Worksheet 11

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Problem 1

(a) Let $U, V, W \subset \mathbb{R}^n$ be open sets. Suppose we have smooth ($\mathcal{C}^\infty$) functions $f_U\colon U \to \mathbb{R}$, $f_V\colon V \to \mathbb{R}$, and $f_V\colon V \to \mathbb{R}$. Define $f_{U,V}\colon U \cap V \to \mathbb{R}$ by $f_V|_{U\cap V} - f_U|_{U\cap V}$. Define $f_{V,W}$ and $f_{U,W}$ similarly. Show that $f_{U,V} + f_{V,W} - f_{U,W} = 0$ (we assume each f is restricted $U \cap V \cap W$ to make sense of the equality). The purpose of this exercise is to prove the converse and generalize the result.

(b) Suppose we are given smooth functions $f_{U,V}\colon U \cap V \to \mathbb{R}$, $f_{V,W}\colon V \cap W \to \mathbb{R}$ and $f_{U,W}\colon U \cap W \to \mathbb{R}$ satisfying $f_{U,V} + f_{V,W} - f_{U,W} = 0$ on $U \cap V \cap W$. Define a partition of unity $\Phi = \{\varphi_U, \varphi_V, \varphi_W\}$ subordinate to the cover $\mathcal{U} = \{U, V, W\}$ (here we can take $A = U \cup V \cup W$ in Spivak's statement).

(i) Define $f_U = -\varphi_V f_{U,V} - \varphi_W f_{U,W}$. Give definitions of f_V and f_W

(ii) Show $f_{U,V} = f_V - f_U$ and similarly for $f_{V,W}$ and $f_{U,W}$.

(c) Now we generalize part (b) slightly. Suppose $X \subset \mathbb{R}^n$ is a smooth manifold and we have a cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ with the property that for each $x \in X$, only finitely many elements of $\mathcal{U}$ that contain x. Moreover, assume we are given a family of smooth functions $f_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathbb{R}$ satisfying

(i) $f_{\alpha\alpha} = 0$

(ii) $f_{\alpha\beta} + f_{\beta\alpha} = 0$

(iii) $f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0$

Show there are smooth functions $f_\alpha\colon U_\alpha \to \mathbb{R}$ satisfying $f_{\alpha\beta} = f_\beta - f_\alpha$ (again with the necessary restrictions for these expressions understood).

(d) **Harder** Now we extend part (c) to account for more than just double overlaps. Let $X \subset \mathbb{R}^n$ be a smooth manifold. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ be an open cover of X. Suppose we are given a collection of smooth functions $f_{\alpha_0\alpha_1\ldots\alpha_p}\colon U_{\alpha_0} \cap \ldots \cap U_{\alpha_p} \to \mathbb{R}$ satisfying

$$\sum_{j=0}^{p+1} (-1)^j f_{\alpha_0,\ldots,\widehat{j},\ldots,\alpha_{p+1}}|_{U_{\alpha_0}\cap\ldots\cap U_{\alpha_{p+1}}} = 0$$

Show there exists smooth functions $f_{\alpha_0,\ldots,\widehat{\alpha_j},\ldots,\alpha_p}\colon U_{\alpha_0} \cap \ldots U_{\alpha_{j-1}} \cap U_{\alpha_{j+1}} \cap \ldots \cap U_{\alpha_n} \to \mathbb{R}$ such that

$$f_{\alpha_0\ldots\alpha_p} = \sum_{j=0}^{p} (-1)^j f_{\alpha_0,\ldots,\widehat{j},\ldots,\alpha_p}|_{U_{\alpha_0}\cap\ldots\cap U_{\alpha_p}}$$

Aside In fancy terms, part (c) can more succinctly be expressed as $H^1(X, \mathcal{C}^\infty) = 0$ (read: the 1^{st} Čech cohomology group of the sheaf of smooth functions on a manifold vanishes) and part (d) can be expressed as $H^p(X, \mathcal{C}^\infty) = 0$. Cohomology is a pervasive tool in geometry and topology and it has many variants. DeRham cohomology is defined for smooth manifolds. It is perhaps the easiest to define and most accessible variant. By the end of this course we will be able to define it though we will not do so. Singular cohomology is a topological invariant which is defined more generally. Tools like homology and cohomology are usually introduced in a course in algebraic topology such as MAT1301. Sheaves are objects which encode the data of functions on a space. Sheaf or Čech cohomology arise in the context of algebriac geometry and complex geometry.

Problem 2: Cutoff functions Let $M \subset \mathbb{R}^n$ be a manifold, and let A be closed subset of M. Let $U \subset M$ be any subset containing A. Show there is a $\mathcal{C}^\infty$ function $f\colon\, M \to \mathbb{R}$ such that

- $0 \leq f \leq 1$ on M
- $f \equiv 1$ on A
- $\operatorname{Supp} f \subset U$

Recall We say $U \subset M$ is open if there exists an open set $V \subset \mathbb{R}^n$ such that $U = V \cap M$.

Problem 3: Smooth Urysohn Suppose A and B are disjoint closed subsets of a manifold $M \subset \mathbb{R}^n$. Show there is a $\mathcal{C}^\infty$ function $f\colon\, M \to \mathbb{R}$ such that

- $0 \leq f \leq 1$ on M
- $f^{-1}(0) = A$ and $f^{-1}(1) = B$