International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Spring 2011.**

1. The faces of a convex polyhedron are similar triangles. Prove that this polyhedron has two pairs of congruent faces.

2. Worms grow at the rate of 1 metre per hour. When they reach their maximum length of 1 metre, they stop growing. A full-grown worm may be dissected into two new worms of arbitrary lengths totalling 1 metre. Starting with 1 full-grown worm, can one obtain 10 full-grown worms in less than 1 hour?

3. Along a circle are 100 white points. An integer k is given, where $2 \leq k \leq 50$. In each move, we choose a block of k adjacent points such that the first and the last are white, and we paint both of them black. For which values of k is it possible for us to paint all 100 points black after 50 moves?

4. Four perpendiculars are drawn from four vertices of a convex pentagon to the opposite sides. If these four lines pass through the same point, prove that the perpendicular from the fifth vertex to the opposite side also passes through this point.

5. In a country, there are 100 towns. Some pairs of towns are joined by roads. The roads do not intersect one another except meeting at towns. It is possible to go from any town to any other town by road. Prove that it is possible to pave some of the roads so that the number of paved roads at each town is odd.

Note: The problems are worth 3, 4, 4, 5 and 5 points respectively.

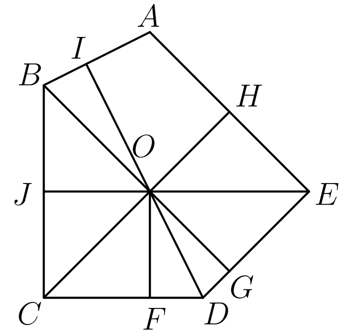
Solution to senior O-Level Spring 2011

1. If each face is an equilateral triangle, then all faces must be congruent. Since a polyhedron must have at least four faces, we easily have two pairs of congruent faces. Suppose the faces are not equilateral triangles. Then not all sides of the polyhedron are of the same length. Pick a side which is the longest. Then the two triangles sharing this side must be congruent in order to be similar. Otherwise, blowing one of them up to be larger will produce a side of greater length. Similarly, the two triangles sharing a side which is the shortest are also congruent. If the longest side and the shortest side appear in the same face, then all faces must again be congruent in order to be similar.

2. We divide 1 metre into 1024 sillimetres and 1 hour into 1024 sillihours. Then an under-sized worm grows at the rate of 1 sillimetre per sillihour. At the start, we cut the full-grown worm into two, one of length 1 sillimetre and the other 1023 sillimetres. After 1 sillihour, the shorter worm has length 2 sillimetres and the longer worm is full-grown. It is then dissected into two, so that the shorter one is also of length 2 sillimetres. After another 2 sillihours, the shorter worms have length 4 sillimetres and the longer worm is full-grown. It is then dissected into two so that the shorter one is also of length 4 sillimetres. Continuing in this manner, we will have 10 full-grown worms after $1+2+4+\cdots+512 = 1023$ sillihours, just beating the deadline of 1 hour.

3. For each k, $2 \leq k \leq 50$, construct a graph with the 100 points as vertices. Two vertices are joined by an edge if and only if they are the end vertices of a block of k adjacent vertices. In each move, we paint the two vertices of some edge. Now the graph may be a cycle or a union of disjoint cycles of the same length. If the length is even, choose every other edge and paint the vertices of the chosen edges. Then all 100 points are painted. If the length is odd, then there is an unpaintable vertex on each cycle. The number of cycles is given by the greatest common divisor d of 100 and $k-1$, which may be any of 1, 2, 4, 5, 10, 20 and 25. The respective lengths are 100, 50, 25, 20, 10, 5 and 4. So the only cases for which the task fails is when $d = 4$ or 20. The former yields $k = 5, 9, 13, 17, 25, 29, 33, 37, 45$ and 49. and the latter yields $k = 21$ or 41. For all other values of k, $2 \leq k \leq 50$, the task is possible.

4. **Solution by Adrian Tang.**
Let $ABCDE$ be the pentagon. Let BG, CH, DI and EJ be the altitudes concurrent at O. Let F be the foot of perpendicular from O to CD and let F' be the foot of perpendicular from A to CD.



By Pythagoras' Theorem,

$$\begin{aligned} BD^2 - OD^2 &= (BG^2 + GD^2) - (OG^2 + GD^2) \\ &= BG^2 - OG^2 \\ &= (BG^2 + GE^2) - (OG^2 + GE^2) \\ &= BE^2 - OE^2. \end{aligned}$$

Consideration of the vertices B, C, D and E in turn yields

$$\begin{aligned} OE^2 - OD^2 &= BE^2 - BD^2, \\ OA^2 - OE^2 &= CA^2 - CE^2, \\ OB^2 - OA^2 &= DB^2 - DA^2, \\ OC^2 - OB^2 &= EC^2 - EB^2. \end{aligned}$$

Adding and simplifying, we have $OC^2 - OD^2 = CA^2 - DA^2$. Now

$$\begin{aligned} F'C^2 - F'D^2 &= (CA^2 - F'A^2) - (DA^2 - F'A^2) \\ &= CA^2 - DA^2 \\ &= OC^2 - OD^2 \\ &= (OF^2 + FC^2) - (OF^2 + FD^2) \\ &= FC^2 - FD^2. \end{aligned}$$

It follows that $F' = F$ and AF is the fifth altitude passing through O.

5. **First Solution.**
We say that a part of the country is connected if we can go from any town to any other town by roads within this part of the country. We prove by induction on n that the task is always possible if the number of towns is $2n$. For $n = 1$, there are 2 towns, and there must be a road between them. Paving this road completes the task. Assume that the task is possible up to a certain value of n. Consider the next case with 2 more towns. We define a tour as going from town to town along the roads, without visiting any town more than once. Since we have a finite number of tours, there is a longest one. With at least 4 towns, this tour consists of at least 2 roads. Let it start from A and continue onto B, C and so on. If we excommunicate town A, the rest of the country must still be connected. Otherwise, there will be some town Z which is linked to B only via A. Then it could have been added to the tour before A and make it longer. Suppose we excommunicate both A and B. If the rest of the country is connected, we can pave the road between A and B, and use the induction hypothesis to finish the task. Suppose that this is not the case. Let X be a town which is now inaccessible from C. It must have been linked to B, and possibly to A, but cannot be linked to some other town Y. Otherwise, we could have added Y and X before B instead of A, again lengthening what is supposed to be the longest tour. If we excommunicate all towns like X in addition to A and B, the rest of the country is connected. Pave all the roads from B to towns like X, including A. Then there is an odd number of paved roads at those towns. If this is also the case for B, we can finish the task using the induction hypothesis. If not, we can still finish the task by putting B back into the rest of the country.

Second Solution by Adrian Tang.
Let $\mathcal{F}$ be the set of towns with an odd number of paved roads and $\mathcal{G}$ be the set of towns with an even number of paved roads. Note that $|\mathcal{F}|$ is even at any time. Initially, $|\mathcal{F}| = 0$. If we have $|\mathcal{F}| = 100$ at some point, the task is accomplished. Suppose $|\mathcal{F}| < 100$. Then there are at least 2 towns A and B in $\mathcal{G}$. Since the graph is connected, there exists a tour from A to B, going along the roads without visiting any town more than once. Interchange the status of each road on this tour, from paved to unpaved and vice versa. (This is of course done on the planning map, before any actual paving is carried out.) Then A and B move from $\mathcal{G}$ to $\mathcal{F}$ while all other towns stay in $\mathcal{F}$ or $\mathcal{G}$ as before. Hence we can make $|\mathcal{F}|$ increase by 2 at a time, until it reaches 100.