Worksheet 10

Adriano Pacifico – Houdini magic with Fubini

> Fubini's Theorem is like sliced bread.
>
> Charles Pugh

Problem 1 We wish to define an "uncoutable sum of nonnegative numbers" (nonnegative to avoid issues of order of summation). If A is uncountable, a conceivable definition is

$$\sum_{j\in A} x_j := \sup\{\sum_{j\in B} x_j \ : \ B \subset A \text{ and } B \text{ is countable}\}$$

Show that if $\sum_{j\in A} x_j < \infty$ then the set $C := \{j \in A \ : \ x_j \neq 0\}$ is countable.

Problem 2 Use Fubini's theorem to show the volume of the rectangle $R = [0, r] \times [0, s]$ is rs. That is, prove

$$\int_R 1 = rs$$

Problem 3 Use Fubini to compute

(a) $\displaystyle\int_R x^2 y$ where $R = [-1, 3] \times [2, 4]$

(b) $\displaystyle\int_R f(x_1)f(x_2)\cdots f(x_n)dx_1dx_2\cdots dx_n$ where $F'(x) = f(x)$ and $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$

(c) $\displaystyle\int_1^2 \int_{\frac{1}{x}}^1 ye^{xy}dydx$

(d) $\displaystyle\int_0^1 \int_x^{\sqrt{x}} \frac{\sin y}{y}dydx$

Problem 4 If $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous, then

$$\int_a^x \int_a^s f(s,t)dtds = \int_a^x \int_t^x f(s,t)dsdt$$

Prolem 5: (Buyer beware!)

(a) Show $\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dy = \frac{1}{1+x^2}$. (One approach is to write $\frac{x^2-y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2}+\frac{-2y^2}{(x^2+y^2)^2}$ and integrate the second term by parts with $u = y$ and $v' = \frac{-y}{x^2+y^2}$.)

(b) Compute $\int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dxdy$ and $\int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dydx$. Are they equal?

(c) Show $\int_0^1 \int_0^1 \left|\frac{x^2-y^2}{(x^2+y^2)^2}\right| dxdy = \infty$

Remark This problem shows that it is *essential* that you check the function you are integrating is positive or the iterated integral of the function's absolute value is finite!

Problem 6: (The hard one) Compute $\int_0^\infty \frac{\sin x}{x}dx$.