International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Spring 2010.**[1]

1. Each of six baskets contains some pears, plums and apples. The number of plums in each basket is equal to the total number of apples in the other five baskets, and the number of apples in each basket is equal to the total number of pears in the other five baskets. Prove that the total number of fruit in the six baskets is a multiple of 31.

2. Karlsson and Lillebror are dividing a square cake. Karlsson chooses a point P of the cake which is not on the boundary. Lillebror makes a straight cut from P to the boundary of the cake, in any direction he chooses. Then Karlsson makes a straight cut from P to the boundary, at a right angle to the first cut. Lillebror will get the smaller of the two pieces. Can Karlsson prevent Lillebror from getting at least one quarter of the cake?

3. An angle is given in the plane, and a compass is the only available tool.
   (a) Use the compass the minimum number of times to determine if the angle is acute or obtuse.
   (b) Use the compass any number of times to determine if the angle is exactly 31°.

4. At a party, each person knows at least three other people. Prove that an even number of them, at least four, can sit at a round table such that each knows both neighbours.

5. On the blackboard are the squares of the first 101 positive integers. In each move, we can replace two of them by the absolute value of their differences. After 100 moves, only one number remains. What is the minimum value of this number?

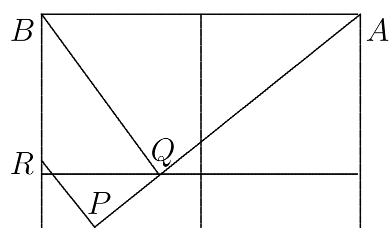
Note: The problems are worth 3, 3, 2+2, 5 and 5 points respectively.

[1] Courtesy of Andy Liu.

Solution to Junior O-Level Spring 2010

1. In counting the total number of apples, we have counted each pear five times. Hence the total number of apples is five times the total number of pears. Similarly, in counting the total number of plums, we have counted each apple five times, the total number of plums is five times the total number of apples, and twenty-five times the total number of pears. It follows that the total number of fruit is equal to the total number of pears times 1+5+25=31, and is therefore a multiple of 31.

2. Lillebror can always get at least one quarter of the cake. Imagine the cake divided into quadrants by a horizontal gridline and a vertical gridline. By symmetry, we may assume that P is in the southwest quadrant. Then Lillebror makes a straight cut towards the northeast corner A, intersecting the horizontal grid line at the point Q. Karlsson will cut towards a point R below the northwest corner B, so that Lillebror gets the quadrilateral $ABRP$. Now the circle with AB as diameter is tangent to the horizontal grid line. Hence Q is either on or outside this circle, so that $\angle BQA \le 90°$. Since $\angle RPA = 90°$, the segments BQ and RP cannot cross, so that triangle ABQ lies entirely within the quadrilateral $ABRP$. However, the area of ABQ is exactly one quarter that of the square. Karlsson's only way to keep Lillebror from getting more than one quarter of the cake is to choose P at the centre of the cake.



3. (a) **Solution by Olga Ivrii.**
   Let O be the vertex of the given angle. Let P be any point on one arm of the angle other than O. Draw a circle with centre P and radius OP. If the other arm is tangent to the circle, then the given angle is a right angle. If the other arm intersects the circle in two points, then the given angle is acute. If the other arm misses the circle, then the given angle is obtuse. Hence the task can be accomplished using the compass only once.

   (b) **Solution by Wen-Hsien Sun.**
   Let O be the vertex of the given angle. Draw a circle Ω with centre O and arbitrary radius, cutting the two arms of the angle at A_0 and A_1 respectively. Using A_1A_2 as radius, mark off on Ω successive points $A_2,\ A_3,\ \ldots$ so that $A_0A_1 = A_1A_2 = A_2A_3 = \cdots$. Then $\angle A_0OA_1 = 31°$ if and only if $A_{360} = A_0$ but $A_k \ne A_0$ for $1 \le k \le 359$, and we have gone around Ω exactly 31 times.

4. Construct a graph where the vertices represent the people, and two vertices are joined by an edge if and only if the people they represent know each other. We have to show that there exists an even cycle. We use mathematical induction on the number of vertices. The base with four vertices is trivial since we must have a complete graph. In general, since the average degree of each vertex is at least three, there are more edges than vertices, so that the graph must contain a cycle $\mathcal{C}$. If it is an even cycle, there is nothing further to prove. Suppose it is an odd cycle. Select any vertex V on $\mathcal{C}$. Since the degree of V is at least three, it is incident with an edge e not on $\mathcal{C}$. We leave $\mathcal{C}$ from V along e and try to return to any vertex on $\mathcal{C}$ without using any edge twice. There are three possibilities.

Case 1. Suppose it is impossible to return to $\mathcal{C}$.
Then the removal of e will separate the graph into two components. We identify the other endpoint of e with V, obtaining a graph with one less vertex which nevertheless satisfies the condition that each vertex is of degree at least three. By the induction hypothesis, this graph has an even cycle. Since the removal of V will also separate this graph into two components, the cycle cannot pass through V, so that it must lie entirely on one side of it. Hence the same cycle is in the original graph before the contraction.
Case 2. Suppose we return to $\mathcal{C}$ for the first time at another vertex U.
Then there are three disjoint paths joining U and V, and two of them will form an even cycle.
Case 3. We always return to $\mathcal{C}$ at the vertex V.
This has to happen with every vertex on $\mathcal{C}$, as otherwise we can leave $\mathcal{C}$ from a different vertex so that either Case 1 or Case 2 applies. Hence each vertex in $\mathcal{C}$ is incident with at least two edges not on $\mathcal{C}$. Let d and f be the edges on $\mathcal{C}$ incident with V. We identify the other endpoint of d with V and remove f, obtaining a graph with one less vertex which nevertheless satisfies the condition that each vertex is of degree at least three. By the induction hypothesis, the resulting graph has an even cycle. Since the removal of V will separate this graph into three components, the cycle cannot pass through V, so that it must lie entirely on one of the three pieces. Hence the same cycle is in the original graph before the contraction.

5. There are 51 odd numbers and 50 even numbers on the blackboard. Each move either keeps the number of odd numbers unchanged, or reduces it by 2. It follows that the last number must be odd, and its minimum value is 1. The squares of four consecutive integers can be replaced by a 4 because $(n+2)^2+(n-1)^2-(n+1)^2-n^2=4$. Hence the squares of eight consecutive integers can be replaced by 0. Taking the squares off from the end eight at a time, we may be left with 1, 4, 9, 16 and 25. However, the best we can get out of these five numbers is 3. Hence we must include 36, 49, 64, 81, 100, 121, 144 and 169. The sequence of combinations may be $169-144=25$, $25-25=0$, $100-0=100$, $100-64=36$, $36-36=0$, $121-0=121$, $121-81=40$, $49-40=9$, $16-9=7$, $9-7=2$, $4-2=2$ and $2-1=1$.