International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2010.**[1]

1. There are 100 points on the plane. All 4950 pairwise distances between two points have been recorded.
   (a) A single record has been erased. Is it always possible to restore it using the remaining records?
   (b) Suppose no three points are on a line, and k records were erased. What is the maximum value of k such that restoration of all the erased records is always possible?
2. At a circular track, $2n$ cyclists started from some point at the same time in the same direction with different constant speeds. If any two cyclists are at some point at the same time again, we say that they meet. No three or more of them have met at the same time. Prove that by the time every two cyclists have met at least once, each cyclist has had at least n^2 meetings.
3. For each side of a given polygon, divide its length by the total length of all other sides. Prove that the sum of all the fractions obtained is less than 2.
4. Two dueling wizards are at an altitude of 100 above the sea. They cast spells in turn, and each spell is of the form "decrease the altitude by a for me and by b for my rival" where a and b are real numbers such that $0 < a < b$. Different spells have different values for a and b. The set of spells is the same for both wizards, the spells may be cast in any order, and the same spell may be cast many times. A wizard wins if after some spell, he is still above water but his rival is not. Does there exist a set of spells such that the second wizard has a guaranteed win, if the number of spells is
   (a) finite;
   (b) infinite?
5. The quadrilateral $ABCD$ is inscribed in a circle with center O. The diagonals AC and BD do not pass through O. If the circumcentre of triangle AOC lies on the line BD, prove that the circumcentre of triangle BOD lies on the line AC.
6. Each cell of a 1000×1000 table contains 0 or 1. Prove that one can either cut out 990 rows so that at least one 1 remains in each column, or cut out 990 columns so that at least one 0 remains in each row.
7. A square is divided into congruent rectangles with sides of integer lengths. A rectangle is important if it has at least one point in common with a given diagonal of the square. Prove that this diagonal bisects the total area of the important rectangles.

Note: The problems are worth 2+3, 6, 6, 2+5, 8, 12 and 14 points respectively.

[1]Courtesy of Andy Liu.

Solution to Senior A-Level Fall 2010

1. **Solution by Central Jury.**

   (a) This is not always possible. Suppose the record of the distance AB is lost. If the other 98 points all lie on a line ℓ, we cannot tell whether A and B are on the same side of ℓ or on opposite sides of ℓ. Thus the lost record cannot be restored from the remaining ones.

   (b) The answer is 96. Suppose 97 records are erased. All of them may be associated with a point A so that we only know the distances AB and AC, where B and C are 2 of the other 99 points. A does not lie on BC as no three of the 100 points lie on a line. Now we cannot determine whether A is on one side or the other side of the line BC. Suppose at most 96 records are erased. Construct a graph with 100 vertices representing the 100 points. Two vertices are joined by an edge if the record of the distance between the two points they represent is erased. The graph has at most 96 edges, and therefore at least 4 components. Take four vertices A, B, C and D, one from each component. The pairwise distances between the points A, B, C and D are on record, so that their relative position can be determined. For any other vertex P, it is in the same component with only one of these four vertices. Hence the distance between the point P and three of the points A, B, C and D are on record. This is enough to determine the position of the point relative to the points A, B, C and D. It follows that all erased records may be restored.

2. For $1 \le i \le 2n$, let the constant speed of cyclist C_i be v_i, where $v_1 < v_2 < \cdots < v_{2n}$. Let $u = \min\{v_2 - v_1, v_3 - v_2, \ldots, v_{2n} - v_{2n-1}\}$. Then $v_j - v_i \ge (j-i)u$ for all $j > i$. Let d be the length of the track. Then the meeting between the last pair of cyclists occurs at time $\frac{d}{u}$. Now C_i and C_j meet once in each time interval of length $\frac{d}{v_j - v_i}$. They would have met at least $j-i$ times by the time of the meeting of the last pair, because $(j-i)\frac{d}{v_j-v_i} \le \frac{d}{u}$. For C_i, this means at least $1+2+\cdots+(i-1)$ meetings with $C_1, C_2, \ldots, C_{i-1}$ and at least $1+2+\cdots+(2n-i)$ meetings with $C_{i+1}, C_{i+2}, \ldots, C_{2n}$. The total is at least

$$\frac{i(i-1)+(2n-i)(2n-i+1)}{2} = (i-n)(i-(n+1)) + n^2 \ge n^2.$$

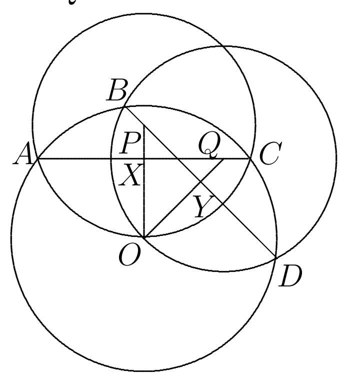
3. Let the side lengths be $a_1 \le a_2 \le \cdots \le a_n < p - a_n$ where $p = a_1 + a_2 + \cdots + a_n$. We have $\frac{a_1}{p-a_1} + \frac{a_2}{p-a_2} + \cdots + \frac{a_n}{p-a_n} \le \frac{a_1}{p-a_n} + \frac{a_2}{p-a_n} + \cdots + \frac{a_n}{p-a_n} = \frac{p}{p-a_n} < 2$ since $p > 2a_n$.

4. (a) The answer is no. With a finite number of spells, there is one for which $b-a$ is maximum. If the first wizard keeps casting this spell, the best that the second wizard can do is to maintain status quo by casting the same spell. Hence the second wizard will hit the water first, giving the first wizard a win..

   (b) The answer is yes. In the n-th spell, let $a = \frac{1}{n}$ and $b = 100 - \frac{1}{n}$. By symmetry, we may assume that the first wizard casts the n-th spell. He is then $100 - \frac{1}{n}$ above water while the second wizard is $\frac{1}{n}$ above water. However, the second wizard wins immediately by casting the $(n+1)$-st spell. He will still be $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$ above water while the first wizard is submerged in water since $(100 - \frac{1}{n}) - (100 - \frac{1}{n+1}) = -\frac{1}{n(n+1)}$.

5. Let P be the circumcentre of triangle OAC. Then PO is perpendicular to AC, intersecting AC at X. Let the line through O perpendicular to BD intersect BD at Y and AC at Q. We claim that Q is the circumcentre of triangle OBD. By Pythagoras' Theorem,

$$\begin{aligned} QD^2 &= QY^2 + DY^2 \\ &= QY^2 + (OD^2 - OY^2) \\ &= QY^2 + OA^2 - (OP^2 - PY^2) \\ &= OA^2 - AP^2 + PQ^2 \\ &= QX^2 + OA^2 - (AP^2 - PX^2) \\ &= QX^2 + (OA^2 - AX^2) \\ &= QX^2 + OX^2 \\ &= QO^2. \end{aligned}$$



6. **Solution by Brian Chen.**
Let $S(p,q)$ denote the following statement.
"In any binary $a \times b$ table with $ab \le p$, one of the following is true.
(A) There exists an $a \times q$ subtable with at least one 0 in each row.
(B) There exists a $q \times b$ subtable with at least one 1 in each column.
The q rows or columns may be chosen arbitrarily."
We wish to prove that $S(1000000, 10)$ is true. We first examine $S(4,1)$. The relevant tables are 1×4, 1×3, 1×2, 1×1, 2×1, 3×1, 4×1 and 2×2. In the first four, if there is at least one 0, then (A) is true. Otherwise, (B) is true. In the next three, if there is at least one 1, then (B) is true. Otherwise, (A) is true. In the last one, if there are no 0s in the first row, then (B) is true. Suppose there is at least one 0 in the first row. If there is at least one 0 in the second row as well, then (A) is true. Otherwise, (B) is true. It follows that $S(4,1)$ is true. We claim that $S(p,q)$ implies $S(4p, q+1)$. Consider any $a \times b$ table with $ab \le 4p$. Let x be the minimum number of 1s in any row and y be the minimum number of 0s in any column. Then the total number of 1s is at least ax and the total number of 0s is at least by. It follows that $ax + by \le ab$. By the Arithmetic-Geometric Means Inequality, $\sqrt{(ax)(by)} \le \frac{ax+by}{2} \le \frac{ab}{2}$. Hence $(ax)(by) \le \frac{(ab)^2}{4}$ so that $xy \le p$. Let R be a row with exactly x 1s and C be a column with exactly y 0s. Consider the $y \times x$ table whose rows have 0s at the intersections with C and whose columns have 1s at the intersections with R. We are assuming that $S(p,q)$ is true, so that either (A) or (B) holds for this table. If (A) holds, adding C would make (A) hold for the $a \times b$ table. If (B) holds, adding R would make (B) hold for the $a \times b$ table. This justifies the claim. From $P(4,1)$, we can deduce in turns $S(16,2)$, $S(64,3)$ and so on, up to $S(1048576, 10)$. This clearly implies $S(1000000, 10)$.

Solution by Central Jury.

Let us choose rows and columns one by one. We also will classify rows and columns as "good" and "bad": the "bad" column intersects all chosen rows by 0s and the "bad" row intersects all chosen columns by 1s. At the initial moment no row or column is chosen and therefore all rows and columns in the table are "bad".

Assume that the table contains at least as many 1s as 0s. Let us choose a row containing at least as many 1s as 0s. Then the columns that intersect this row at 1s will become "good" and therefore the number of "bad" columns decrease at least twice.

If in the table exists a row which has in the intersection with the "bad" columns at least as many 1s as 0s, we choose it. Then again some of the former "bad" columns become "good". Let us continue to choose rows in this way while it is possible. We arrive to the following scenarios:

1) We have chosen $m < 10$ rows and there are no more "bad" columns left. Then we add any rows to the chosen ones to have ten rows in total.

2) We have chosen 10 rows. Then we have no more than $1000 : 2^{10} < 1$ "bad" columns (so we have no "bad" columns left).

In both cases we constructed a subtable 10×1000, each column of which contains at least one 1s.

3) After choosing $m < 10$ rows we stopped: in intersection with the "bad" columns each other row has more 0s than 1s . Therefore, we have more 0s than 1s in "bad" columns. In this case let us restart our process, albeit choosing "bad" columns following the same principle as we used before: we choose column if in the intersection with the "bad" rows it contains at least as many 0s as 1s.

Thus we either constructed a subtable 1000×1000 each row of which contains at least one 0s or we have a situation when "bad" columns and "bad" rows left. However this is impossible.

Really, consider a subtable which is intersection of "bad" columns and "bad" rows. On one hand in this subtable the number of 0s is no less than half (if we count them by columns) while on the other hand it is less than half (if we count them by rows). Contradiction.

7. **Solution by Central Jury.**
Let the rectangles be of dimensions $m \times n$ or $n \times m$. Divide the whole square into unit squares. Put the label 0 on each square on the chosen diagonal. Put the label 1 on each square of the next $m + n - 1$ diagonals above and parallel to the given diagonal, and put the label -1 on each square of the next $m + n - 1$ diagonals below and parallel to the given diagonal. For each important rectangle, all squares have been labelled, and their sum is equal to the area of the part of the rectangle above the given diagonal minus the area of the part below. For each unimportant rectangle, at least one square is unlabelled. We now proceed to complete the labelling, diagonal by diagonal away from and parallel to the given one. Place an $m \times n$ or $n \times m$ rectangle on the board so that the square we are trying to label is the only unlabelled square in that rectangle. We choose a label so that the sum of all the labels in this rectangle is 0. Because labels on diagonals parallel to the given one are the same, the choice of position or orientation of this rectangle is immaterial. The completed labelling of a 12×12 board, with $m = 2$ and $n = 3$, is shown below.

0	1	1	1	1	−5	7	−5	1	1	1	−5
−1	0	1	1	1	1	−5	7	−5	1	1	1
−1	−1	0	1	1	1	1	−5	7	−5	1	1
−1	−1	−1	0	1	1	1	1	−5	7	−5	1
−1	−1	−1	−1	0	1	1	1	1	−5	7	−5
5	−1	−1	−1	−1	0	1	1	1	1	−5	7
−7	5	−1	−1	−1	−1	0	1	1	1	1	−5
5	−7	5	−1	−1	−1	−1	0	1	1	1	1
−1	5	−7	5	−1	−1	−1	−1	0	1	1	1
−1	−1	5	−7	5	−1	−1	−1	−1	0	1	1
−1	−1	−1	5	−7	5	−1	−1	−1	−1	0	1
5	−1	−1	−1	5	−7	5	−1	−1	−1	−1	0

Each label off the given diagonal is the negative of the label symmetric to it about the given diagonal. It follows that the sum of all the labels in the whole square is 0. The sum of the labels in each unimportant rectangle is chosen to be 0. Hence the sum of the labels in all the important rectangles is also 0. This means that the given diagonal bisects their total area.