International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2015**

Problem 1. (a) The integers x, x^2 and x^3 begin with the same digit. Does it imply that this digit is 1?

(b) The same question for the integers $x, x^2, x^3, \dots, x^{2015}$.

Answer. No.

Solution. (a) *Example:* $x = 99$, $x^2 = 99^2 = 9801$, $x^3 = 99^3 = 970299$.

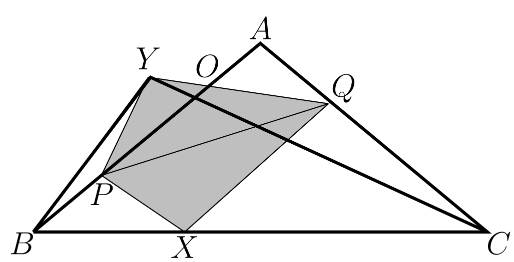
(b) *Solution (Ben Wei).* We use Bernoulli inequality $(1-\varepsilon)^k \geq (1-k\varepsilon)$ for $0 < \varepsilon < 1$ and $k \geq 1$ (It can be proved by induction). Consider $x = 99999$ in the form $x = 10^5(1-\varepsilon)$ with $\varepsilon = 10^{-5}$. Then $x^k = 10^{5k}(1-\varepsilon)^k \geq 10^{5k}(1-k\varepsilon) \geq 0.9 \cdot 10^{5k}$.

Therefore $10^{5k} > x^k \geq 0.9 \cdot 10^{5k}$ for all $k = 1, 2, \dots, 2015$, meaning that all given integers start with digit 9.

□

Problem 2. A point X is marked on the base BC of an isosceles triangle ABC, and points P and Q are marked on the sides AB and AC so that $APXQ$ is a parallelogram. Prove that the point Y symmetrical to X with respect to line PQ lies on the circumcircle of the triangle ABC.

Solution (Richard Chow). Consider triangles PYO and OAQ. Note that $\angle PYQ = \angle PXQ = \angle PAQ$ and $\angle YOP = \angle AOQ$. Then $\angle YPO = \angle AQO$ which implies that $\angle BPY = \angle YQC$. Since triangle ABC is isosceles and $PX \parallel AC$, triangle BPX is also isosceles and since $PX = PY$, triangle BPY is isosceles as well. In similar way we can rove that triangle YQC is also isosceles. Then triangles BPY and YQC are similar. If follows that $\angle BYC = \angle BAC$ and therefore quadrilateral $BYAC$ is cyclic.



□

Problem 3. (a) A $2 \times n$-table (with $n > 2$) is filled with numbers so that the sums in all the columns are different. Prove that it is possible to permute the numbers in the table so that the sums in the columns would still be different and the sums in the rows would also be different.

(b) A 100×100-table is filled with numbers such that the sums in all the columns are different. Is it always possible to permute the numbers in the table so that the sums in the columns would still be different and the sums in the rows would also be different?

Solution. (a) (Dima Paramonov). If sums of the numbers in rows are different, then the statement holds. Assume that these sums are the same. If some column contains different numbers, then exchanging these numbers we satisfy the requirement. Assume that in each column the top and the bottom numbers are the same (so that both rows of the table are identical). We can always change the order of columns in the table so that $a_1 < a_2 < a_3 < \ldots < a_k$. Let us consider the following permutation which affects only the first three columns:

a_1	a_1	a_2	a_4	$\ldots$	a_k
a_2	a_3	a_3	a_4	$\ldots$	a_k

It is clear now that sums of the numbers in rows are different. Indeed, $a_1+a_1+a_2 < a_1+a_2+a_3$. Since $a_1 + a_2 < a_1 + a_3 < a_2 + a_3 < 2a_4 < \ldots. < 2a_k$, sums of the numbers in each column remain different.

(b) Counterexample (Sina Abbasi). Consider a 100×100 table, filled with "0"s and "1"s as follows: column i, $i = 1, ..., 99$ contains $i-1$ of 1s, the last column is completely filled with 1s while the remaining cells are filled with "0"s.

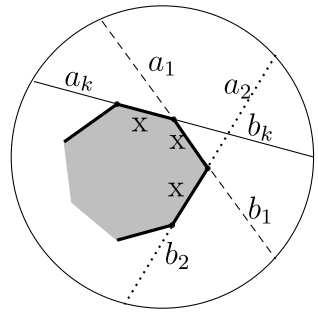
Since the total sum of elements in a table is the same no matter if it is calculated by columns or by rows and all possible values of sums in rows are between 0 and 100 (101 different values in total) we have: $0+1+2+\cdots+98+100 = 0+1+2+\cdots+100-x$ where x is a value of sum in a row we must exclude (assuming that all other sums of the numbers in rows are different). Given that $0 \le x \le 100$, $x = 99$ is an unique solution. Therefore the possible values of sums in rows are $0, 1, \ldots, 98$ and 100. However the existence of a row with value 0 contradicts to the existence of a column with value 100. □

Problem 4. A convex N-gon with equal sides is located inside a circle. Each side is extended in both directions up to the intersection with the circle so that it contains two new segments outside the polygon. Prove that one can paint some of these new $2N$ segments in red and the rest in blue so that the sum of lengths of all the red segments would be the same as for the blue ones.

Combined solution of Victor Rong and Frieda Rong. Standing at some point inside of the polygon and looking towards its side x_i, $(i = 1, ..., k, |x_i| = x)$ denote the segments to the left and to the right of the side as a_i and b_i respectively.

Consequently applying Intersecting Chord Theorem to each vertex of the polygon we get

$$\begin{aligned}
a_1(x+b_1) &= b_k(a_k+x),\\
a_2(x+b_2) &= b_1(a_1+x),\\
a_3(x+b_3) &= b_2(a_2+x),\\
&\ldots\\
a_k(x+b_k) &= b_{k-1}(a_{k-1}+x).
\end{aligned}$$



Summing up the equations and simplifying we get

$$a_1 + a_2 + \ldots + a_k = b_1 + b_2 + \ldots + b_k.$$

By colouring a_i and b_i in red and blue we prove the statement. $\square$

Problem 5. Do there exist two polynomials with integer coefficients such that each polynomial has a coefficient with an absolute value exceeding 2015 but all coefficients of their product have absolute values not exceeding 1?

Answer. Yes, such two polynomials exist.

Solution of Central Committee. Let us call a polynomial $P(x)$ *simple* if every coefficient is either 0 or 1. Observe that if $P(x)$ is a simple polynomial of degree n then $(x^m + 1)P(x)$ with $m > n$ is also a simple polynomial. Therefore, starting with the polynomial $(x + 1)$ and multiplying it recursively 2016 times by polynomials in the form of $(x^m + 1)$ with increasing odd m we obtain a simple polynomial $f(x)$. Note that $f(x)$ is divisible by $g(x) = (x + 1)^{2017} = x^{2017} + 2017x^{2016} + \ldots$. (Indeed, $(x^m + 1)$ with odd m is divisible by $(x+1)$). Let $f(x) = g(x)h(x)$ where $h(x) = x^k + ax^{k-1} + \ldots$. Consider the second coefficient of $f(x)$. On one hand it equals to 0 or 1, on the other hand it equals to $2017 + a$. Thus, $a = 2017$ or $a = 2016$, in any case exceeding 2015. $\square$

Problem 6. An Emperor invited 2015 wizards to a festival. Each of the wizards knows who of them is good and who is evil, however the Emperor doesn't know this. A good wizard always tells the truth, while an evil wizard can tell the truth or lie at any moment. The Emperor gives each wizard a card with a single question, maybe different for different wizards, and after that listens to the answers of all wizards which are either "yes" or "no". Having listened to all the answers, the Emperor expels a single wizard through a magic door which shows if this wizard is good or evil. Then the Emperor makes new cards with questions and repeats the procedure with the remaining wizards, and so on. The Emperor may stop at any moment, and after this the Emperor may expel or not expel a wizard. Prove that the Emperor can expel all the evil wizards having expelled at most one good wizard.

Solution of Sina Abbasi. Step 1. The Emperor lists all wizards $W_1, W_2, \ldots, W_N$ and asks everyone but W_1 if W_1 is evil (It does not no matter what he asks W_1).

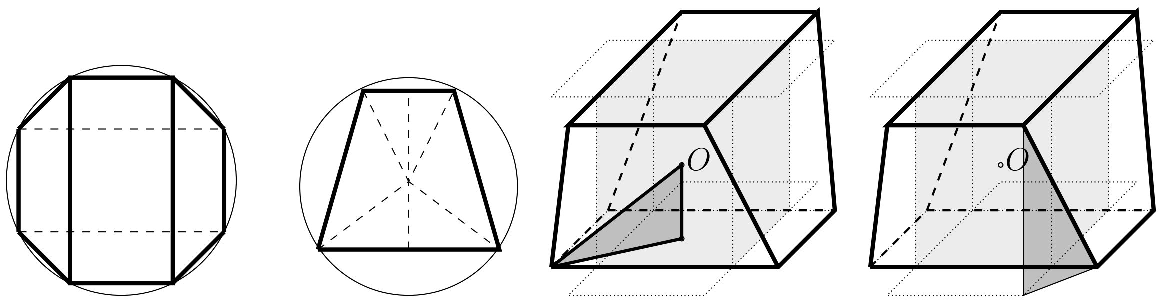
- If all of them confirm that W_1 is evil, W_1 is expelled. If W_1 occurs to be evil, the number of evil wizards is decreased by 1 (so the Emperor can start the next round). If W_1 occurs to be good then all the remaining wizards are evil, and they are expelled one by one in next rounds.

- If wizard W_i answers "no", then he is expelled. If W_i occurs to be evil the number of evil wizards is decreased by 1. If W_i occurs to be good we find one non-expelled good wizard (W_1) and we go to Step 2.

Step 2. Assume that all wizards are lined up, starting with good wizard W_1. The Emperor asks W_i ($i = 1, ...$) if W_{i+1} is evil (It does not no matter what he would ask the last wizard). If W_i answers "yes", W_{i+1} is expelled and the Emperor can start the next round. If all wizards said "no", then all evil wizards are expelled and Emperor stops the trial. $\square$

Problem 7. It is well-known that if a quadrilateral has the circumcircle and the incircle with the same centre then the quadrilateral is a square. Is the similar statement true in 3 dimensions: namely, if a cuboid is inscribed into a sphere and circumscribed around a sphere and the centres of the spheres coincide, does it imply that the cuboid is a cube? (A cuboid is a polyhedron with 6 quadrilateral faces such that each vertex belongs to 3 edges.)

Answer. Such cuboid is not necessary a cube.



(a) top view (b) side view (c) (d)

Solution of Central Committee. Consider two 6×8 rectangles with a common centre. Rotate them ninety degrees about each other and shift at the at distance $4\sqrt{3}$ so that the line connecting their centres is perpendicular to both rectangles.

Let O be a midpoint of this line. Connecting the corresponding vertices of top and bottom rectangles we get a cuboid, with 8 vertices, 6 faces and three edges emitting from each vertex. The distance between O and each of eight vertices is the same, equaled to $R = \sqrt{(2\sqrt{3})^2 + 3^2 + 4^2} = \sqrt{37}$. Therefore the cuboid is inscribed into sphere with centre O and radius R.

Now observe that each of six faces of the cuboid (2 congruent rectangles and 4 congruent trapezoids) is inscribed in a circle of radius 5. Indeed, radius of the circle circumscribed about rectangle equals a half of it diagonal $\sqrt{6^2 + 8^2}/2 = 5$. A lateral side of the trapezoid equals to $\sqrt{48 + 2} = \sqrt{50}$ while a heigh of the trapezoid equals $\sqrt{50 - 1} = 7$. It is easy to check that such a trapezoid is also inscribed into a circle of radius 5. Since each face of the solid is inscribed in circles of equal radius, the centres of these circles ($O_i, i = 1, \ldots, 6$) are equidistant from O. This implies that a sphere with centre O and radius $r = OO_i$ touches each face at point O_i. $\square$