International Mathematics
TOURNAMENT OF TOWNS

Senior A-Level Paper **Fall 2012.**

1. Given an infinite sequence of numbers $a_1, a_2, a_3, \dots$ For each positive integer k there exists a positive integer $t = t(k)$ such that $a_k = a_{k+t} = a_{k+2t} = \dots$ Is this sequence necessarily periodic? That is, does a positive integer T exist such that $a_k = a_{k+T}$ for each positive integer k?

Solution. The answer is no. For example, let m_k be the highest degree of 2 that divides k, $a_k = 0$ if m_k is even and $a_k = 1$ if m_k is odd, and $t(k) = 2m_k$.

2. Chip and Dale play the following game. Chip starts by splitting 1001 nuts between three piles, so Dale can see it. In response, Dale chooses some number N from 1 to 1001. Then Chip moves nuts from the piles he prepared to a new (fourth) pile until there will be exactly N nuts in any one or more piles. When Chip accomplishes his task, Dale gets an exact amount of nuts that Chip moved. What is the maximal number of nuts that Dale can get for sure, no matter how Chip acts? (Naturally, Dale wants to get as many nuts as possible, while Chip wants to lose as little as possible).

Solution. Consider a line segment of length 1001 on which we mark points A, B and C corresponding to the piles with a, b and c nuts in them. Let us also mark the points $A + B$, $A + B$ and $B + C$, corresponding to two combined piles, the point O, corresponding to an empty pile, and the point $A + B + C$, corresponding to the pile of 1001 nuts. If Dale choses a number n then Chip's strategy is to look for the closest marked point to this number and to move nuts from the corresponding pile (or combined piles) to the pile 0. It is clear that if the points are marked uniformly (with the distance 143 between each pair of subsequent points) then the maximal difference between n and the closest number is 71, therefore Chip can loose at most 71 nuts.

On the other hand, since the maximal distance between the subsequent points is at least 143, Dale can always choose a number such that he can guarantee at least 71 nuts.

3. A car rides along a circular track in the clockwise direction. At noon Peter and Paul took their positions at two different points of the track. Some moment later they simultaneously ended their duties and compared their notes. The car passed each of them at least 30 times. Peter noticed that each circle was passed by the car 1 second faster than the preceding one while Paul's observation was opposite: each circle was passed 1 second slower than the preceding one.

Prove that their duty was at least an hour and a half long.

Solution. Each observer has noticed at least 29 circles. For Peter, the car passed consecutive circles in $m+14, m+13, \ldots, m-14$ seconds, and for Paul in $p-14, p-13, \ldots, p+14$ seconds. The total time for passing 29 circles is equal to $29m$ and $29p$ respectively. First 15 Paul's circles cover 14 Peter's circles, either from 1st to 14th (if the car passed Peter for the first time before Paul), or from 2nd to 15th (otherwise). In any case

$$(p-14)+(p-13)+\cdots+p>(m+13)+(m+12)+\cdots+m.$$

On the other hand, the last 15 Paul's circles cover 14 Peter's circles, either from 16th to 29th, or from 15th to 28th, hence

$$(m-14)+(m-13)+\cdots+m>(p+13)+(p+12)+\cdots+p.$$

Summing up the inequalities and collecting terms, we get $p+m>392$, hence $29p+29m>29\cdot 392$. Thus the total time for at least one observer is at least $29\cdot 196=5684$. This is greater than an hour and a half (5400 seconds).

4. In a triangle ABC two points, C_1 and A_1 are marked on the sides AB and BC respectively (the points do not coincide with the vertices). Let K be the midpoint of A_1C_1 and I be the incentre of the triangle ABC. Given that the quadrilateral A_1BC_1I is cyclic, prove that the angle AKC is obtuse.

Solution. Let M be the midpoint of AC, and A_2, B_2 and C_2 are touching points of the inscribed circle with sides BC, AC and AB respectively. Since $\angle A_1IC_1=180^\circ-\angle B=\angle A_2IC_2$, the right-angled triangles A_1A_2I and C_1C_2I are equal (by a cathetus and an acute angle). One of them is inside, and the other one outside the rectangle BA_2IC_2. Hence $AC_1+CA_1=AC_2+CA_2=AB_2+CB_2=AC$.

Construct parallelograms AC_1KD and CA_1KE. Then $ADCE$ is also a parallelogram (possibly degenerate) and M is its centre, that is, the midpoint of segment DE. As is well-known, a median is less than the half-sum of the adjacent sides, hence $KM<1/2(KD+KE)=1/2(AC_1+CA_1)=1/2AC$. This means that point K is inside the circle with diameter AC, hence angle AKC is obtuse.

5. Peter and Paul play the following game. First, Peter chooses some positive integer a with the sum of its digits equal to 2012. Paul wants to determine this number; he knows only that the sum of the digits of Peter's number is 2012. On each of his moves Paul chooses a positive integer x and Peter tells him the sum of the digits of $|x-a|$. What is the minimal number of moves in which Paul can determine Peter's number for sure?

Solution. The answer is 2012. Let $S(n)$ be the sum of the digits of n.

Algorithm. At the first step, Paul chooses 1. If a ends with k zeroes then $S(a-1) = 2011+9k$. Thus Paul gets to know the position of the rightmost nonzero digit in a. Set $a_1 = a - 10^k$. Paul knows that $S(a_1) = 2011$. At the second step Paul chooses x such that $a - x = a_1 - 1$ and gets to know the number m of zeroes at the end of a_1. Set $a_2 = a_1 - 10^m$ and so on. After the 2012th step Paul obtains $S(a_{2012}) = 0$, thus having determined a.

Estimate. Suppose all digits in a are 0 and 1, that is, $a = 10^{k_{2012}} + 10^{k_{2011}} + \cdots + 10^{k_1}$ where $k_{2012} > k_{2011} > \cdots > k_1$. It is possible that at the first step Paul chooses an integer $x < 10^{k_1}$. Then $S(a - x) = S(10^{k_1} - x) + 2011$ independently of values of $k_{2012}, \ldots, k_{i+1}$. So Paul gets no new information about $k_{2012}, \ldots, k_{i+1}$. Similarly, it is possible that at the second step Paul chooses an integer smaller than 10^{k_2}, and so on. Then after 2011 steps Paul does not know k_{2012}.

6. (a) A point A is marked inside a sphere. Three perpendicular lines drawn through A intersect the sphere at six points. Prove that the centre of gravity of these six points does not depend on the choice of such three lines.

(b) An icosahedron with the centre A is placed inside a sphere (its centre does not necessarily coincide with the centre of the sphere). The rays going from A to the vertices of the icosahedron mark 12 points on the sphere. Then the icosahedron is rotated about its centre. New rays mark new 12 points on the sphere. Let O and N be the centres of mass of old and new points respectively. Prove that $O = N$.

(An icosahedron is a regular polyhedron with 20 triangular faces; each vertex emits 5 edges).

Solution. Let C be the centre of the sphere, O the centre of mass in question. It is also the centre of mass for the midpoints of chords in the sphere, cut by the drawn lines. (For an icosahedron, by its central symmetry, pairs of opposite rays may be replaced by the lines containing the main diagonals.)

a) The midpoints of the chords (say K, L, M) are projections of C to the lines drawn, hence $\overline{AC} = \overline{AK} + \overline{AL} + \overline{AM}$ (a vector is the sum of its projections to three perpendicular axes). Hence $\overline{AO} = \frac{1}{3}\overline{AC}$.

b) Let $\overline{AO} = \mathbf{a}$, $\overline{AC} = \mathbf{c}$. It suffices to show that $\mathbf{a} = \alpha\mathbf{c}$ where α is independent of $\mathbf{c}$ and of the position of the icosahedron.

The midpoints of the chords are projections of C to the diagonals of the icosahedron. Let $\mathbf{e}_i$ be the unit vector directed along the ith diagonal, A_i be the corresponding projection. Then $\overline{AA_i} = |\mathbf{c}|\cos\varphi \cdot \mathbf{e}_i = (\mathbf{c}, \mathbf{e}_i)\mathbf{e}_i$ where brackets denote the scalar product and φ is the angle between $\mathbf{c}$ and $\mathbf{e}_i$, and $6\mathbf{a} = (\mathbf{c}, \mathbf{e}_1)\mathbf{e}_1 + \cdots + (\mathbf{c}, \mathbf{e}_6)\mathbf{e}_6$. The last expression depends on $\mathbf{c}$ linearly, hence it suffices to prove the equation

$$(\mathbf{c}, \mathbf{e}_1)\mathbf{e}_1 + \cdots + (\mathbf{c}, \mathbf{e}_6)\mathbf{e}_6 = 6\alpha\mathbf{c} \qquad (*)$$

for any three non-complanar vectors, for instance for $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

The results does not change if we replace some of $\mathbf{e}_i$ by opposite ones. Hence proving $(*)$ for $\mathbf{c} = \mathbf{e}_1$ we may assume that $\mathbf{e}_2, \dots, \mathbf{e}_6$ are directed to the vertices closest to that one where $\mathbf{e}_1$ is directed. Then by symmetry we have $(\mathbf{e}_1, \mathbf{e}_2) = \cdots = (\mathbf{e}_1, \mathbf{e}_6)$. But $\mathbf{e}_2 + \cdots + \mathbf{e}_6 = \beta \mathbf{e}_1$ where β is a constant.

The equation $(*)$ for $\mathbf{c} = \mathbf{e}_2$ and $\mathbf{c} = \mathbf{e}_3$ is proved similarly.

Remark. As is clear from the above, $6\alpha = 1 + 5\cos^2\psi$ where ψ is the angle between two neighbouring diagonals of an icosahedron.

7. There are 1 000 000 soldiers in a line. The sergeant splits the line into 100 segments (the length of different segments may be different) and permutes the segments (not changing the order of soldiers in each segment) forming a new line. The sergeant repeats this procedure several times (splits the new line in segments of the same lengths and permutes them in exactly the same way as the first time). Every soldier originally from the first segment recorded the number of performed procedures that took him to return to the first segment for the first time. Prove that at most 100 of these numbers are different.

Solution. Let us mark 99 borders between segments by flags; during iterations, flags remain on their spots. We call a pair of soldiers *special* if originally they were neighbours in the first segment but returned to this segment after different number of iterations. Then clearly these soldiers at some moment went to different segments. Let us consider the first moment when it happened. Until this moment they were neighbours in the line, so they are still neighbours but now there is a flag between them.

We claim that *each special pair is served by its own dedicated flag* (so no flag can serve two special pairs). Assume that some flag F first separated two special pairs A and B; pair A was separated after k iterations and pair B was separated after $m > k$ iterations.

Note that our operations are invertible: positions of the soldiers on the previous step are uniquely defined. So let us pull k iterations back from the moment when flag F separated pair A the first time. Then soldiers from A return to their positions in the first segment.

Now pull k operations back from the moment m when the second pair got separated. Since the pair B occupied the same places at moment m as pair A at the moment k, this pair B also will return to the same place, which is in the first segment. However, time is $0 < m - k < m$ which contradicts to our conjecture that m was the first moment when it happened.

Therefore, the number of special pairs does not exceed 99. This means that going from one soldier to the next one along the first segment, the recorded numbers could change no more than 99 times and therefore there are at most 100 recorded numbers.

Remark. One can prove easily that each soldier from the first segment really returns to it. However it is not necessary: if a soldier never returns to the first segment we can define the return time equal to ∞ and this does not affect the above arguments.