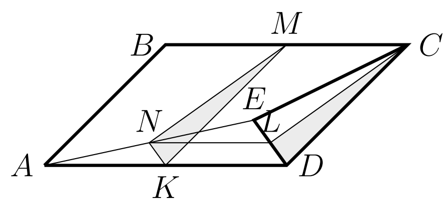
International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2015**

Problem 1. A point is chosen inside a parallelogram $ABCD$ so that $CD = CE$. Prove that the segment DE is perpendicular to the segment connecting the midpoints of the segments AE and BC.

Solution. Denote by M, N, K and L the midpoints of BC, AE, AD and ED respectively. Since triangle ECD is isosceles, the median CL is also an altitude and therefore $\angle CLD = 90°$. Since NK is the midline of triangle AED, NK is parallel to ED and $NK = LD$. Then triangles MKN and CDL are congruent. ($NK = LD, MK = CD$ and $\angle MKN = \angle CDL$, as angles between parallel sides). Hence, $\angle MNK = 90°$ implying that ED is perpendicular to MN.



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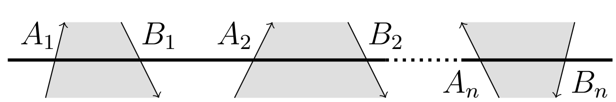
Problem 2. Area 51 has the shape of a non-convex polygon. It is protected by a chain fence along its perimeter and is surrounded by a minefield so that a spy can only move along the fence. The spy went around the Area once so that the Area was always on his right. A straight power line with 36 poles crosses this area so that some of the poles are inside the Area, and some are outside it. Each time the spy crossed the power line, he counted the poles to the left of him (he could see all the poles). Having passed along the whole fence, the spy had counted 2015 poles in total. Find the number of poles inside the fence.

Answer. 1.

Solution. Let $A_1, B_1, A_2, B_2, \ldots, A_n, B_n$ be consecutive points where the power line enters and exits the Area; a Spy goes along the fence so that the Area is on his right. Let us orient the line so that when it enters the Area the Spy goes "up" and when it exits the Area, the Spy goes "down" (see the figure). Then passing through A_k and B_k (A_k and B_k are not necessary consecutive for the Spy) the Spy counts all poles to the left from A_k and all poles to the right of B_k, therefore he counts $36 - a_k$ poles (skipping the poles a_k between A_k and B_k). Then coming back to the point he started the Spy counts $36n - x = 2015$ poles where $x = a_1 + a_2 + \cdots + a_n$ $(0 \le x \le 36)$. Since$2016 = 2015 + 1$ is divisible by 36 this equation has an unique solution $x = 1$.



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Problem 3. (a) The integers x, x^2 and x^3 begin with the same digit. Does it imply that this digit is 1?

(b) The same question for the integers $x, x^2, x^3, \ldots, x^{2015}$.

Answer. No.

Solution. (a) *Example:* $x = 99$, $x^2 = 99^2 = 9801$, $x^3 = 99^3 = 970299$.

(b) *Solution (Ben Wei).* We use Bernoulli inequality $(1-\varepsilon)^k \geq (1-k\varepsilon)$ for $0 < \varepsilon < 1$ and $k \geq 1$ (It can be proved by induction). Consider $x = 99999$ in the form $x = 10^5(1-\varepsilon)$ with $\varepsilon = 10^{-5}$. Then $x^k = 10^{5k}(1-\varepsilon)^k \geq 10^{5k}(1-k\varepsilon) \geq 0.9 \cdot 10^{5k}$.

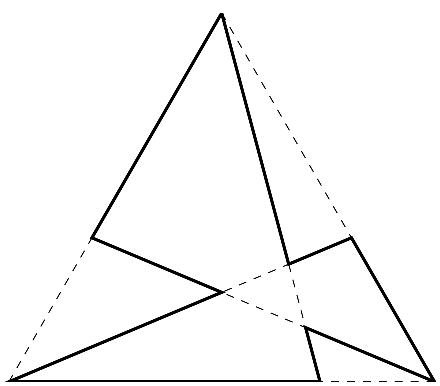
Therefore $10^{5k} > x^k \geq 0.9 \cdot 10^{5k}$ for all $k = 1, 2, \ldots, 2015$, meaning that all given integers start with digit 9.

$\square$

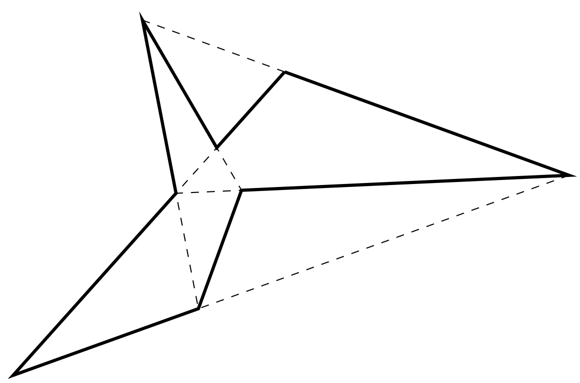
Problem 4. For each side of some polygon, the line containing it contains at least one more vertex of this polygon. Is it possible that the number of vertices of this polygon is

(a) ≤ 9?

(b) ≤ 8?



(a)



(b)

Solution (a) (Ben Wei), (b) (Halim Howard). $\square$

Problem 5. (a) A $2 \times n$-table (with $n > 2$) is filled with numbers so that the sums in all the columns are different. Prove that it is possible to permute the numbers in the table so that the sums in the columns would still be different and the sums in the rows would also be different.

(b) A 10×10-table is filled with numbers such that the sums in all the columns are different. Is it always possible to permute the numbers in the table so that the sums in the columns would still be different and the sums in the rows would also be different?

Solution. (a) (Dima Paramonov). If sums of the numbers in rows are different, then the statement holds. Assume that these sums are the same. If some column contains different numbers, then exchanging these numbers we satisfy the requirement. Assume that in each column the top and the bottom numbers are the same (so that both rows of the table are identical). We can always change the order of columns in the table so that $a_1 < a_2 < a_3 < \ldots < a_k$. Let us consider the following permutation which affects only the first three columns:

a_1	a_1	a_2	a_4	$\ldots$	a_k
a_2	a_3	a_3	a_4	$\ldots$	a_k

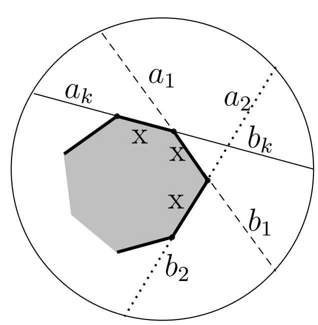
It is clear now that sums of the numbers in rows are different. Indeed, $a_1+a_1+a_2 < a_1+a_2+a_3$. Since $a_1 + a_2 < a_1 + a_3 < a_2 + a_3 < 2a_4 < \ldots . < 2a_k$, sums of the numbers in each column remain different.

(b) Counterexample (Sina Abbasi). Consider a 10×10 table, filled with "0"s and "1"s as follows: column i, $i = 1, ..., 9$ contains $i - 1$ of 1s, the last column is completely filled with 1swhile the remaining cells are filled with "0"s.

Since the total sum of elements in a table is the same no matter if it is calculated by columns or by rows and all possible values of sums in rows are between 0 and 10 (11 different values in total) we have: $0+1+2+3+4+5+6+7+8+10 = 0+1+2+\cdots+10-x$ where x is a value of sum in a row we must exclude (assuming that all other sums of the numbers in rows are different). Given that $0 \leq x \leq 10$, $x = 9$ is an unique solution. Therefore the possible values of sums in rows are $0, 1, \ldots, 8$ and 10. However the existence of a row with value 0 contradicts to the existence of a column with value 10. $\square$

Problem 6. A convex N-gon with equal sides is located inside a circle. Each side is extended in both directions up to the intersection with the circle so that it contains two new segments outside the polygon. Prove that one can paint some of these new $2N$ segments in red and the rest in blue so that the sum of lengths of all the red segments would be the same as for the blue ones.

Combined solution of Victor Rong and Frieda Rong. Standing at some point inside of the polygon and looking towards its side x_i, $(i = 1, ..., k, |x_i| = x)$ denote the segments to the left and to the right of the side as a_i and b_i respectively.



Consequently applying Intersecting Chord Theorem to each vertex of the polygon we get

$$
\begin{aligned}
a_1(x+b_1) &= b_k(a_k+x),\\
a_2(x+b_2) &= b_1(a_1+x),\\
a_3(x+b_3) &= b_2(a_2+x),\\
&\dots\\
a_k(x+b_k) &= b_{k-1}(a_{k-1}+x).
\end{aligned}
$$

Summing up the equations and simplifying we get

$$a_1 + a_2 + \ldots + a_k = b_1 + b_2 + \ldots + b_k.$$

By colouring a_i and b_i in red and blue we prove the statement. $\square$

Problem 7. An Emperor invited 2015 wizards to a festival. Each of the wizards knows who of them is good and who is evil, however the Emperor doesn't know this. A good wizard always tells the truth, while an evil wizard can tell the truth or lie at any moment. The Emperor asks each wizard (in an order of his choice) a single question, maybe different for different wizards, and listens to the answer which is either "yes" or "no". Having listened to all the answers, the Emperor expels a single wizard through a magic door which shows if this wizard is good or evil. Then the Emperor repeats the procedure with the remaining wizards, and so on. The Emperor may stop at any moment, and after this the Emperor may expel or not expel a wizard. Prove that the Emperor can expel all the evil wizards having expelled at most one good wizard.

Solution of Sina Abbasi. Step 1. The Emperor lists all wizards $W_1, W_2, \ldots, W_N$ and asks everyone but W_1 if W_1 is evil (It does not no matter what he asks W_1).

- If all of them confirm that W_1 is evil, W_1 is expelled. If he occurs to be evil, the number of evil wizards is decreased by 1 (so the Emperor can start the next round). If W_1 occurs to be good then all the remaining wizards are evil, and they are expelled in next rounds one by one.

- If wizard W_i answers "no", then he is expelled. If he occurs to be evil the number of evil wizards is decreased by 1. If he occurs to be good we find one non-expelled good wizard (W_1) and we go to Step 2.

Step 2. Assume that all wizards are lined up, starting with good wizard W_1. The Emperor asks W_i ($i = 1, \ldots$) if W_{i+1} is evil (It does not no matter what he asks the last wizard). If W_i answers "yes", W_{i+1} is expelled and the Emperor can start the next round. If all wizards said "no", then all evil wizards are expelled and Emperor stops the trial. $\square$