(x).$$

(Note your answer may depend on x).

Problem 6. **(Hard)** Let F_n denote the n^{th} Fibonacci number. Compute $\displaystyle\prod_{n=1}^{\infty}\left(1 + \frac{(-1)^n}{F_n^2}\right)$.