Worksheet 12

Adriano Pacifico – A change of perspective

Theorem (Change of Variables). Let $A \subset \mathbb{R}^n$ and suppose we have an integrable function $f\colon A \to \mathbb{R}$. Suppose we have change of variables from some domain B to A, using coordinates x on A, u on B and the notation $x(u)$ to denote the change of variable. Then

$$\int_A f(x)dx = \int_B f(x(u)) \left|\det \frac{\partial x}{\partial u}\right| du$$

Problem 1 Compute

(i) $\displaystyle\int_S \frac{(x+y)^4}{(x-y)^5} dxdy$ on $S = \{-1 \leq x+y \leq 1, 1 \leq x-y \leq 3\}$.

(ii) $\displaystyle\int_D x dxdy$ where $D = \{0 \leq x, 1 \leq xy \leq 2, 1 \leq \frac{y}{x} \leq 2\}$.

(iii) the area of the region R in th 1^{st} quadrant bounded by $y = x^2$, $y = \frac{x^2}{5}$, $xy = 2$ and $xy = 4$.

Problem 2 Show $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ as follows:

(i) Show $\displaystyle\int_0^1 \int_0^1 \frac{dxdy}{1 - x^2y^2} = \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}$. (Hint: This problem is the reason for the tear drops on my guitar.)

(ii) Let Δ denote the simplex $\{(u,v) \in \mathbb{R}^2 \ : u, v > 0, u+v < \frac{\pi}{2}\}$. Show $T\colon \Delta \to [0,1] \times [0,1]$ defined by $T(u,v) = (\frac{\sin u}{\cos v}, \frac{\sin v}{\cos u})$ defines a change of variables.

(iii) Show $\displaystyle\int_0^1 \int_0^1 \frac{dxdy}{1 - x^2y^2} = \frac{\pi^2}{8}$. Conclude $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Problem 3 Compute $\displaystyle\int_B \frac{x^4 + 2y^4}{x^4 + 4y^4 + z^4} dxdydz$ where $B = \{(x,y,z) \in \mathbb{R}^3 \ : x^2 + y^2 + z^2 \leq 1\}$ denotes the unit ball.