International Mathematics
TOURNAMENT OF TOWNS

Junior A-Level Paper **Fall 2012.**

1. The decimal representation of an integer uses only two different digits. The number is at least 10 digits long, and any two neighbouring digits are distinct. What is the greatest power of two that can divide this number?

Solution. The answer is 6. Let N be the given number. Consider the case when the number of digits of N is $2m$. Then N can be represented as $N = xyxy \dots xy = 1010 \dots 1 \times xy$. Since the first factor is odd, the greatest power of two that can divide N coincides with the greatest power of two that can divide xy, which is 6 (then $xy = 64 = 2^6$).

If N contains $2m + 1$ digits then $N = yxy \dots xy = y \cdot 10^{2m} + xy \dots xy$ where $m \geq 5$ and therefore $2m \geq 10$. Then $y \cdot 10^{2m}$ is divisible by 2^7; therefore if N is divisible by 2^7 so is $xy \dots xy$, but it is divisible by at most 2^6 as is shown above. Hence the answer is 6.

2. Chip and Dale play the following game. Chip starts by splitting 222 nuts between two piles, so Dale can see it. In response, Dale chooses some number N from 1 to 222. Then Chip moves nuts from the piles he prepared to a new (third) pile until there will be exactly N nuts in any one or two piles. When Chip accomplishes his task, Dale gets an exact amount of nuts that Chip moved. What is the maximal number of nuts that Dale can get for sure, no matter how Chip acts?

(Naturally, Dale wants to get as many nuts as possible, while Chip wants to lose as little as possible).

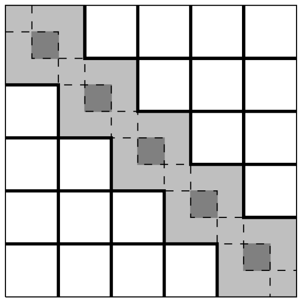
Solution The answer is 37. *Upper estimate.* If Chip puts 74 and 148 nuts in the piles, he can move not more 37 nuts for any N. Indeed, represent N as $74k + r$ where k equals 0, 1, 2, or 3, and $-37 \leq r < 37$. If $r = 0$ then $k > 0$, N equals 74, 148, or 222, and this is the number of nuts in one or two piles. If $r > 0$ then $74k$ equals 0, 74, or 148. Then Chip moves r nuts from an appropriate pile to a new pile and presents this pile to Dale, adding if necessary the pile that was not changed.

Lower estimate. For any initial splitting, there exists N such that at least 37 nuts must be moved. Indeed, let the numbers of nuts in the initial piles be p and q, $q \geq p$. If $p \geq 74$ then for $N = 37$ it is necessary to move 37 nuts. If $p < 74$ then $q > 148$. For $N = 111$ it is necessary either to add more than 37 to p nuts or to remove more than 37 from q nuts.

3. Some cells of a 11×11 table are filled with pluses. It is known that the total number of pluses in the given table and in any of its 2×2 sub-tables is even. Prove that the total number of pluses on the main diagonal of the given table is also even.

(2×2 sub-table consists of four adjacent cells, four cells around a common vertex).

Solution. Let us split the given square into 2×2 squares and a grey diagonal part as shown in the picture.



Since the given 11×11 square as well as any 2×2 square contains an even number of pluses, the diagonal part contains an even number of pluses.

We can obtain the main diagonal from the diagonal part by excluding two diagonal rows of 2×2 squares shown in dashed and including the double sum of every second square in the main diagonal (shown in dark grey).

Since the number of pluses in the part we include and in the part we exclude is even, the number of pluses in the main diagonal is also even.

4. Given a triangle ABC. Suppose I is its incentre, and X, Y, Z are the incentres of triangles AIB, BIC and AIC respectively. The incentre of triangle XYZ coincides with I. Is it necessarily true that triangle ABC is regular?

Solution. The answer is yes. Suppose K is the point of intersection of segments XY and BI, L is the intersection of YZ and CI, and M is the intersection of XZ and AI. By condition, segment XI bisects angles KIM and KXM, hence triangles IKX and IMX are congruent. Similarly, triangles IKY and ILY are congruent as well as ILZ and IMZ. Hence $\angle IKY = \angle ILY = 180° - \angle ILZ = 180° - \angle IMZ = \angle IMX = \angle IKX$, thus $BI \perp XY$.

In triangle XBY, segment BK is a bisector and an altitude, hence it is also a median, so line BI is the midperpendicular for segment XY. Hence $\angle XIK = \angle YIK$. But $\angle XIK = 1/2\angle AIB = 1/2(90° + 1/2\angle C)$. Similarly $\angle YIK = 1/2(90° + 1/2\angle A)$, thus $\angle A = \angle C$. Similarly $\angle A = \angle B$. Hence the triangle ACB is equilateral.

5. A car rides along a circular track in the clockwise direction. At noon Peter and Paul took their positions at two different points of the track. Some moment later they simultaneously ended their duties and compared their notes. The car passed each of them at least 30 times. Peter noticed that each circle was passed by the car 1 second faster than the preceding one while Paul's observation was opposite: each circle was passed 1 second slower than the preceding one.

Prove that their duty was at least an hour and a half long.

Solution. Each observer has noticed at least 29 circles. For Peter, the car passed consecutive circles in $m+14, m+13, \dots, m-14$ seconds, and for Paul in $p-14, p-13, \dots, p+14$ seconds. The total time for passing 29 circles is equal to $29m$ and $29p$ respectively. First 15 Paul's circles cover 14 Peter's circles, either from 1st to 14th (if the car passed Peter for the first time before Paul), or from 2nd to 15th (otherwise). In any case

$$(p-14)+(p-13)+\cdots+p > (m+13)+(m+12)+\cdots+m.$$

On the other hand, the last 15 Paul's circles cover 14 Peter's circles, either from 16th to 29th, or from 15th to 28th, hence

$$(m-14)+(m-13)+\cdots+m > (p+13)+(p+12)+\cdots+p.$$

Summing up the inequalities and collecting terms, we get $p+m > 392$, hence $29p+29m > 29 \cdot 392$. Thus the total time for at least one observer is at least $29 \cdot 196 = 5684$. This is greater than an hour and a half (5400 seconds).

6. **(a)** A point A is marked inside a circle. Two perpendicular lines drawn through A intersect the circle at four points. Prove that the centre of mass of these four points does not depend on the choice of the lines.

(b) A regular $2n$-gon ($n \geq 2$) with centre A is drawn inside a circle (A does not necessarily coincide with the centre of the circle). The rays going from A to the vertices of the $2n$-gon mark $2n$ points on the circle. Then the $2n$-gon is rotated about A. The rays going from A to the new locations of vertices mark new $2n$ points on the circle. Let O and N be the centres of gravity of old and new points respectively. Prove that $O = N$.

Solution. If A coincides with O, the assertion is obvious. Otherwise we will prove that the centre of mass is the midpoint of the segment OA.

a) Two perpendicular lines cut off two perpendicular chords. The centre of mass of a chord is its midpoint. If one of two chords is a diameter then the midpoint of the other one is A, so the centre of mass is the midpoint of OA. Otherwise let B and C be the midpoints of the chords. Then $OABC$ is a rectangle,and the centre of mass is the midpoint of BC which coincides with the midpoint of OA.

b) Connect the $2n$ points on the circle with A to obtain n chords such that the angle between two neighbouring ones is $180°/n$. The centre of mass of the endpoints of chords is the same as the centre of mass of their midpoints. These midpoints belong to a smaller circle with diameter OA. Equal inscribed angles correspond to equal arcs, thus these midpoints are vertices of a regular polygon inscribed in the smaller circle. Hence their centre of mass is the centre of this circle, that is, the midpoint of OA.

7. Peter and Paul play the following game. First, Peter chooses some positive integer a with the sum of its digits equal to 2012. Paul wants to determine this number; he knows only that the sum of the digits of Peter's number is 2012. On each of his moves Paul chooses a positive integer x and Peter tells him the sum of the digits of $|x - a|$. What is the minimal number of moves in which Paul can determine Peter's number for sure?

Solution. The answer is 2012. Let $S(n)$ be the sum of the digits of n.

Algorithm. At the first step, Paul chooses 1. If a ends with k zeroes then $S(a-1) = 2011+9k$. Thus Paul gets to know the position of the rightmost nonzero digit in a. Set $a_1 = a - 10^k$. Paul knows that $S(a_1) = 2011$. At the second step Paul chooses x such that $a - x = a_1 - 1$ and gets to know the number m of zeroes at the end of a_1. Set $a_2 = a_1 - 10^m$ and so on. After the 2012th step Paul obtains $S(a_{2012}) = 0$, thus having determined a.

Estimate. Suppose all digits in a are 0 and 1, that is, $a = 10^{k_{2012}} + 10^{k_{2011}} + \cdots + 10^{k_1}$ where $k_{2012} > k_{2011} > \cdots > k_1$. It is possible that at the first step Paul chooses an integer $x < 10^{k_1}$. Then $S(a - x) = S(10^{k_1} - x) + 2011$ independently of values of $k_{2012}, \ldots, k_{i+1}$. So Paul gets no new information about $k_{2012}, \ldots, k_{i+1}$. Similarly, it is possible that at the second step Paul chooses an integer smaller than 10^{k_2}, and so on. Then after 2011 steps Paul does not know k_{2012}.