International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2013**

1 [**5**] There is a 8×8 table, drawn in a plane and painted in a chess board fashion. Peter mentally chooses a square and an interior point in it. Basil can draws any polygon (without self-intersections) in the plane and ask Peter whether the chosen point is inside or outside this polygon. What is the minimal number of questions sufficient to determine whether the chosen point is black or white?

2 [**6**] Find all positive integers n for which the following statement holds:

For any two polynomials $P(x)$ and $Q(x)$ of degree n there exist monomials ax^k and bx^ℓ, $0 \le k, \ell \le n$, such that the graphs of $P(x) + ax^k$ and $Q(x) + bx^\ell$ have no common points.

3 [**6**] Let ABC be an equilateral triangle with centre O. A line through C meets the circumcircle of triangle AOB at points D and E. Prove that points A, O and the midpoints of segments BD, BE are concyclic.

4 [**7**] Is it true that every integer is a sum of finite number of cubes of distinct integers?

5 Do there exist two integer-valued functions f and g such that for every integer x we have

(a) [**3**] $f(f(x)) = x, \quad g(g(x)) = x, \quad f(g(x)) > x, \quad g(f(x)) > x$?

(b) [**5**] $f(f(x)) < x, \quad g(g(x)) < x, \quad f(g(x)) > x, \quad g(f(x)) > x$?

6 [**9**] On a table, there are 11 piles of ten stones each. Pete and Basil play the following game. In turns they take 1, 2 or 3 stones at a time: Pete takes stones from any single pile while Basil takes stones from different piles but no more than one from each. Pete moves first. The player who cannot move, loses.

Which of the players, Pete or Basil, has a winning strategy?

7 [**14**] A closed broken self-intersecting line is drawn in the plane. Each of the links of this line is intersected exactly once and no three links intersect at the same point. Further, there are no self-intersections at the vertices and no two links have a common segment. Can it happen that every point of self-intersection divides both links in halves?