Worksheet 14

Adriano Pacifico – The air seems tensor today!

Reminder: Let V be a vector space over $\mathbb{R}$. Recall for multilinear maps $f\colon V^k \to \mathbb{R}$ and $g\colon V^\ell \to \mathbb{R}$, we define $f \otimes g\colon V^{k+\ell} \to \mathbb{R}$ by $f \otimes g(v, w) = f(v)g(w)$ where $v \in V^k$ and $w \in V^\ell$.

Problem 1 Let $e_1 = (1, 0), e_2 = (0, 1)$ be standard basis on $\mathbb{R}^2$ and e_1^*, e_2^* be the corresponding dual basis for $(\mathbb{R}^2)^*$. Compute

(a) $e_1^*(17, 19)$

(b) $e_1^*(x, y) + e_2^*(a, y)$

(c) $e_1^* \otimes e_2^*\big((x, y), (u, v)\big)$

(d) $e_2^* \otimes e_1^*\big((x, y), (u, v)\big)$

(e) $e_1^* \otimes e_1^*\big((x, y), (u, v)\big)$

(f) $\frac{1}{2}\Big(e_1^* \otimes e_2^*\big((x, y), (u, v)\big) + e_2^* \otimes e_1^*\big((x, y), (u, v)\big)\Big)$

(g) $\frac{1}{2}\Big(e_1^* \otimes e_2^*\big((x, y), (u, v)\big) - e_2^* \otimes e_1^*\big((x, y), (u, v)\big)\Big)$

Remarks

- Compare (c) and (d) above. This tells us tensor products are not symmetric (i.e. $f \otimes g \neq g \otimes f$) in general.
- (f) describes the symmetrization, $\operatorname{Sym} e_1^* \otimes e_2^*$, of $e_1^* \otimes e_2^*$. It is a "symmetric version" of $e_1^* \otimes e_2^*$. Since $\operatorname{Sym} e_1^* \otimes e_2^*$ is symmetric but $e_1^* \otimes e_2^*$ is not, we cannot hope for the two expressions to be equal, though we can as that the weaker condition
$$\operatorname{Sym} e_1^* \otimes e_2^*(v, v) = e_1^* \otimes e_2^*(v, v)$$
hold for all $v \in \mathbb{R}^2$. This explains the somewhat curious $\frac{1}{2}$ factor in the expression. It turns out that this equality and symmetry uniquely determine $\operatorname{Sym} e_1^* \otimes e_2^*$. A similar statement is true in general.
- Similarly, part (g) describes the skew-symmetriztion, $\operatorname{Alt} e_1^* \otimes e_2^*$. This is the more useful construction for us. Differential forms and determinants are modeled on it.

Problem 2

(a) Express the standard inner product on $\mathbb{R}^2$, $\langle (x, y), (u, v)\rangle = xu + yv$ in terms of e_1^*, e_2^* and their tensor products.

(b) Generalize part (a) to $\mathbb{R}^n$

Problem 3

(a) Consider $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let $v = \begin{pmatrix} a \\ c \end{pmatrix}$ and $w = \begin{pmatrix} b \\ d \end{pmatrix}$. Describe $\det M$ in terms of v, w and e_1^*, e_2^* and their tensor products.

(b) Generalize part (a) to $\mathbb{R}^3$

(c) How could you do this for $\mathbb{R}^n$ in general? As tensors, what kinds of properties do determinants satisfy?

Problem 4 Let e_1^*, e_2^*, e_3^* be the basis on $(\mathbb{R}^3)^*$ dual to the standard basis on $\mathbb{R}^3$. Show $e_1^* \otimes e_2^* \otimes e_3^*$ is not the sum of a symmetric tensor and an alternating tensor.

Problem 5 Let V be an n–dimensional vector space. Compute the dimension of the space of symmetric k–tensors on V. Do the same for the space of alternating k–tensors.

Problem 6: (Harder) Let V be a finite dimensional vector space and ω a skew-symmettric (alternating/anti-symmetric) 2–tensor on V with the property that if $\omega(v, w) = 0$ for all $w \in V$, then $v = 0$. Show V has even dimension.