International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2012[1].**

1. It is possible to place an even number of pears in a row such that the masses of any two neighbouring pears differ by at most 1 gram. Prove that it is then possible to put the pears two in a bag and place the bags in a row such that the masses of any two neighbouring bags differ by at most 1 gram.

2. One hundred points are marked in the plane, with no three in a line. Is it always possible to connect the points in pairs such that all fifty segments intersect one another?

3. In a team of guards, each is assigned a different positive integer. For any two guards, the ratio of the two numbers assigned to them is at least 3:1. A guard assigned the number n is on duty for n days in a row, off duty for n days in a row, back on duty for n days in a row, and so on. The guards need not start their duties on the same day. Is it possible that on any day, at least one in such a team of guards is on duty?

4. Each entry in an $n \times n$ table is either $+$ or $-$. At each step, one can choose a row or a column and reverse all signs in it. From the initial position, it is possible to obtain the table in which all signs are $+$. Prove that this can be accomplished in at most n steps.

5. Let p be a prime number. A set of $p+2$ positive integers, not necessarily distinct, is called *interesting* if the sum of any p of them is divisible by each of the other two. Determine all interesting sets.

6. A bank has one million clients, one of whom is Inspector Gadget. Each client has a unique PIN number consisting of six digits. Dr. Claw has a list of all the clients. He is able to break into the account of any client, choose any n digits of the PIN number and copy them. The n digits he copies from different clients need not be in the same n positions. He can break into the account of each client, but only once. What is the smallest value of n which allows Dr. Claw to determine the complete PIN number of Inspector Gadget?

7. Let AH be an altitude of an equilateral triangle ABC. Let I be the incentre of triangle ABH, and let L, K and J be the incentres of triangles ABI, BCI and CAI respectively. Determine $\angle KJL$.

Note: The problems are worth 4, 4, 6, 6, 8, 8 and 8 points respectively.

[1]Courtesy of Professor Andy Liu.

Solution to Junior A-Level Spring 2012

1. Let the pears be $P_1,\ P_2,\dots,P_{2n}$, with respective masses $a_1\le a_2\le\cdots\le a_{2n}$. For $1\le k\le n$, put P_k and P_{2n-k+1} in bag B_k and place the bags in a row in numerical order. We claim that the difference in the masses of B_k and B_{k+1} is at most 1 gram for $1\le k<n$. Now the mass of B_k is a_k+a_{2n-k+1} amd the mass of B_{k+1} is $a_{k+1}+a_{2n-k}$. Both $a_{k+1}-a_k$ and $a_{2n-k+1}-a_{2n-k}$ are non-negative, and we claim that each is at most 1 gram. The desired result will then follow. Consider P_k and P_{k+1} in the original line-up. If they are in fact neighbours, the claim follows from the given condition. Suppose there are other pears in between. Moving from P_k to P_{k+1}, let P_j be the first pear not lighter than P_k and P_i be the pear before P_j. Then $a_i\le a_k\le a_j$ and $a_j-a_i\le 1$. Hence $a_j-a_k\le 1$. However, we must have $a_{k+1}\le a_j$. It follows that $a_{k+1}-a_k\le 1$. Similarly, we can prove that $a_{2n-k+1}-a_{2n-k}\le 1$, justifying the claim.

2. **Solution by Olga Ivrii.**
Let $P_1,\ P_2,\ \dots,P_{99}$ be evenly spaced points on a circle with centre P_{100}. We may assume that when the points are connected in pairs, P_{100} is connected to P_1. Let P_{50} be connected to P_k where $k\ne 1,50,100$. If $2\le k\le 49$, $P_{50}P_k$ and P_1P_{100} are in opposite semicircles divided by the diameter passing through P_{50}. If $51\le k\le 99$, $P_{50}P_k$ and P_1P_{100} are in opposite semicircles divided by the diameter passing through P_{49}. In either case, $P_{50}P_k$ and P_1P_{100} do not intersect.

3. Let the guards be $G_1,\ G_2,\ \dots,\ G_k$ and let $n_1>n_2>\cdots>n_k\ge 1$ be the numbers assigned to them. In fact, $n_i\ge 3n_{i+1}$ for $1\le i<k$. There is an interval of $3n_2$ days during which G_1 is not on duty. Within this interval, there is a subinterval of $n_2\ge 3n_3$ days during which G_2 is not on duty either. Repeating this argument until we reach G_k, we will have an interval of n_k days in which none of the guards are on duty.

4. **Solution by Olga Ivrii.**
Suppose $n\ge 2$. Let $1\le i<k\le n$ and $1\le j<\ell\le n$. We claim that the number of +s among $a_{i,j},\ a_{i,\ell},\ a_{k,j}$ and $a_{k,\ell}$ is even. This is because each step reverses the signs of an even number (0 or 2) of these 4 signs. The claim follows since we can make all signs +s. We now use induction on n. The basis $n=1$ is trivial. Suppose the result holds for some $n\ge 1$. Consider an $(n+1)\times(n+1)$ table. If there are no −s, there is nothing to prove. We may assume that $a_{n+1,n+1}$ is −. By the induction hypothesis, we can make all signs +s except possibly for those in the last row and those in the last column, in n steps. Consider $a_{1,n+1}$. Suppose it is +. By our earlier claim, taking $i=1,\ k=n+1$ and $\ell=n+1$, we can conclude that $a_{n+1,j}$ is − for all j. By the claim again, with $k=n+1,\ j=1$ and $\ell=n+1$, we can conclude that $a_{i,n+1}$ is + for all $i<n+1$. Reversing the signs in the last row will complete the desired transformation in $n+1$ moves. Suppose $a_{1,n+1}$ is −. We can prove as above that all entries in the last column are −s and all entries in the last row except the last one are +s. Reversing the signs in the last column will complete the desired transformation, again in $n+1$ steps.

5. By factoring it out if necessary, we may assume that the greatest common divisor of the $p+2$ numbers in an interesting set is 1. Let $S = a_1 + a_2 + \cdots + a_{p+2}$. For $1 \le k \le p+2$, a_k divides $S - a_j$ for $j \ne k$. Hence a_k divides $(p+1)S - (S - a_k)$, so that a_k divides pS. We consider two cases.
   Case 1. None of a_k is divisible by p.
   Then a_k divides S. For any $j \ne k$, a_k divides $S - a_j$, so that it divides a_j. It follows that all the numbers are equal. Since their greatest common divisor is 1, they are all equal to 1.
   Case 2. At least one a_k is divisible by p.
   Not all of them can be divisible by p since their greatest common divisor is 1. We may assume that a_k is not divisible by p for $1 \le k \le n$ and divisible by p for $n+1 \le k \le p+2$, where $n \le p+1$. Suppose $n \le p$. Let $T = a_1 + a_2 + \cdots + a_n$. If T is not divisible by p, then $a_1 + a_2 + \cdots + a_p$ is not divisible by a_{p+2}. If T is divisible by p, then $a_2 + a_3 + \cdots + a_{p+1}$ is not divisible by a_{p+2}. It follows that $n = p+1$, so that a_{p+2} is the only number divisible by p. As in Case 1, $a_1 = a_2 = \cdots = a_{p+1} = 1$ so that $a_{p+2} = p$.
   In summary, all interesting sets are of the form $a_1 = a_2 = \cdots = a_{p+1} = d$ and $a_{p+2} = d$ or pd for an arbitrary positive integer d.

6. In order to be sure of knowing a digit of Inspector Gadget's PIN number, Dr. Claw either must check it or check that digit of every other client. For $n = 3$, Dr. Claw can find out the first three digits of Inspector Gadget's PIN number, and deduce the last three digits by checking those of every other client. There is no solution for $n \le 2$ since Dr. Claw can know at most 2 digits of Inspector Gadget's PIN number and deduce 2 more by checking those of every other client.

7. **Solution by Central Jury:**
   Since K is the incentre of triangle BCI, $\angle BKI = 90^\circ + \frac{1}{2}\angle BCI = 97.5^\circ$. Let O be the centre of triangle ABC and let M be the point symmetric to I about OA. Note that K lies on BM. We have $\angle MAJ = 7.5^\circ = \angle OAJ$. Since $\angle AOJ = 60^\circ = \angle MOJ$, J is the incentre of triangle MAO. Hence $\angle MJO = 90^\circ + \frac{1}{2}\angle MAO = 97.5^\circ = \angle BKI$. Hence $IJMK$ is a cyclic quadrilateral and $\angle IJK = \angle IMK = 15^\circ$. Since L is symmetric to K about BO, $\angle KJL = 2\angle IJK = 30^\circ$.