International Mathematics
TOURNAMENT OF THE TOWNS SOLUTIONS

Senior A-Level Paper **Spring 2013.**

1 [**3**] Several positive integers are written on a blackboard. The sum of any two of them is a positive integer power of two (for example, $2, 4, 8, \ldots$). What is the maximal possible number of different integers on the blackboard?

Answer. Two.

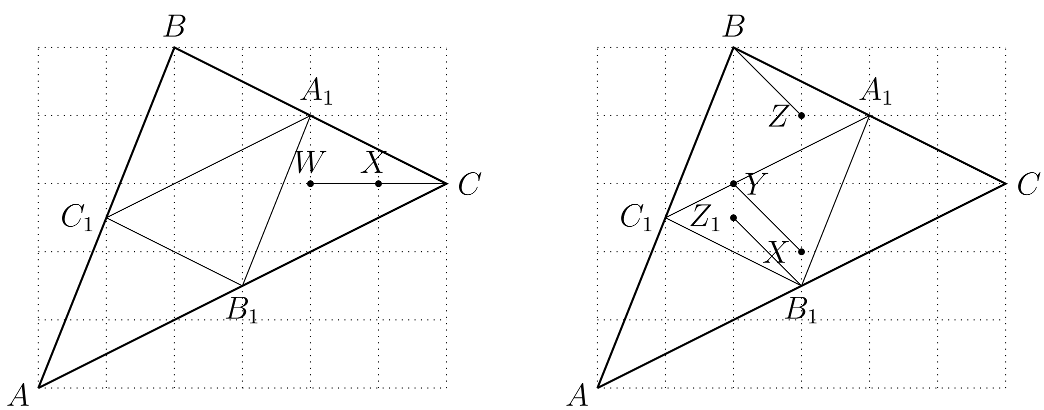
Solution. See Juniors 1.

2 [**4**] A boy and a girl were sitting on a long bench. Then twenty more children one after another came to sit on the bench, each taking a place between already sitting children. Let us call a girl brave if she sat down between two boys, and let us call a boy brave if he sat down between two girls. It happened, that in the end all girls and boys were sitting in the alternating order. Is it possible to uniquely determine the number of brave children?

Solution. Divide the bench into segments occupied by boys or girls only. These segments alternate. Notice that if a not brave child comes to the bench then the number of segments does not change. If a brave child comes to the bench then the number of segments increases by 2. Initially there were two segments. In the end there were 22 segments. Therefore, the number of brave children is $(22-2):2=10$.

3 [**6**] A point in the plane is called a node if both its coordinates are integers. Consider a triangle with vertices at nodes containing at least two nodes inside. Prove that there exists a pair of internal nodes such that a straight line connecting them either passes through a vertex or is parallel to side of the triangle.

Solution. Let A_1, B_1, and C_1 be midpoints of sides BC, CA, and AB of triangle ABC respectively.



(a) (b)

Consider two arbitrary nodes X and Y inside the triangle. Suppose one of them lies outside triangle $A_1B_1C_1$; assume that node X belongs to triangle A_1B_1C (see Figure (a)). Construct segment CW so that point X is the midpoint of CW. Note that point W is also a node, and that W belongs to the interior of triangle ABC. Hence there are two interior nodes, X and W such that line CW passes through a vertex (C).

Suppose that both nodes X and Y belong to triangle $A_1B_1C_1$ (see Figure (b)). If line XY is parallel to one of the sides of the given triangle then the statement holds. Otherwise, let us apply the lemma (see solution of Problem 5, Juniors) to the triangle $A_1B_1C_1$. According to the lemma there exists a segment (B_1Z_1) equal and parallel to segment XY, with one endpoint at a vertex (B_1) and the other endpoint (Z_1) inside triangle $A_1B_1C_1$. Consider segment BZ symmetrical to B_1Z_1 about the midpoint of A_1C_1. By construction segment BZ is also equal and parallel to the segment XY. Hence Z is a node which belongs to the interior of triangle A_1BC_1. Then, as is shown above, line BZ contains another node inside triangle ABC.

4 [**6**] Integers $1, 2, \ldots, 100$ are written on a circle, not necessarily in that order. Can it be that the absolute value of the difference between any two adjacent integers is at least 30 and at most 50?

ANSWER. No.

SOLUTION. Suppose we can place the numbers on a circle so that the condition holds. Let us call the integers from 26 to 75 *medial*, and all the others *extreme*. Two extreme integers cannot be consecutive (their difference is either less than 25 or greater than 50). Note that the numbers of the extreme and medial integers are the same and therefore they must alternate. However the medial number 26 can be adjacent to only one extreme integer 76. Contradiction.

5 [**7**] On an initially colourless plane three points are chosen and marked in red, blue and yellow. At each step two points marked in different colours are chosen. Then one more point is painted in the third colour so that these three points form a regular triangle with the vertices coloured clockwise in "red, blue, yellow". A point already marked may be marked again so that it may have several colours. Prove that for any number of moves all the points containing the same colour lie on the same line.

SOLUTION. In what follows, *simple rotation* would mean "rotation by 60° clockwise". Denote the given points by R, B, and Y. Construct a point R' corresponding to B and Y, and a point B' corresponding to Y and R. Then the simple rotation about Y transforms segment $R'R$ into BB'. (If at least one of these segments degenerates into a point then the triangle RBY is regular and its vertices are listed clockwise; in this case no new points arise.)

Thus line RR' is transformed into a line BB' by a simple rotation about their common point O. Let us call the first line *red*, the second line *blue*, and the line obtained by simple rotation of blue line about O *yellow*. Observe that a simple rotation about any point R_1 on the red line transforms blue line into yellow line because the distances from R_1 to these lines are equal. Consequently, if we construct a point Y_1 corresponding to two arbitrary points B_1 and R_1 on blue and red lines respectively, it gets on yellow line. Similarly, given two points on lines of different colours, a point constructed according to the rule will be on the line of the third colour.

6 There are five distinct real positive numbers. It is known that the total sum of their squares and the total sum of their pairwise products are equal.

(a) [**4**] Prove that we can choose three numbers such that it would not be possible to make a triangle with sides' lengths equal to these numbers.

(b) [**5**] Prove that the number of such triples is at least six (triples which consist of the same numbers in different order are considered the same).

SOLUTION. Let us place the numbers in increasing order: $a < b < c < d < e$.

(a) Suppose the assertion fails, then $a + b > e$. Hence

$$a^2 + b^2 + c^2 + d^2 + e^2 < ab + bc + cd + de + (a + b)e.$$

Contradiction.

(b) Let us consider the following cases:

(1) $b + c \leq d$.
Then each of six triples in which two numbers are from the set $\{a, b, c\}$ and the third number is from the set $\{d, e\}$ does not form a triangle.

(2) $c + d \leq e$. Then each of six triples which includes e does not form a triangle.

(3) $b + d \leq e$ and $a + b \leq d$. Then each of six triples $\{a, b, d\}$, $\{a, b, e\}$, $\{a, c, e\}$, $\{a, d, e\}$, $\{b, c, e\}$, $\{b, d, e\}$ does not form a triangle.

Suppose that neither of above cases takes place, that is, $b + c > d$, $c + d > e$ and at least one of inequalities $b + d > e$ and $a + b > d$ holds. We shall show that this is impossible. Indeed,

(4) If $b + c > d$, $b + d > e$ then

$$a^2 + b^2 + c^2 + d^2 + e^2 < ab + bc + ce + (b + c)d + (b + d)e.$$

Contradiction.

(5) If $c + d > e$, $a + b > d$ then

$$a^2 + b^2 + c^2 + d^2 + e^2 < ab + bc + cd + (a + b)d + (c + d)e.$$

Contradiction.

Remark. More than six "bad triples" cannot be guaranteed. Indeed, consider the numbers a, b, c, d close to 1, and find e as a root of a quadratic equation (since the constant term is close to 2, it has a positive root). Then each triple from the set $\{a, b, c, d\}$ is "good".

7 The King decided to reduce his Council consisting of thousand wizards. He placed them in a line and placed hats with numbers from 1 to 1001 on their heads not necessarily in this order (one hat was hidden). Each wizard can see the numbers on the hats of all those before him but not on himself or on anyone who stayed behind him. By King's command, starting from the end of the line each wizard calls one integer from 1 to 1001 so that every wizard in the line can hear it. No number can be repeated twice.

In the end each wizard who fails to call the number on his hat is removed from the Council. The wizards knew the conditions of testing and could work out their strategy prior to it.

(a) [5] Can the wizards work out a strategy which guarantees that more than 500 of them remain in the Council?

(b) [7] Can the wizards work out a strategy which guarantees that at least 999 of them remain in the Council?

Answer. Yes, they can (in both cases).

Solution

(a) Let the 1000-th wizard calculate the sum of the numbers he sees and call the remainder of that sum when it is divided by 1001 (if the remainder is 0, he calls 1001). Then 999-th wizard can compute his number by subtracting the sum of the numbers he sees from what he has heard. In this way the wizards proceed until someone computes the number that coincides with the number called by the 1000th wizard. According to the preliminary agreement, this *looser* calls the number of the first wizard in line. It signals to the others that one who called the number of the first wizard has indeed calculated the number called by the 1000-th wizard; therefore, they proceed accordingly. As the result, all the wizards but the last one, the looser, and the first wizard will deliver correct answers.

(b) The 1000-th wizard does not know two numbers, his and on the hidden hat. There are two ways to order these two numbers. In one case the permutation of all 1001 numbers is even, in the other case it is odd. According to the preliminary agreement the 1000-th wizard calls the number that makes the permutation even (with 50% probability to be wrong). Now 999-th wizard also does not know only two numbers, his own number and the number on the hidden hat. He arranges these numbers so that the resulting permutation is even and calls the corresponding number. His answer is indeed correct since the permutation he made in his mind coincides with the permutation defined by the last wizard, which correctly reflects the number of the 999-th wizard. In a similar way, all the others wizards calculate their numbers. Consequently all wizards but probably the last one will deliver correct answers.

Here we used

Definition

(1) *Permutation* of order n is $(a_1, a_2, \ldots, a_n)$ where $a_1, \ldots, a_n$ are numbers from 1 to n in some order (each number is included exactly once).

(2) *Inversion* is every occurrence when $a_k < a_l$ while $k > l$;

(3) *Parity* of the permutation is a parity of the total number of inversions. One can observe that if two numbers in the permutation exchange their places the parity of permutation changes (from odd to even and from even to odd).