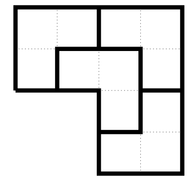
37th International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2015**

1 A polygon on a grid is called *amazing* if it is not a rectangle and several its copies form a polygon similar to it. For instance, a corner consisting of three cells is an amazing polygon (see the figure at the right).



(a) [**2**] Find an amazing polygon consisting of 4 cells.

(b) [**3**] For which $n > 4$ there exists an amazing polygon consisting of n cells?

2 From a set of integers $\{1, \ldots, 100\}$, k integers were deleted. Is it always possible to choose k distinct integers from the remaining set such that their sum is 100 if

(a) [**2**] $k = 9$?

(b) [**4**] $k = 8$?

3 Let P be the perimeter of an arbitrary triangle. Prove that sum of the lengths of any two medians is

(a) [**3**] no greater than $\frac{3P}{4}$;

(b) [**4**] no less than $\frac{3P}{8}$.

4 [**8**] A 9×9 grid square is made of matches, so that the side of any cell is one match. In turns Pete and Basil remove matches, one match at a time. The player who destroys the last cell wins. Pete starts first. Which of the players has a winning strategy, no matter how his opponent plays? (A cell is destroyed if it has less than 4 matches on its perimeter).

5 [**8**] In triangle ABC, medians AA_0, BB_0 and CC_0 intersect at point M. Prove that the circumcenters of triangles MA_0B_0, MCB_0, MA_0C_0, MBC_0 and point M are concyclic.

6 Several distinct real numbers are written on a blackboard. Peter wants to create an algebraic expression such that among its values there would be these and only these numbers. He may use any real numbers, brackets, signs $+$, $-$, $\times$ and a special sign $\pm$. Usage of $\pm$ is equivalent to usage of $+$ and $-$ in all possible combinations. For instance, the expression 5 ± 1 results in $\{4, 6\}$, while $(2 \pm 0.5) \pm 0.5$ results in $\{1, 2, 3\}$. Can Peter construct an expression if the numbers on the blackboard are:

(a) [**3**] 1, 2, 4?

(b) [**7**] any 100 distinct numbers (not necessarily integer)?

7 [**9**] Santa Claus had n sorts of candies, k candies of each sort. He distributed them at random between k gift bags, n candies per a bag and gave a bag to everyone of k children at Christmas party. The children learned what they had in their bags and decided to trade. Two children trade one candy for one candy in case if each of them gets the candy of the sort which was absent in his/her bag. Prove that they can organize a sequence of trades so that finally every child would have candies of each sort.