International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2010.**[1]

1. There are 100 points on the plane. All 4950 pairwise distances between two points have been recorded.
   (a) A single record has been erased. Is it always possible to restore it using the remaining records?
   (b) Suppose no three points are on a line, and k records were erased. What is the maximum value of k such that restoration of all the erased records is always possible?
2. At a circular track, $2n$ cyclists started from some point at the same time in the same direction with different constant speeds. If any two cyclists are at some point at the same time again, we say that they meet. No three or more of them have met at the same time. Prove that by the time every two cyclists have met at least once, each cyclist has had at least n^2 meetings.
3. For each side of a given polygon, divide its length by the total length of all other sides. Prove that the sum of all the fractions obtained is less than 2.
4. Two dueling wizards are at an altitude of 100 above the sea. They cast spells in turn, and each spell is of the form "decrease the altitude by a for me and by b for my rival" where a and b are real numbers such that $0 < a < b$. Different spells have different values for a and b. The set of spells is the same for both wizards, the spells may be cast in any order, and the same spell may be cast many times. A wizard wins if after some spell, he is still above water but his rival is not. Does there exist a set of spells such that the second wizard has a guaranteed win, if the number of spells is
   (a) finite;
   (b) infinite?
5. The quadrilateral $ABCD$ is inscribed in a circle with center O. The diagonals AC and BD do not pass through O. If the circumcentre of triangle AOC lies on the line BD, prove that the circumcentre of triangle BOD lies on the line AC.
6. Each cell of a 1000×1000 table contains 0 or 1. Prove that one can either cut out 990 rows so that at least one 1 remains in each column, or cut out 990 columns so that at least one 0 remains in each row.
7. A square is divided into congruent rectangles with sides of integer lengths. A rectangle is important if it has at least one point in common with a given diagonal of the square. Prove that this diagonal bisects the total area of the important rectangles.

Note: The problems are worth 2+3, 6, 6, 2+5, 8, 12 and 14 points respectively.

[1]Courtesy of Andy Liu